A WEAK-STRONG UNIQUENESS PRINCIPLE FOR THE MULLINS–SEKERKA EQUATION

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ABSTRACT. We establish a weak-strong uniqueness principle for the two-phase Mullins–Sekerka equation in the plane: As long as a classical solution to the evolution problem exists, any weak De Giorgi type varifold solution (see for this notion the recent work of Stinson and the second author, Arch. Ration. Mech. Anal. **248**, 8, 2024) must coincide with it. In particular, in the absence of geometric singularities such weak solutions do not introduce a mechanism for (unphysical) non-uniqueness. We also derive a stability estimate with respect to changes in the data. Our method is based on the notion of relative entropies for interface evolution problems, a reduction argument to a perturbative graph setting (which is the only step in our argument exploiting in an essential way the planar setting), and a stability analysis in this perturbative regime relying crucially on the gradient flow structure of the Mullins–Sekerka equation.

CONTENTS

1. Introduction

1.1. **Context.** This work addresses the uniqueness and stability properties of the Mullins–Sekerka equation, a free boundary problem modeling volume-preserving phase separation and coarsening processes for various physical situations. The model was first introduced by Mullins and Sekerka in [22] and can be derived from thermodynamic principles [11]. It also arises as the sharp-interface limit of the Cahn–Hilliard equation ([1], [3] and [20]).

The Mullins–Sekerka equation can be viewed as a geometric third-order partial differential equation. This viewpoint is useful for constructing classical solutions ([28] and [5]), studying long-term asymptotics via a center manifold [6], and it also directly predicts the typical scaling behavior of the induced coarsening process. Naturally, as a higher-order evolution equation, the Mullins–Sekerka equation does not admit a comparison principle, which makes uniqueness questions subtle. Additionally, certain symmetric singularities can lead to physical non-uniqueness. However, it is conjectured that before the onset of singularities, the evolution is unique. We prove this uniqueness property in a large class of weak solutions by controlling a suitable relative entropy functional. As our relative entropy controls the distance of the weak and strong solutions, we additionally derive the stability of solutions with respect to perturbations of their initial data.

Our viewpoint is variational, interpreting the Mullins–Sekerka equation as a gradient flow. Albeit a well-appreciated fact in the community and obvious for smooth solutions, the rigorous definition and an existence proof of weak solutions based on the gradient flow structure has only been given recently by Stinson and the second author [14]. In the present work, we show that it is this very structure that guarantees the weak-strong uniqueness of the flow. It is natural that the gradient flow structure of the Mullins–Sekerka equation plays a crucial role in its uniqueness properties as it also lies at the heart of many other properties of the equation, such as the long-term asymptotics ([15] and [23]) or its (formal) recovery as a sharp interface limit of the Cahn–Hilliard equation [20].

Next to [14], there are several other weak solution concepts for the Mullins–Sekerka equation. The pioneering work [21] of Luckhaus and Sturzenhecker constructs weak solutions via an implicit time discretization. They rely on an assumption on the convergence of energies that was later be removed by Röger [24], using a result of Schätzle [25]. We point out that our results also apply to the well-known weak solutions constructed by Luckhaus and Sturzenhecker satisfying a sharp energy-dissipation inequality.

1.2. **Weak-strong uniqueness principles for interface evolution.** In general, there are two theoretical frameworks for proving weak-strong uniqueness for interface evolution equations. Evans and Spruck [7] and independently Chen, Giga and Goto [4] provided a notion of viscosity solution for mean curvature flow. This powerful framework relies on the comparison principle and is therefore only applicable to certain second-order geometric evolution equations.

More recently, the authors have developed a novel framework based on the dissipative nature of many interface evolution problems. The key idea is to monitor a suitable relative entropy functional that controls the distance between a weak and a strong solution. For the two-phase Navier–Stokes equation, this was carried out in [8], and for multiphase mean curvature flow in [9]. Also for other geometric motions that do not admit a geometric comparison principle, this method has been shown to be applicable, cf. ([17] and [18]), and it even gives new results for classical problems which admit a comparison principle ([13] and [19]).

In the present work, we face several novel difficulties due to the nonlocal structure of the Mullins–Sekerka equation. To overcome them, we first derive a preliminary version of our stability estimate of the relative entropy. When estimating the right-hand side, a key challenge is to define an appropriate chemical potential to compare the dissipation of the weak solution to the one of the strong solution. To this end, we distinguish two cases. If at a given time, the weak solution cannot be parametrized as a graph over the strong solution, we show that the dissipation of the weak solution dominates all other terms. On the other hand, at times when the weak solution can be written as a graph over the strong solution, we are in a perturbative setting. We construct an auxiliary chemical potential which is harmonic in the two phases of the weak solution but attains (an extension of) the mean curvature of the strong solution as its boundary condition on the weak interface. Finally, we use the stability of the Dirichlet-to-Neumann operator to derive a stability estimate for the relative entropy also in the perturbative case.

1.3. **Description of the model.** Let $d \in \{2,3\}$, let $T \in (0,\infty)$ be a finite time horizon, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary $\partial\Omega$, and let $A := \bigcup_{t\in[0,T)} A(t)\times\{t\}$ be the space-time track of an evolving family of subsets $A(t) \subset \Omega$, all of which are sets of finite perimeter in $\mathbb{R}^d$ such that it holds $\partial A(t)\cap\Omega = \partial^* A(t)\cap\Omega$ for all $t \in [0,T)$. We then consider the Mullins–Sekerka problem

$$-\Delta u = 0 \qquad \text{in } \bigcup_{t\in(0,T)} (\Omega \setminus \partial A(t)) \times \{t\}, \tag{1a}$$

$$-[\![(\mathrm{n}_{\partial A}\cdot\nabla)u]\!]\mathrm{n}_{\partial A} = \mathrm{V}_{\partial A} \qquad \text{on } \bigcup_{t\in(0,T)} (\partial A(t) \cap \Omega) \times \{t\}, \tag{1b}$$

$$u\mathrm{n}_{\partial A} = \mathrm{H}_{\partial A} \qquad \text{on } \bigcup_{t\in(0,T)} (\partial A(t) \cap \Omega) \times \{t\}, \tag{1c}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)u = 0 \qquad \text{on } \bigcup_{t\in(0,T)} \partial\Omega \times \{t\}. \tag{1d}$$

Here, for all $t \in [0,T)$ we denote by $\mathrm{n}_{\partial A}(\cdot,t)$ the unit normal vector field along the interface $\partial A(t)\cap\Omega$ pointing inside $A(t)$, by $\mathrm{H}_{\partial A}(\cdot,t)$ the mean curvature vector field along $\partial A(t)\cap\Omega$, by $\mathrm{V}_{\partial A}(\cdot,t)$ the normal velocity of $\partial A(t)\cap\Omega$, and by $\mathrm{n}_{\partial\Omega}$ the unit normal vector field along the domain boundary $\partial\Omega$ pointing inside Ω. Furthermore, for all $t \in [0,T)$ the quantity $[\![(\mathrm{n}_{\partial A}\cdot\nabla)u]\!](\cdot,t)$ denotes the jump of $(\mathrm{n}_{\partial A}\cdot\nabla)u$ across $\partial A(t)\cap\Omega$, with the orientation chosen such that for all sufficiently regular vector fields $f\colon \overline{\Omega} \to \mathbb{R}^d$ and scalar fields $\eta\colon \overline{\Omega} \to \mathbb{R}$ it holds (dropping for notational simplicity the time variable)

$$\begin{aligned} -\int_{\Omega\setminus\partial A} (\nabla\cdot f)\eta\,dx = \int_\Omega f\cdot\nabla\eta\,dx + \int_{\partial A\cap\Omega} [\![\mathrm{n}_{\partial A}\cdot f]\!]\eta\,d\mathcal{H}^{d-1} \\ + \int_{\partial\Omega} (\mathrm{n}_{\partial\Omega}\cdot f)\eta\,d\mathcal{H}^{d-1}. \end{aligned} \tag{2}$$

The associated energy is given by the interface area functional

$$E[A(t)] := \mathcal{H}^{d-1}(\partial A(t)\cap\Omega).$$

Assuming that the evolving interface intersects the domain boundary only orthogonally[1], we may compute at least on a formal level (dropping again for notational simplicity the time variable)

$$\begin{aligned}\frac{d}{dt}E[A] &= -\int_{\partial A\cap\Omega} \mathrm{H}_{\partial A}\cdot \mathrm{V}_{\partial A}\,d\mathcal{H}^{d-1}\\ &\stackrel{(1c)}{=} -\int_{\partial A\cap\Omega} u(\mathrm{n}_{\partial A}\cdot \mathrm{V}_{\partial A})\,d\mathcal{H}^{d-1}\\ &\stackrel{(1b)}{=} \int_{\partial A\cap\Omega} u[\![(\mathrm{n}_{\partial A}\cdot\nabla u)]\!]\,d\mathcal{H}^{d-1}\\ &\stackrel{(2),(1a),(1d)}{=} -\int_\Omega |\nabla u|^2\,dx,\end{aligned}$$

which reveals the formal gradient flow structure of the equation. Next to this energy dissipation relation, we also have formally

$$\begin{aligned}\frac{d}{dt}\int_A \zeta\,dx &= \int_A \partial_t\zeta\,dx - \int_{\partial A\cap\Omega}\zeta(\mathrm{n}_{\partial A}\cdot \mathrm{V}_{\partial A})\,d\mathcal{H}^{d-1}\\ &\stackrel{(1b)}{=} \int_A \partial_t\zeta\,dx + \int_{\partial A\cap\Omega}\zeta[\![(\mathrm{n}_{\partial A}\cdot\nabla u)]\!]\,d\mathcal{H}^{d-1}\\ &\stackrel{(2),(1a),(1d)}{=} \int_A \partial_t\zeta\,dx - \int_\Omega \nabla u\cdot\nabla\zeta\,dx\end{aligned}\tag{3}$$

for all sufficiently regular test functions $\zeta\colon \overline{\Omega}\times[0,T)\to\mathbb{R}$. In particular, it follows that the volume of the evolving phase is preserved under the flow:

$$\int_{A(t)} 1\,dx = \int_{A(0)} 1\,dx \quad \text{for all } t\in[0,T).$$

Definition 1 below provides a suitable weak formulation of the Mullins–Sekerka problem (1a)–(1d) in the framework of sets of finite perimeter and varifolds. Strong solutions of (1a)–(1d) will be modelled on the basis of a smoothly evolving phase $\mathscr{A} = \bigcup_{t\in[0,T^*)}\mathscr{A}(t)\times\{t\}$ with time horizon $T^*\in(0,T)$, where for each $t\in[0,T^*)$ the open set $\mathscr{A}(t)\subset\Omega$ is assumed to consist of finitely many simply connected components each of which is compactly supported within Ω.

2. Main result

Our main result establishes the uniqueness of classical solutions in a large class of weak solutions. As our method is a relative-entropy inequality, we additionally obtain stability for perturbations of the initial data.

Theorem 1 (Weak-strong uniqueness and quantitative stability for the Mullins–Sekerka equation)**.** *Let $d=2$. Consider $A=\bigcup_{t\in(0,T)}A(t)\times\{t\}$ and $\mu=(\mu_t)_{t\in(0,T)}$ such that (A,μ) is a De Giorgi type varifold solution for the Mullins-Sekerka problem (1a)–(1d) in the sense of Definition 1 with time horizon $T\in(0,\infty)$ and initial data $A(0)$. Furthermore, let a smoothly evolving phase $\mathscr{A}=\bigcup_{t\in[0,T_*)}\mathscr{A}(t)\times\{t\}$ with time horizon $T_*\in(0,T)$ be given, the motion law of which is subject to*

[1]At the level of a strong solution, we will always assume in the present work that the evolving interface is compactly supported within the domain. We will therefore be able to neglect almost all problems in connection with potential contact point dynamics.

Mullins–Sekerka flow (1a)–(1d). *We finally assume that for all* $t \in [0, T_*)$ *the open set* $\mathscr{A}(t) \subset \Omega$ *is compactly supported within* Ω.

There exists an error functional $(0, T_*) \ni t \mapsto E[A, \mu|\mathscr{A}](t) \in L^\infty_{loc}((0, T_*); [0, \infty))$ *between the two solutions such that*

$$
\begin{aligned}
&E[A, \mu|\mathscr{A}](t) = 0 \\
&\iff \mathcal{L}^2\big(A(t)\Delta\mathscr{A}(t)\big) = 0,\ \mu_t = \mathcal{H}^1\llcorner\big(\partial^* A(t) \cap \Omega\big) \otimes (\delta_{\mathrm{n}_{\partial^* A}(x,t)})_{x \in (\partial^* A(t) \cap \Omega)}
\end{aligned}
\tag{4}
$$

and there exists a non-negative map $C \in L^1_{loc}((0, T_*); [0, \infty))$ *such that for almost every* $T' \in (0, T_*)$ *there exists a constant* $C(T') > 0$ *such that*

$$
E[A, \mu|\mathscr{A}](T') \leq E[A, \mu|\mathscr{A}](0) + C(T') \int_0^{T'} E[A, \mu|\mathscr{A}](t)\, dt. \tag{5}
$$

In particular,

$$
\begin{aligned}
&E[A, \mu|\mathscr{A}](0) = 0 \\
&\implies \mathcal{L}^2\big(A(t)\Delta\mathscr{A}(t)\big) = 0,\ \mu_t = \mathcal{H}^1\llcorner\big(\partial^* A(t) \cap \Omega\big) \otimes (\delta_{\mathrm{n}_{\partial^* A}(x,t)})_{x \in (\partial^* A(t) \cap \Omega)} \\
&\quad \textit{for a.e. } t \in (0, T_*).
\end{aligned}
\tag{6}
$$

The class of weak solutions in which our weak-strong uniqueness statement holds is described in detail in the following definition. Those solutions were constructed recently by Stinson and the second author [14, Theorem 1 and Lemma 3]. These De Giorgi type varifold solutions are a suitable version of curves of maximal slope for general gradient flows, cf. [2], in the context of the Mullins–Sekerka equation. That means, they characterize the flow by an optimal energy dissipation inequality, see (7f) below. It is worth mentioning that any weak solution in the sense of Luckhaus and Sturzenhecker [21] that satisfies the sharp energy dissipation inequality is such a De Giorgi type varifold solution (with density $\rho = 1$ and equal potentials $u = w$), which means that our main result shows in particular the weak-strong uniqueness in this smaller class as well.

Definition 1 (De Giorgi type varifold solutions of the Mullins–Sekerka equation)**.** Let $d \in \{2, 3\}$, let $T \in (0, \infty)$ be a finite time horizon, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary $\partial\Omega$, let $A(0) \subset \Omega$ be an open subset of Ω such that $A(0)$ has finite perimeter in $\mathbb{R}^d$, and define the associated canonical oriented varifold $\mu_0 := \mathcal{H}^1\llcorner\big(\partial^* A(0) \cap \Omega\big) \otimes (\delta_{\mathrm{n}_{\partial^* A(0)}})_{x \in (\partial^* A(0) \cap \Omega)} \in \mathrm{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$.

Consider a family $(A(t))_{t \in (0,T)}$ of open subsets of Ω, define χ_A as the characteristic function associated with the space-time set $A := \bigcup_{t \in (0,T)} A(t) \times \{t\}$, and consider a family $\mu = (\mu_t)_{t \in (0,T)}$ of oriented varifolds $\mu_t \in \mathrm{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$, $t \in (0, T)$, such that μ_t is $(d{-}1)$-integer rectifiable for a.e. $t \in (0, T)$.[2] We call (A, μ) a *De Giorgi type varifold solution of the Mullins–Sekerka problem* (1a)–(1d) *with time horizon* T *and initial data* $A(0)$ if (χ_A, μ) is an admissible pair in the precise sense of [14, Definition 3] (with respect to $\alpha = \frac{\pi}{2}$) and

i) (Finite surface area and preserved mass) It holds

$$
\chi_A \in L^\infty(0, T; BV(\mathbb{R}^d; \{0, 1\})),\ \int_{A(t)} 1\, dx = \int_{A(0)} 1\, dx \quad \text{for a.e. } t \in (0, T); \tag{7a}
$$

[2] An oriented varifold μ is called integer rectifiable if its canonically associated general varifold $\widehat{\mu}$, cf. for an example (195), is integer rectifiable.

ii) (Kinetic potential) There exists a potential $u \in L^2(0,T;H^1(\Omega))$ with $\fint_\Omega u(\cdot,t)\,dx = 0$ for a.e. $t \in (0,T)$ such that

$$\begin{aligned}&\int_{A(T')} \zeta(\cdot,T')\,dx - \int_{A(0)} \zeta(\cdot,0)\,dx \\ &= \int_0^{T'} \int_{A(t)} \partial_t\zeta\,dxdt - \int_0^{T'} \int_\Omega \nabla u\cdot\nabla\zeta\,dxdt\end{aligned} \tag{7b}$$

for a.e. $T' \in (0,T)$ and all $\zeta \in C^\infty_{cpt}(\overline{\Omega}\times[0,T))$;

iii) (Curvature potential for mean curvature) there exists a potential $w \in L^2(0,T;H^1(\Omega))$ and $C_w = C_w(d,\Omega,T,A(0)) > 0$ such that it holds

$$\|w(\cdot,t)\|_{H^1(\Omega)} \le C_w(1 + \|(\nabla w)(\cdot,t)\|_{L^2(\Omega)}), \tag{7c}$$

$$\Delta w(\cdot,t) = 0 \text{ in } \Omega\setminus\overline{\partial^* A(t)}, \quad (\mathrm{n}_{\partial\Omega}\cdot\nabla)w(\cdot,t) = 0 \text{ on } \partial\Omega, \tag{7d}$$

for a.e. $t \in (0,T)$, and such that w satisfies the isotropic Gibbs–Thomson law (1c) in the sense that

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-\mathrm{p}\otimes\mathrm{p}) : \nabla B(x)\,d\mu_t(x,\mathrm{p}) = \int_{A(t)} \nabla\cdot\big(w(\cdot,t)B\big)\,dx \tag{7e}$$

for a.e. $t \in (0,T)$ and all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ such that $B\cdot\mathrm{n}_{\partial\Omega} = 0$ along $\partial\Omega$;

iv) (Energy dissipation inequality) and finally the energy dissipation property in the form of

$$E[\mu_{T'}] + \int_0^{T'}\int_\Omega \frac{1}{2}|\nabla u|^2\,dxdt + \int_0^{T'}\int_\Omega \frac{1}{2}|\nabla w|^2\,dxdt \le E[\mu_0] \tag{7f}$$

is satisfied for a.e. $T' \in (0,T)$, where the energy functional is defined by

$$E[\mu_t] := |\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) \tag{7g}$$

for all $t \in [0,T)$, where $|\mu_t|_{\mathbb{S}^{d-1}} \in \mathrm{M}(\overline{\Omega})$ denotes the mass measure of the oriented varifold μ_t.

Remark 2. That μ_t is indeed integer rectifiable for a.e. $t \in (0,T)$ in the precise sense of Footnote 2 — in contrast to the slightly weaker statement of [14, Definition 3, item i)], i.e., that the mass measure of μ_t is $(d-1)$-integer rectifiable for a.e. $t \in (0,T)$ — is in fact proven in [14, Proof of Theorem 1, Step 6].

3. Overview of the strategy

For the rest of the paper, we consider $d = 2$ and fix both a De Giorgi type varifold solution (A,μ) with initial data $A(0)$ and time horizon $T \in (0,\infty)$ and a smoothly evolving solution $\mathscr{A} = \bigcup_{t\in[0,T_*)} \mathscr{A}(t)\times\{t\}$ with $T_* \in (0,T)$ for the Mullins-Sekerka problem (1a)–(1d).

Recall the following structural assumptions on $\mathscr{A}$. For every $t \in (0,T_*)$, the phase $\mathscr{A}(t)$ consists of finitely many simply connected components (the number of which is constant in time), all of which are open sets with smooth boundary and being compactly supported within Ω. We also assume that there exists $\ell \in C^1_{loc}([0,T_*);(0,1))$ such that for every $t \in (0,T_*)$ the set $B_{\ell(t)}(\partial\mathscr{A}(t)) = \{x \in \mathbb{R}^d\colon \operatorname{dist}(x,\partial\mathscr{A}(t)) < \ell(t)\}$ is a regular tubular neighborhood for the interface $\partial\mathscr{A}(t) = \partial\mathscr{A}(t)\cap\Omega$ satisfying

$$B_{\ell(t)}(\partial\mathscr{A}(t)) \subset \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) > \ell(t)\}. \tag{8}$$

As usual, the nearest point projection onto $\partial\mathscr{A}(t)$, denoted by $P_{\partial\mathscr{A}}(\cdot,t)$, together with the associated signed distance function, denoted by $s_{\partial\mathscr{A}}(\cdot,t)$, yield a smooth change of variables

$$B_{\ell(t)}(\partial\mathscr{A}(t)) \ni x \mapsto (P_{\partial\mathscr{A}}(\cdot,t), s_{\partial\mathscr{A}}(\cdot,t)) \in \partial\mathscr{A}(t)\times(-\ell(t),\ell(t)), \tag{9}$$

where the signed distance function is oriented according to

$$\nabla s_{\partial\mathscr{A}} = \mathrm{n}_{\partial\mathscr{A}}. \tag{10}$$

Analogously, we assume that $\ell(t)$ is an admissible tubular neighborhood width for $\partial\Omega$ for all $t \in (0,T_*)$.

In order to introduce a suitable error functional between the two solutions A and $\mathscr{A}$, we will later choose extensions of the strong normal

$$\xi \in W^{1,\infty}_{loc}((0,T_*);W^{1,\infty}(\overline{\Omega};\mathbb{R}^2)) \cap L^{\infty}_{loc}((0,T_*);W^{2,\infty}(\overline{\Omega};\mathbb{R}^2)) \tag{11}$$

and the signed distance function, a weight (which one should think of as a truncated version of the signed distance function)

$$\vartheta \in W^{1,\infty}_{loc}((0,T_*);W^{1,\infty}(\overline{\Omega})) \cap L^{\infty}_{loc}((0,T_*);W^{2,\infty}(\overline{\Omega})) \tag{12}$$

satisfying as a bare minimum

$$\xi = \nabla s_{\partial\mathscr{A}} \text{ in } \bigcup_{t\in[0,T_*)} B_{\frac{\ell(t)}{2}}(\partial\mathscr{A}(t))\times\{t\}, \qquad |\xi| \le 1 \text{ on } \overline{\Omega}\times[0,T_*), \tag{13}$$

$$\vartheta = -\frac{s_{\partial\mathscr{A}}}{\ell^2} \text{ in } \bigcup_{t\in[0,T_*)} B_{\frac{\ell(t)}{2}}(\partial\mathscr{A}(t))\times\{t\}, \tag{14}$$

$$\vartheta < 0 \text{ in } \bigcup_{t\in[0,T_*)} \mathscr{A}(t)\times\{t\}, \qquad \vartheta > 0 \text{ in } \bigcup_{t\in[0,T_*)} \big(\Omega \setminus \overline{\mathscr{A}(t)}\big)\times\{t\}, \tag{15}$$

$$\xi = 0 \text{ in } \bigcup_{t\in[0,T_*)} \big(\Omega\setminus B_{\frac{3}{4}\ell(t)}(\partial\mathscr{A}(t))\big)\times\{t\}. \tag{16}$$

Based on such a pair (ξ,ϑ), we define for all $t \in [0,T_*)$

$$E_{rel}[A,\mu|\mathscr{A}](t) := E[\mu_t] - \int_{\partial^*A(t)\cap\Omega} \mathrm{n}_{\partial^*A}(\cdot,t)\cdot\xi(\cdot,t)\,d\mathcal{H}^1, \tag{17}$$

$$E_{vol}[A|\mathscr{A}](t) := \int_{A(t)\Delta\mathscr{A}(t)} |\vartheta(\cdot,t)|\,dx, \tag{18}$$

$$E[A,\mu|\mathscr{A}](t) := E_{rel}[A,\mu|\mathscr{A}](t) + E_{vol}[A|\mathscr{A}](t). \tag{19}$$

Defining the Radon–Nikodým derivative

$$\varrho_t := \frac{\mathcal{H}^{d-1}\llcorner(\partial^*A(t)\cap\Omega)}{|\mu_t|_{\mathbb{S}^1}\llcorner\Omega} \in [0,1], \tag{20}$$

it follows from [14, Definition 3, item ii)] that $\varrho_t = 0$ outside $\partial^*A(t)\cap\Omega$ and

$$\varrho_t \in (2\mathbb{N}-1)^{-1} \quad \text{on } \partial^*A(t)\cap\Omega, \tag{21}$$

so that we may rewrite the relative entropy (17) in the form of

$$\begin{aligned} E_{rel}[A,\mu|\mathscr{A}](t) = |\mu_t|_{\mathbb{S}^1}(\partial\Omega) + \int_{\Omega\cap\{\varrho_t\leq\frac{1}{3}\}} (1-\rho_t)\,d|\mu_t|_{\mathbb{S}^1} \\ + \int_{\partial^*A(t)\cap\Omega} (1-\mathrm{n}_{\partial^*A}\cdot\xi)(\cdot,t)\,d\mathcal{H}^1. \end{aligned} \tag{22}$$

Furthermore, due to (16) it follows again from [14, Definition 3, item ii)] that

$$E_{rel}[A,\mu|\mathscr{A}](t) = \int_{\overline{\Omega}\times\mathbb{S}^1} 1-\mathrm{p}\cdot\xi(x,t)\,d\mu_t(x,\mathrm{p}). \tag{23}$$

In order to suitably control the time evolution of the error functional $E[A,\mu|\mathscr{A}]$, a third construction will play a prominent role given by a vector field

$$B\in L^\infty_{loc}((0,T_*);W^{2,\infty}(\overline{\Omega};\mathbb{R}^2)) \tag{24}$$

satisfying at least

$$B=0 \text{ in } \bigcup_{t\in[0,T_*)} \big(\Omega\setminus B_{\frac{3}{4}\ell(t)}(\partial\mathscr{A}(t))\big)\times\{t\}. \tag{25}$$

One should think of B as an extension of the normal velocity of the smoothly evolving solution $\mathscr{A}$:

$$B(\cdot,t) = \mathrm{V}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(\cdot,t),t) \text{ on } B_{\frac{\ell(t)}{2}}(\partial\mathscr{A}(t)),\ t\in[0,T_*). \tag{26}$$

With these definitions in place, we will now give an overview on our strategy for the proof of Theorem 1.

3.1. **Preliminary relative entropy inequality.** The first ingredient is a preliminary estimate for the time evolution of the error functional $E[A,\mu|\mathscr{A}]$ defined in (19), just working under the minimal structural assumptions (11)–(16) and (24)–(25) for the associated constructions (ξ,ϑ,B). The goal here is to identify the structure of the terms we will need to estimate by the relative entropy.

Lemma 3 (Preliminary relative entropy inequality). *For a.e. $T'\in(0,T)$, it holds*

$$\begin{aligned} &E_{rel}[A,\mu|\mathscr{A}](T') \\ &\leq E_{rel}[A,\mu|\mathscr{A}](0) + \int_0^{T'} R_{dissip}(t)+R_{\partial_t\xi}(t)+R_{\nabla B}(t)+R_{varifold/BV}(t)\,dt \end{aligned} \tag{27}$$

and

$$E_{vol}[A|\mathscr{A}](T') = E_{vol}[A|\mathscr{A}](0) + \int_0^{T'} A_{dissip}(t)+A_{\partial_t\vartheta}(t)+A_{\nabla B}(t)\,dt, \tag{28}$$

where we defined

$$\begin{aligned} R_{dissip}(t) = &-\int_\Omega \frac{1}{2}|\nabla u(\cdot,t)|^2\,dx - \int_\Omega \frac{1}{2}|\nabla w(\cdot,t)|^2\,dx \\ &-\int_\Omega \nabla u(\cdot,t)\cdot\nabla(\nabla\cdot\xi)(\cdot,t)\,dx \\ &+\int_{\partial^*A(t)\cap\Omega} (B\cdot\mathrm{n}_{\partial^*A})(\cdot,t)\big(w+(\nabla\cdot\xi)\big)(\cdot,t)\,d\mathcal{H}^1, \end{aligned} \tag{29}$$

$$R_{\partial_t\xi}(t) = -\int_{\partial^*A(t)\cap\Omega} \big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^{\mathsf{T}}\xi\big)(\cdot,t)\cdot(\mathrm{n}_{\partial^*A}-\xi)(\cdot,t)\,d\mathcal{H}^1 \tag{30}$$

$$- \int_{\partial^* A(t)\cap\Omega} \big(\partial_t \xi + (B\cdot\nabla)\xi\big)(\cdot,t)\cdot\xi(\cdot,t)\,d\mathcal{H}^1,$$

$$R_{\nabla B}(t) = -\int_{\partial^* A(t)\cap\Omega} \big((\mathrm{n}_{\partial^* A}-\xi)\otimes(\mathrm{n}_{\partial^* A}-\xi)\big)(\cdot,t) : \nabla B(\cdot,t)\,d\mathcal{H}^1 - \int_{\partial^* A(t)\cap\Omega} (\mathrm{n}_{\partial^* A}\cdot\xi - 1)(\cdot,t)(\nabla\cdot B)(\cdot,t)\,d\mathcal{H}^1, \tag{31}$$

$$R_{varifold/BV}(t) = -\int_{\partial^* A(t)\cap\Omega} (\mathrm{Id} - \mathrm{n}_{\partial^* A}\otimes\mathrm{n}_{\partial^* A})(\cdot,t) : \nabla B(\cdot,t)\,d\mathcal{H}^1 + \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-\mathrm{p}\otimes\mathrm{p}) : \nabla B(\cdot,t)\,d\mu_t(\cdot,\mathrm{p}) \tag{32}$$

as well as

$$U_{dissip}(t) = -\int_\Omega \nabla u(\cdot,t)\cdot\nabla\vartheta(\cdot,t)\,dx + \int_{\partial^* A(t)\cap\Omega} \vartheta(\cdot,t)(\mathrm{n}_{\partial^* A}\cdot B)(\cdot,t)\,d\mathcal{H}^1, \tag{33}$$

$$U_{\partial_t\vartheta}(t) = \int_\Omega (\chi_A - \chi_{\mathscr{A}})(\cdot,t)(\partial_t\vartheta + (B\cdot\nabla)\vartheta)(\cdot,t)\,dx, \tag{34}$$

$$U_{\nabla B}(t) = \int_\Omega (\chi_A - \chi_{\mathscr{A}})(\cdot,t)\vartheta(\cdot,t)(\nabla\cdot B)(\cdot,t)\,dx. \tag{35}$$

3.2. **Time splitting argument: Definition of good and bad times.** In order to suitably estimate the right hand sides of (27) and (28), respectively, we essentially aim to reduce the task to a perturbative setting, where the interface of the weak solution can be represented as a graph over the interface of the smooth solution (with arbitrarily small C^1 norm). This reduction argument is implemented based on a case distinction for times $t \in (0,T)$, with the non-perturbative regime corresponding to either disproportionally large dissipation of energy of the weak solution or a lower bound for the error $E[A,\mu|\mathscr{A}]$. We formalize this as follows.

Let $\Lambda \in (0,\infty)$ and $M \in (1,\infty)$ constants we will determine later. We then construct a disjoint decomposition

$$(0,T_*) = \mathscr{T}_{\mathrm{bad}}(\Lambda,M) \cup \mathscr{T}_{\mathrm{good}}(\Lambda,M) \tag{36}$$

by means of

$$\mathscr{T}_{\mathrm{bad}}(\Lambda,M) := \mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda) \cup \mathscr{T}^{(2)}_{\mathrm{bad}}(\Lambda,M), \tag{37}$$

$$\mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda) := \Big\{ t \in (0,T_*) \colon \int_\Omega \frac{1}{2}|\nabla w(\cdot,t)|^2\,dx > \Lambda \Big\}, \tag{38}$$

$$\mathscr{T}^{(2)}_{\mathrm{bad}}(\Lambda,M) := \Big\{ t \in (0,T_*) \setminus \mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda) \colon E[A,\mu|\mathscr{A}](t) > \frac{\ell(t)}{M} \Big\}, \tag{39}$$

$$\mathscr{T}_{\mathrm{good}}(\Lambda,M) := (0,T_*) \setminus \mathscr{T}_{\mathrm{bad}}(\Lambda,M). \tag{40}$$

The merit of these definitions is that on one side, the estimate for the right hand sides of (27) and (28) is easily closed for times $t \in \mathscr{T}_{\mathrm{bad}}(\Lambda,M)$, whereas on the other side for times $t \in \mathscr{T}_{\mathrm{good}}(\Lambda,M)$ one can actually *prove* that one has to be in a perturbative setting; cf. Proposition 5 below. The latter is a powerful tool to estimate the (not yet fully unraveled) "non-local contributions" in the right hand sides of (27) and (28) originating from the $(\mathrm{V}_{\partial A}, \mathrm{H}_{\partial A}) \in H^{-1/2}_{MS}(\partial A\cap\Omega)\times H^{1/2}_{MS}(\partial A\cap\Omega)$ gradient flow structure of Mullins–Sekerka flow (1a)–(1d).

3.3. **Stability estimates at bad times.** In a first step, we will now estimate the right hand sides of (27) and (28) in the non-perturbative setting.

Lemma 4 (Stability estimates at bad times)**.** *For all $T' \in (0,T_*)$ there exists $\Lambda = \Lambda(\mathscr{A}, A(0), T') \in (0,\infty)$ such that for all $M \in (1,\infty)$ and all $t \in \mathscr{T}_{\mathrm{bad}}(\Lambda, M)\cap(0,T')$, it holds*

$$R_{dissip}(t) + R_{\partial_t\xi}(t) + R_{\nabla B}(t) + R_{varifold/BV}(t) \tag{41}$$

$$\leq \begin{cases} -\int_\Omega \frac{3}{8}|\nabla u(\cdot,t)|^2 + \frac{3}{8}|\nabla w(\cdot,t)|^2\,dx, & t \in \mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda), \\ -\int_\Omega \frac{3}{8}|\nabla u(\cdot,t)|^2 + \frac{1}{2}|\nabla w(\cdot,t)|^2\,dx + \frac{\Lambda}{4}\frac{M}{\ell(t)}E[A,\mu|\mathscr{A}](t), & \textit{else}, \end{cases}$$

and

$$\begin{aligned} &U_{dissip}(t) + U_{\partial_t\vartheta}(t) + U_{\nabla B}(t) \\ &\leq \begin{cases} \int_\Omega \frac{1}{8}|\nabla u(\cdot,t)|^2 + \frac{1}{8}|\nabla w(\cdot,t)|^2\,dx, & t \in \mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda), \\ \int_\Omega \frac{1}{8}|\nabla u(\cdot,t)|^2\,dx + \frac{\Lambda}{4}\frac{M}{\ell(t)}E[A,\mu|\mathscr{A}](t), & \textit{else}. \end{cases} \end{aligned} \tag{42}$$

3.4. **Stability estimates at good times.** As already said, a key ingredient for our approach to weak-strong uniqueness and stability of the Mullins–Sekerka problem (1a)–(1d) consists of a reduction argument to the interface of the weak solution being given as a graph over the interface of the smoothly evolving solution with arbitrarily small C^1 norm. It is a non-trivial result that our definition (40) of good times is actually a sufficient condition to imply such a perturbative graph setting, an implication which we formalize in the following result.

Proposition 5 (Perturbative graph setting at good times)**.** *Fix $T' \in (0,T_*)$, and let $\Lambda = \Lambda(\mathscr{A}, A(0), T') \in (0,\infty)$ be the constant from Lemma 4. For every $C \in (1,\infty)$ there exists a constant*

$$M = M(\Omega, A(0), \mathscr{A}, C, \Lambda, T') \in (1,\infty) \tag{43}$$

such that for almost every $t \in \mathscr{T}_{\mathrm{good}}(\Lambda, M)\cap(0,T')$ there exists a height function

$$h(\cdot,t) \in W^{2,p}(\partial\mathscr{A}(t)), \quad p \in [2,\infty), \tag{44}$$

such that

$$\partial^* A(t) = \big\{x + h(x,t)\mathrm{n}_{\partial\mathscr{A}}(x,t) \colon x \in \partial\mathscr{A}(t)\big\} \subset \{x \in \Omega\colon\ \mathrm{dist}(x,\partial\Omega) > \ell(t)\}, \tag{45}$$

$$|\mu_t|_{\mathbb{S}^1} = \mathcal{H}^1 \llcorner \partial^* A(t), \tag{46}$$

and

$$\frac{1}{\ell(t)}\|h(\cdot,t)\|_{L^\infty(\partial\mathscr{A}(t))} + \|\nabla^{tan} h(\cdot,t)\|_{L^\infty(\partial\mathscr{A}(t))} \leq \frac{1}{16C}. \tag{47}$$

We continue on how to close the stability estimate for times at which the previous result holds, treating it as a black-box. The first essential step consists of suitably rewriting the terms constituting R_{dissip} and U_{dissip} from (29) and (33), respectively.

To this end, one may draw inspiration from the gradient flow structure of Mullins–Sekerka flow (1a)–(1d), which, for the smoothly evolving $\partial\mathscr{A}$, is given as follows. Denote by $H^{1/2}_{MS}(\partial\mathscr{A})$ the set of all maps $f\colon \partial\mathscr{A} \to \mathbb{R}$ arising as traces of maps from $H^1(\mathscr{A})$ such that $\fint_{\partial\mathscr{A}} f\,d\mathcal{H}^1 = 0$. One then defines a Hilbert space structure on $H^{1/2}_{MS}(\partial\mathscr{A})$ by means of the minimal Dirichlet energy extension to Ω:

$$\langle f, \widetilde{f}\rangle_{H^{1/2}_{MS}(\partial\mathscr{A})} := \int_\Omega \nabla u_f \cdot \nabla u_{\widetilde{f}}\,dx, \tag{48}$$

where, for given $f \in H^{1/2}_{MS}(\partial\mathscr{A})$, the associated potential $u_f \in H^1(\Omega)$ satisfies

$$\begin{aligned} \Delta u_f &= 0 && \text{in } \Omega \setminus \partial\mathscr{A}, \\ u_f &= f && \text{on } \partial\mathscr{A}, \\ (\mathrm{n}_{\partial\Omega}\cdot\nabla)u_f &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Denoting by $H^{-1/2}_{MS}(\partial\mathscr{A})$ the dual of $H^{1/2}_{MS}(\partial\mathscr{A})$, it is immediate to recognize that the Riesz isomorphism $\mathcal{R}\colon H^{-1/2}_{MS}(\partial\mathscr{A}) \to H^{1/2}_{MS}(\partial\mathscr{A})$ is formally realized through the inverse of the (two-phase) Dirichlet-to-Neumann operator:

$$\langle F, f\rangle_{H^{-1/2}_{MS}(\partial\mathscr{A})\times H^{1/2}_{MS}(\partial\mathscr{A})} = \langle \mathcal{R}(F), f\rangle_{H^{1/2}_{MS}(\partial\mathscr{A})} = -\int_{\partial\mathscr{A}} (\mathrm{n}_{\partial\mathscr{A}}\cdot [\![\nabla u_{\mathcal{R}(F)}]\!]) f\,d\mathcal{H}^1,$$

as can be seen similarly to the computation (3). Note that the integration by parts is rigorous for those F such that $\mathcal{R}(F)$ is the trace of a, say, $H^2(\mathscr{A})$ function.

The idea for the rewriting of, say, R_{dissip} is now as follows. What one likes to achieve in (29) is to complete squares in the two quadratic terms originating from the energy dissipation inequality of a weak solution such that the resulting quadratic terms penalize the *difference of curvatures* of the two evolving geometries as measured through the correct gradient flow norm. This is indeed in direct analogy of how we proceeded in the case of (multiphase) mean curvature flow being the $L^2(\partial\mathscr{A})$ gradient flow of the perimeter; cf. Section 3.1 of [9]. The next result implements this procedure for Mullins–Sekerka flow.

Lemma 6 (Structure of dissipative terms in perturbative regime)**.** *Fix $t \in (0, T_*)$, and assume that for t the conclusions (44)–(47) of Proposition 5 hold true. Furthermore, assume that t is a Lebesgue point for $s \mapsto u(\cdot, s)$ in the sense that*

$$\lim_{\tau\searrow 0} \fint_{t-\tau}^{t+\tau} \int_\Omega |\nabla u - \nabla u(\cdot,t)|^2\,dxds = 0. \tag{49}$$

i) Defining a chemical potential $\widetilde{w} = \widetilde{w}(\cdot,t) \in H^1(\Omega)$ by means of

$$-\Delta\widetilde{w} = 0 \qquad \text{in } \Omega\setminus\partial^*A(t), \tag{50}$$

$$\mathrm{tr}_{\partial^*A(t)}\widetilde{w} = -(\nabla\cdot\xi)(\cdot,t) \qquad \text{on } \partial^*A(t)\cap\Omega, \tag{51}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)\widetilde{w} = 0 \qquad \text{on } \partial\Omega, \tag{52}$$

we obtain the following alternative representation for R_{dissip} from (29):

$$\begin{aligned} R_{dissip}(t) = &-\int_\Omega \frac{1}{2}|(\nabla u - \nabla\widetilde{w})(\cdot,t)|^2\,dx - \int_\Omega \frac{1}{2}|(\nabla w - \nabla\widetilde{w})(\cdot,t)|^2\,dx \\ &+ \int_{\partial^*A(t)\cap\Omega} (w-\widetilde{w})(\cdot,t)\big((B\cdot\mathrm{n}_{\partial^*A}) + [\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}]\!]\big)(\cdot,t)\,d\mathcal{H}^1. \end{aligned} \tag{53}$$

ii) Similarly, by defining another chemical potential $\widetilde{w}_\vartheta = \widetilde{w}_\vartheta(\cdot,t)$ through the problem

$$-\Delta\widetilde{w}_\vartheta = 0 \qquad \text{in } \Omega\setminus\partial^*A(t), \tag{54}$$

$$\mathrm{tr}_{\partial^* A(t)}\widetilde{w}_\vartheta = \vartheta(\cdot,t) \qquad \textit{on } \partial^* A(t)\cap\Omega, \tag{55}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)\widetilde{w}_\vartheta = 0 \qquad \textit{on } \partial\Omega, \tag{56}$$

we obtain that U_{dissip} *from* (33) *can be rewritten as follows:*

$$\begin{aligned} U_{dissip}(t) = &\int_{\partial^* A(t)\cap\Omega} \vartheta(\cdot,t)\big((B\cdot \mathrm{n}_{\partial^* A}) + [\![(\mathrm{n}_{\partial^* A}\cdot\nabla)\widetilde{w}]\!]\big)(\cdot,t)\,d\mathcal{H}^1 \\ &+ \int_{\partial^* A(t)\cap\Omega} (u-\widetilde{w})(\cdot,t)[\![(\mathrm{n}_{\partial^* A}\cdot\nabla)\widetilde{w}_\vartheta]\!](\cdot,t)\,d\mathcal{H}^1. \end{aligned} \tag{57}$$

The second, and most straightforward, step is to estimate the terms on the right hand sides of (27) and (28) that only feature local terms. That is, they only depend on the kinematics of the problem and do not need to be treated using the nonlocal gradient flow structure of Mullins-Sekerka flow via the dissipative terms from (53).

Lemma 7 (Stability estimate for local terms in perturbative regime)**.** *Fix* $t \in (0,T_*)$*, and assume that for* t *the conclusions* (44)–(47) *of Proposition* 5 *hold true. For any collection* (ξ,ϑ,B) *satisfying only the assumptions* (11)–(16) *and* (24)–(26)*, we have the estimates*

$$R_{\partial_t\xi}(t) = 0, \tag{58}$$

$$\begin{aligned} R_{\nabla B}(t) \le &\big\|(\nabla\cdot B)(\cdot,t)\big\|_{L^\infty(B_{\ell(t)/2}(\partial\mathscr{A}(t)))} E_{rel}[A,\mu|\mathscr{A}](t) \\ &+ \big\|(\nabla B)(\cdot,t)+(\nabla B)^\mathsf{T}(\cdot,t)\big\|_{L^\infty(B_{\ell(t)/2}(\partial\mathscr{A}(t)))} E_{rel}[A,\mu|\mathscr{A}](t), \end{aligned} \tag{59}$$

$$\begin{aligned} R_{varifold/BV}(t) \le &\big\|(\nabla\cdot B)(\cdot,t)\big\|_{L^\infty(B_{\ell(t)/2}(\partial\mathscr{A}(t)))} E_{rel}[A,\mu|\mathscr{A}](t) \\ &+ 4\big\|(\nabla B)(\cdot,t)+(\nabla B)^\mathsf{T}(\cdot,t)\big\|_{L^\infty(B_{\ell(t)/2}(\partial\mathscr{A}(t)))} E_{rel}[A,\mu|\mathscr{A}](t), \end{aligned} \tag{60}$$

as well as

$$U_{\partial_t\vartheta}(t) \le \frac{|\partial_t\ell(t)|}{\ell(t)} E_{vol}[A|\mathscr{A}](t), \tag{61}$$

$$U_{\nabla B}(t) \le \big\|(\nabla\cdot B)(\cdot,t)\big\|_{L^\infty(B_{\ell(t)/2}(\partial\mathscr{A}(t)))} E_{vol}[A|\mathscr{A}](t). \tag{62}$$

In a third and final step, we have to provide an estimate for the remaining terms given by (53) and (57). Focusing for the moment on R_{dissip}, the idea again draws heavily from the gradient flow structure of Mullins–Sekerka flow and goes as follows.

First, as already discussed above, the motivation for the argument leading to (53) is that $-\int_\Omega \frac{1}{2}|\nabla(w-\widetilde{w})|^2\,dx$ penalizes the difference of curvatures of the two solutions as measured through the gradient flow structure associated with Mullins–Sekerka flow. Indeed, $-\nabla\cdot\xi$ serves as a proxy for $H_{\partial\mathscr{A}}$ away from $\partial\mathscr{A}$ since ξ represents an extension of the unit normal vector field $\mathrm{n}_{\partial\mathscr{A}}$, and $w-\widetilde{w}$ is by construction of $\widetilde{w}$ and the Gibbs–Thomson law (7e) the minimal Dirichlet extension of $H_{\partial^* A\cap\Omega}+\nabla\cdot\xi$ away from the interface $\partial^* A\cap\Omega$. In the perturbative graph setting of Proposition 5, one may thus expect that to leading order

$$-\int_\Omega \frac{1}{2}|\nabla(w-\widetilde{w})|^2\,dx \approx -\frac{1}{2}\|h''\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})} = -\frac{1}{2}\|\nabla u_{h''}\|^2_{L^2(\Omega)}, \tag{63}$$

where as before $u_f \in H^1(\Omega)$ denotes the minimal Dirichlet energy extension of $f \in H^{1/2}_{MS}(\partial\mathscr{A})$ to Ω.

Second, since B is an extension of the normal velocity of the smoothly evolving $\partial\mathscr{A}$ (i.e., an extension of the jump of the Neumann data of the chemical potential of the strong solution, the latter being equal to $H_{\partial\mathscr{A}}$ along $\partial\mathscr{A}$), in the perturbative graph setting of Proposition 5, one should think of the term $\mathrm{n}_{\partial^*A}\cdot(B+[\![\nabla\widetilde{w}]\!])$ along $\partial^*\mathscr{A}$ as a proxy for $\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla u_{(H^2_{\partial\mathscr{A}}h-\fint_{\partial\mathscr{A}}H^2_{\partial\mathscr{A}}h\,d\mathcal{H}^1)}]\!]$ along $\partial\mathscr{A}$ (since to leading order $H_{\partial\mathscr{A}}+\nabla\cdot\xi|_{\partial^*A\cap\Omega}\sim H^2_{\partial\mathscr{A}}h$). Hence, by the very definition (48) of the Hilbert space structure on $H^{1/2}_{MS}(\partial\mathscr{A})$ one may expect that to leading order

$$
\begin{aligned}
&\int_{\partial^*A(t)\cap\Omega}(w-\widetilde{w})(\cdot,t)\big((B\cdot\mathrm{n}_{\partial^*A})+[\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}]\!]\big)(\cdot,t)\,d\mathcal{H}^1\\
&\approx\Big\langle h'',H^2_{\partial\mathscr{A}}h-\fint_{\partial\mathscr{A}}H^2_{\partial\mathscr{A}}h\,d\mathcal{H}^1\Big\rangle_{H^{1/2}_{MS}(\partial\mathscr{A})}\\
&=\int_\Omega\nabla u_{h''}\cdot\nabla u_{(H^2_{\partial\mathscr{A}}h-\fint_{\partial\mathscr{A}}H^2_{\partial\mathscr{A}}h\,d\mathcal{H}^1)}.
\end{aligned}
\tag{64}
$$

Hence, in view of the representation (53) for R_{dissip}, one may expect in the perturbative graph setting of Proposition 5 to obtain to leading order an estimate of the form

$$
\begin{aligned}
R_{dissip}\approx-\int_\Omega\frac{1}{2}|\nabla(u-\widetilde{w})|^2\,dx-\int_\Omega\frac{1}{4}|\nabla(w-\widetilde{w})|^2\,dx\\
+\widetilde{C}\Big\|H^2_{\partial\mathscr{A}}h-\fint_{\partial\mathscr{A}}H^2_{\partial\mathscr{A}}h\,d\mathcal{H}^1\Big\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})}
\end{aligned}
\tag{65}
$$

for some constant $\widetilde{C}>0$.

Finally, in the perturbative graph setting of Proposition 5, it is not hard to show (by a change to tubular neighborhood coordinates for E_{vol}, and by expressing $\mathrm{n}_{\partial^*A}\cdot\xi$ to leading order in terms of the graph function h and its derivative h' for E_{rel}) that our error functional behaves to leading order as

$$
E[A,\mu|\mathscr{A}]\approx\left(\frac{1}{2}\Big\|\frac{h}{\ell}\Big\|^2_{L^2(\partial\mathscr{A})}+\frac{1}{2}\|h'\|^2_{L^2(\partial\mathscr{A})}\right).
\tag{66}
$$

Hence, once one is provided with an interpolation estimate of the form

$$
\|f\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})}\lesssim\frac{1}{\ell^2}\|f\|^2_{L^2(\partial\mathscr{A})}+\|f'\|^2_{L^2(\partial\mathscr{A})},
\tag{67}
$$

one may expect based on (65)–(67)

$$
R_{dissip}\leq-\int_\Omega\frac{1}{2}|\nabla(u-\widetilde{w})|^2\,dx-\int_\Omega\frac{1}{4}|\nabla(w-\widetilde{w})|^2\,dx+\widetilde{C}E[A,\mu|\mathscr{A}],
\tag{68}
$$

which renders the following rigorous version of this heuristic conceivable. Starting point for its proof will be a domain mapping argument (not surprisingly in the form of a Hanzawa transformation), cf. the more technical discussion of Subsection 5.5.

Proposition 8 (Stability estimates for non-local terms in perturbative regime)**.** *Let $\delta\in(0,1)$, fix $t\in(0,T_*)$, and assume that for t the conclusions (44)–(47) of Proposition 5 hold true. In particular, recall that for a given $C\in(1,\infty)$ it holds*

$$
\frac{1}{\ell(t)}\|h(\cdot,t)\|_{L^\infty(\partial\mathscr{A}(t))}+\|\nabla^{tan}h(\cdot,t)\|_{L^\infty(\partial\mathscr{A}(t))}\leq\frac{1}{16C}.
\tag{69}
$$

Furthermore, denote by $\Lambda \in (1,\infty)$ *and* $M \in (1,\infty)$ *two constants such that*

$$\int_\Omega \frac{1}{2}|\nabla w(\cdot,t)|^2 \leq \Lambda, \tag{70}$$

$$E[A,\mu|\mathscr{A}](t) \leq \frac{\ell(t)}{M}. \tag{71}$$

Finally, fix (ξ,ϑ,B) *satisfying the assumptions* (11)–(16) *and* (24)–(26)*, and let the chemical potentials* $\widetilde{w}$ *and* $\widetilde{w}_\vartheta$ *be defined according to Lemma 6.*

Then, one may choose $C \gg_\delta 1$ *and* $M \gg_\delta 1$ *locally uniformly in* $[0,T_*)$ *such that there exists a constant* $\widetilde{C} = \widetilde{C}(\mathscr{A},\Omega,\Lambda,\ell,M,T^*) \in (1,\infty)$ *such that*

$$\begin{aligned}
&\int_{\partial^* A(t)\cap\Omega} (w-\widetilde{w})(\cdot,t)\big((B\cdot \mathrm{n}_{\partial^*A}) + [\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}]\!]\big)(\cdot,t)\,d\mathcal{H}^1 \\
&\leq \delta \int_\Omega |\nabla(w-\widetilde{w})(\cdot,t)|^2\,dx + \frac{\widetilde{C}}{\delta}E[A,\mu|\mathscr{A}](t),
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
&\int_{\partial^* A(t)\cap\Omega} (u-\widetilde{w})(\cdot,t)[\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}_\vartheta]\!](\cdot,t)\,d\mathcal{H}^1 \\
&\leq \delta \int_\Omega |\nabla(u-\widetilde{w})(\cdot,t)|^2\,dx + \frac{\widetilde{C}}{\delta}E[A,\mu|\mathscr{A}](t),
\end{aligned} \tag{73}$$

as well as

$$\int_{\partial^* A(t)\cap\Omega} \vartheta(\cdot,t)\big((B\cdot \mathrm{n}_{\partial^*A}) + [\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}]\!]\big)(\cdot,t)\,d\mathcal{H}^1 \leq \widetilde{C}E[A,\mu|\mathscr{A}](t). \tag{74}$$

4. Proof of Theorem 1

We proceed in three steps.

Step 1: Construction of triple (ξ,ϑ,B) *satisfying* (11)–(16) *and* (24)–(26)*.* Let $\bar\eta \in C^\infty_{cpt}(\mathbb{R};[0,1])$ be such that $\operatorname{supp}\bar\eta \subset [-3/4,3/4]$, $\bar\eta \equiv 1$ on $[-1/2,1/2]$ and $|\bar\eta'| \leq 8$. We then define the vector fields ξ and B by means of

$$\xi(x,t) := \bar\eta\Big(\frac{s_{\partial\mathscr{A}}(x,t)}{\ell(t)}\Big)\nabla s_{\partial\mathscr{A}}(x,t), \tag{75}$$

$$B(x,t) := \bar\eta\Big(\frac{s_{\partial\mathscr{A}}(x,t)}{\ell(t)}\Big)\mathrm{V}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x,t),t), \tag{76}$$

where $(x,t) \in \overline{\Omega}\times[0,T_*)$. We further choose $\bar\vartheta \in C^\infty(\mathbb{R};[-1,1])$ such that $\bar\vartheta(s) = s$ on $[-1/2,1/2]$, $\bar\vartheta \equiv -1$ on $(-\infty,-1]$, $\bar\vartheta \equiv 1$ on $[1,\infty)$, and finally $\bar\vartheta' > 0$ in $(-1,1)$. Based on this auxiliary map, we define the weight ϑ as follows:

$$\vartheta(x,t) := -\frac{1}{\ell(t)}\bar\vartheta\Big(\frac{s_{\partial\mathscr{A}}(x,t)}{\ell(t)}\Big), \tag{77}$$

where $(x,t) \in \overline{\Omega}\times[0,T_*)$. The properties (11)–(16) and (24)–(26) are immediate consequences of the definitions (75)–(77) and the regularity of the smoothly evolving phase $\mathscr{A}$.

Step 2: Proof of stability estimate (5)*.* Given the triple (ξ,ϑ,B) from the previous step, define the functional $(0,T_*) \ni t \mapsto E[A,\mu|\mathscr{A}](t) \in L^\infty_{loc}((0,T_*);[0,\infty))$ by (19).

Starting point for the proof of the stability estimate (5) are the preliminary stability estimates (27) and (28) from Lemma 3. Post-processing these by means of Lemma 4, Proposition 5, Lemma 6, Lemma 7, and Proposition 8, we infer that for almost every $T' \in (0, T_*)$ there is $C(T') > 0$ such that

$$\begin{aligned} &E[A,\mu|\mathscr{A}](T') + \int_0^{T'} \chi_{\mathcal{T}_{\mathrm{bad}}} \bigg(\int_\Omega \frac{1}{4}|\nabla u|^2\,dx + \int_\Omega \frac{1}{4}|\nabla w|^2\,dx \bigg)\,dt \\ &\quad + \int_0^{T'} \chi_{\mathcal{T}_{\mathrm{good}}} \bigg(\int_\Omega \frac{1}{4}|(\nabla u - \nabla \widetilde{w})|^2\,dx + \int_\Omega \frac{1}{4}|(\nabla w - \nabla \widetilde{w})|^2\,dx \bigg)\,dt \\ &\qquad \leq E[A,\mu|\mathscr{A}](0) + C(T') \int_0^{T'} E[A,\mu|\mathscr{A}](t)\,dt. \end{aligned} \tag{78}$$

This of course implies the claim (5).

Step 3: Weak-strong uniqueness. By Grönwall's inequality, it follows from the weak-strong stability estimate (5) that

$$E[A,\mu|\mathscr{A}](T') \leq E[A,\mu|\mathscr{A}](0) e^{\int_0^{T'} C(t)\,dt}$$

for almost every $T' \in (0, T_*)$. Weak-strong uniqueness in the form of (6) thus follows from (4). □

5. Weak-strong stability estimates for Mullins–Sekerka flow

5.1. **Proof of Lemma 3: Preliminary relative entropy inequality.** We proceed in two steps.

Step 1: Control by the dissipation estimate and by testing the weak formulation. Starting point for the derivation of (27) is the following representation of the relative entropy:

$$E_{rel}[A,\mu|\mathscr{A}](T') = E[\mu_{T'}] + \int_{A(T')} (\nabla \cdot \xi)(\cdot, T')\,dx$$

for all $T' \in [0, T_*)$, where we integrated by parts in the definition (17) of the relative entropy and used that $\xi(\cdot, T')$ is compactly supported within Ω, see (16). We thus infer from the dissipation estimate (7f) and the weak formulation (7b) applied to $\nabla \cdot \xi$ that

$$\begin{aligned} &E_{rel}[A,\mu|\mathscr{A}](T') \\ &\leq E_{rel}[A,\mu|\mathscr{A}](0) - \int_0^{T'} \int_\Omega \frac{1}{2}|\nabla u|^2\,dxdt - \int_0^{T'} \int_\Omega \frac{1}{2}|\nabla w|^2\,dxdt \\ &\quad - \int_0^{T'} \int_\Omega \nabla u \cdot \nabla(\nabla \cdot \xi)\,dxdt - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \partial_t \xi \cdot \mathrm{n}_{\partial^* A}\,d\mathcal{H}^1 dt \end{aligned} \tag{79}$$

for a.e. $T' \in [0, T_*)$.

Starting point for the proof of (28) is in turn the observation

$$E_{vol}[A|\mathscr{A}](T') = \int_\Omega (\chi_A - \chi_{\mathscr{A}})(\cdot, T')\vartheta(\cdot, T')\,dx$$

due to the sign conditions (15). As ϑ vanishes on $\partial\mathscr{A}$ by (14), we may use the the weak formulation (7b) applied to the strong solution and ϑ, along with an integration by parts in the potential term, to get

$$\int_0^{T'} \int_\Omega \chi_{\mathscr{A}} \partial_t \vartheta \, dxdt = 0,$$

so that the weak formulation (7b) applied to the weak solution gives

$$E_{vol}[A|\mathscr{A}](T') = E_{vol}[A|\mathscr{A}](0) + \int_0^{T'} \int_\Omega (\chi_A - \chi_{\mathscr{A}}) \partial_t \vartheta \, dxdt - \int_0^{T'} \int_\Omega \nabla u \cdot \nabla \vartheta \, dxdt. \tag{80}$$

Step 2: Inserting expected PDEs satisfied by $\partial_t \xi$ and $\partial_t \vartheta$. Since the vector fields $\xi(\cdot, t)$ and $B(\cdot, t)$ are compactly supported within Ω for all $t \in [0, T_*)$, see (16) and (25), by now standard computations for the relative entropy method in curvature driven interface evolution yield (cf. [12, identity (2.11)])

$$\begin{aligned}
&- \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \partial_t \xi \cdot \mathrm{n}_{\partial^* A} \, d\mathcal{H}^1 dt \\
&= - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\mathrm{Id} - \mathrm{n}_{\partial^* A} \otimes \mathrm{n}_{\partial^* A}) : \nabla B \, d\mathcal{H}^1 dt \\
&\quad + \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (B \cdot \mathrm{n}_{\partial^* A})(\nabla \cdot \xi) \, d\mathcal{H}^1 dt \\
&\quad - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \big(\partial_t \xi + (B \cdot \nabla)\xi + (\nabla B)^\mathsf{T} \xi\big) \cdot (\mathrm{n}_{\partial^* A} - \xi) \, d\mathcal{H}^1 dt \\
&\quad - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \big(\partial_t \xi + (B \cdot \nabla)\xi\big) \cdot \xi \, d\mathcal{H}^1 dt \\
&\quad - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\mathrm{n}_{\partial^* A} - \xi) \otimes (\mathrm{n}_{\partial^* A} - \xi) : \nabla B \, d\mathcal{H}^1 dt \\
&\quad - \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} (\mathrm{n}_{\partial^* A} \cdot \xi - 1)(\nabla \cdot B) \, d\mathcal{H}^1 dt
\end{aligned} \tag{81}$$

for a.e. $T' \in [0, T_*)$. Adding zero, then making use of the isotropic Gibbs–Thomson law (7e) and finally plugging everything back into (79) yields (27).

Next, adding and subtracting $(\chi_A - \chi_{\mathscr{A}})(B \cdot \nabla)\vartheta$, as well as integration by parts seperately for both solutions and using the facts that ϑ vanishes on $\partial \mathscr{A}$ and that B has compact support, as per identities (14) and (25), shows that

$$\begin{aligned}
\int_0^{T'} \int_\Omega (\chi_A - \chi_{\mathscr{A}}) \partial_t \vartheta \, dxdt &= \int_0^{T'} \int_\Omega (\chi_A - \chi_{\mathscr{A}})(\partial_t \vartheta + (B \cdot \nabla)\vartheta) \, dxdt \\
&\quad + \int_0^{T'} \int_\Omega (\chi_A - \chi_{\mathscr{A}}) \vartheta \nabla \cdot B \, dxdt \\
&\quad + \int_0^{T'} \int_{\partial^* A(t) \cap \Omega} \vartheta \mathrm{n}_{\partial^* A} \cdot B \, d\mathcal{H}^1 dt.
\end{aligned} \tag{82}$$

Plugging (82) back into (80) concludes the proof. □

5.2. **Proof of Lemma 4: Stability estimates at bad times.** For notational simplicity, we drop the time dependence in all occurring terms. Looking back to the first step of the proof of Lemma 3 (or alternatively, simply choosing $B \equiv 0$ at bad times), we see that

$$\begin{aligned} R_{dissip} + R_{\partial_t\xi} + R_{\nabla B} = &-\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \int_\Omega \frac{1}{2}|\nabla w|^2\,dx \\ &- \int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx \\ &- \int_{\partial^* A\cap\Omega} \partial_t\xi\cdot \mathrm{n}_{\partial^* A}\,d\mathcal{H}^1, \end{aligned} \tag{83}$$

$$U_{dissip} + U_{\partial_t\vartheta} + U_{\nabla B} = \int_\Omega (\chi_A - \chi_{\mathscr{A}})\partial_t\vartheta\,dx - \int_\Omega \nabla u\cdot\nabla\vartheta\,dx. \tag{84}$$

Step 1: Estimate for times $\mathscr{T}^{(1)}_{\mathrm{bad}}(\Lambda)$. We simply estimate

$$\begin{aligned} &-\int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx - \int_{\partial^* A\cap\Omega} \partial_t\xi\cdot\mathrm{n}_{\partial^* A}\,d\mathcal{H}^1 \\ &\leq \int_\Omega \frac{1}{8}|\nabla u|^2\,dx + 2\mathcal{L}^d(\operatorname{supp}\xi)\|\nabla(\nabla\cdot\xi)\|^2_{L^\infty(\overline{\Omega})} + E[A(0)]\|\partial_t\xi\|_{L^\infty(\overline{\Omega})} \end{aligned} \tag{85}$$

and

$$\begin{aligned} &\int_\Omega (\chi_A - \chi_{\mathscr{A}})\partial_t\vartheta\,dx - \int_\Omega \nabla u\cdot\nabla\vartheta\,dx \\ &\leq \int_\Omega \frac{1}{8}|\nabla u|^2\,dx + 2\mathcal{L}^d(\operatorname{supp}\nabla\vartheta)\|\nabla\vartheta\|^2_{L^\infty(\overline{\Omega})} + \mathcal{L}^d(\operatorname{supp}\partial_t\vartheta)\|\partial_t\vartheta\|^2_{L^\infty(\overline{\Omega})}. \end{aligned} \tag{86}$$

Hence, in view of (83) and (84), any $\Lambda > 0$ satisfying

$$\Lambda \geq 4\operatorname*{ess\,sup}_{t\in(0,T')} \big(2\mathcal{L}^d(\operatorname{supp}\xi)\|\nabla(\nabla\cdot\xi)\|^2_{L^\infty(\overline{\Omega})} + E[A(0)]\|\partial_t\xi\|_{L^\infty(\overline{\Omega})}\big)(t),$$

$$\Lambda \geq 4\operatorname*{ess\,sup}_{t\in(0,T')} \big(2\mathcal{L}^d(\operatorname{supp}\nabla\vartheta)\|\nabla\vartheta\|^2_{L^\infty(\overline{\Omega})} + \mathcal{L}^d(\operatorname{supp}\partial_t\vartheta)\|\partial_t\vartheta\|^2_{L^\infty(\overline{\Omega})}\big)(t)$$

does the job for (41) and (42) due to definition (38).

Step 2: Estimate for times $\mathscr{T}^{(2)}_{\mathrm{bad}}(\Lambda, M)$. Just looking back at (83)–(86) and recalling the admissible choices for $\Lambda > 0$ from the previous display, we see that (41) and (42) are immediate consequences of definition (39). □

5.3. **Proof of Lemma 6: Structure of dissipative terms.** Before we derive the representation formulas (53) and (57), we state two helpful intermediate results whose proofs are postponed until the end of this subsection. The first infers from the motion law (7b) harmonicity of the chemical potential $u(\cdot,t)$ away from the interface.

Lemma 9 (Harmonicity of the chemical potential in perturbative regime)**.** *Fix* $t \in (0, T_*)$ *subject to the assumptions of Lemma 6. Then* $u(\cdot,t)$ *satisfies*

$$\int_{A(t)} \nabla u(\cdot,t)\cdot\nabla\zeta\,dx = 0 \tag{87}$$

for all $\zeta \in C^\infty_{cpt}(A(t))$*, and*

$$\int_{\Omega\setminus A(t)} \nabla u(\cdot,t)\cdot\nabla\zeta\,dx = 0 \tag{88}$$

for all $\zeta \in C^\infty_{cpt}(\overline{\Omega} \setminus \overline{A(t)})$.

The second auxiliary result exploits the perturbative setting to facilitate a rigorous integration by parts involving the jumps of the Neumann data for the chemical potentials along the interface $\partial^* A(t) \cap \Omega$, which in turn then allows, among other things, to "smuggle in" the chemical potential $\widetilde{w}(\cdot,t)$ by means of its boundary condition (51).

Lemma 10 (Integration by parts formulas)**.** *Fix* $t \in (0, T_*)$ *subject to the assumptions of Lemma 6. Then*

$$-\int_\Omega \nabla u(\cdot,t) \cdot \nabla(\nabla\cdot\xi)(\cdot,t)\,dx = \int_\Omega \nabla u(\cdot,t)\cdot\nabla\widetilde{w}(\cdot,t)\,dx \tag{89}$$

and

$$\begin{aligned} &-\int_\Omega \nabla\widetilde{w}(\cdot,t)\cdot\nabla(w-\widetilde{w})(\cdot,t)\,dx \\ &= \int_{\partial^* A(t)\cap\Omega} (w-\widetilde{w})(\cdot,t)[\![(\mathrm{n}_{\partial^* A}\cdot\nabla)\widetilde{w}]\!](\cdot,t)\,d\mathcal{H}^1, \end{aligned} \tag{90}$$

as well as

$$-\int_\Omega \nabla\widetilde{w}\cdot\nabla\vartheta\,dx = \int_{\partial^* A\cap\Omega} \vartheta[\![(\mathrm{n}_{\partial^* A}\cdot\nabla)\widetilde{w}]\!]\,d\mathcal{H}^1, \tag{91}$$

$$-\int_\Omega (\nabla u - \nabla\widetilde{w})\cdot\nabla\vartheta\,dx = \int_{\partial^* A\cap\Omega} (u-\widetilde{w})[\![(\mathrm{n}_{\partial^* A}\cdot\nabla)\widetilde{w}_\vartheta]\!]\,d\mathcal{H}^1. \tag{92}$$

We now combine the information from the previous two results to a proof of the representation formulas (53) and (57).

Proof of Lemma 6. The proof is split into two steps. For notational ease, we drop in the notation the time dependence of all quantities.

Step 1: Formula for R_{dissip}. For convenience, let us restate the definition of R_{dissip} from (29):

$$\begin{aligned} R_{dissip} = &-\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \int_\Omega \frac{1}{2}|\nabla w|^2\,dx - \int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx \\ &+ \int_{\partial^* A\cap\Omega} (B\cdot\mathrm{n}_{\partial^* A})\big(w + (\nabla\cdot\xi)\big)\,d\mathcal{H}^1. \end{aligned} \tag{93}$$

Inserting the boundary condition (51), we observe that

$$\int_{\partial^* A\cap\Omega} (B\cdot\mathrm{n}_{\partial^* A})\big(w + (\nabla\cdot\xi)\big)\,d\mathcal{H}^1 = \int_{\partial^* A\cap\Omega} (B\cdot\mathrm{n}_{\partial^* A})(w-\widetilde{w})\,d\mathcal{H}^1. \tag{94}$$

Next, by inserting the integration by parts formula (89) and completing squares, we get

$$\begin{aligned} &-\int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \int_\Omega \frac{1}{2}|\nabla w|^2\,dx - \int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx \\ &= -\int_\Omega \frac{1}{2}|\nabla u - \nabla\widetilde{w}|^2\,dx - \int_\Omega \frac{1}{2}|\nabla w - \nabla\widetilde{w}|^2\,dx - \int_\Omega \nabla\widetilde{w}\cdot\nabla(w-\widetilde{w})\,dx. \end{aligned} \tag{95}$$

Hence, (53) follows from (93)–(95) and the integration by parts formula (90).

Step 2: Formula for U_{dissip}. Recall first from (33) that

$$U_{dissip} = -\int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx + \int_{\partial^* A\cap\Omega} \vartheta(\mathrm{n}_{\partial^* A}\cdot B)\,d\mathcal{H}^1. \tag{96}$$

To upgrade this to (57), we simply start by adding zero in the form of

$$
\begin{aligned}
&-\int_\Omega \nabla u\cdot\nabla\vartheta\,dx+\int_{\partial^*A\cap\Omega}\vartheta\mathrm{n}_{\partial^*A}\cdot B\,d\mathcal{H}^1\\
&=-\int_\Omega(\nabla u{-}\nabla\widetilde{w})\cdot\nabla\vartheta\,dx-\int_\Omega\nabla\widetilde{w}\cdot\nabla\vartheta\,dx+\int_{\partial^*A\cap\Omega}\vartheta\mathrm{n}_{\partial^*A}\cdot B\,d\mathcal{H}^1.
\end{aligned}
$$

Hence, (57) follows from the integration by parts formulas (91) and (92). □

Proof of Lemma 9. Let $\zeta\in C^\infty_{cpt}(\overline{\Omega})$ be a nonnegative test function supported in $\overline{\Omega}\setminus\overline{A(t)}$. We then have by the nonnegativity of ζ, the property $\zeta\equiv 0$ in $A(t)$, and the weak formulation of the Mullins-Sekerka equation (7b)

$$
0\le\int_{A(\tilde t)}\zeta\,dx=\int_{A(\tilde t)}\zeta\,dx-\int_{A(t)}\zeta\,dx\overset{(7b)}{=}-\int_t^{\tilde t}\int_\Omega\nabla u\cdot\nabla\zeta\,dxds.
$$

Dividing by $|\tilde t-t|$ and adding zero, we obtain

$$
\begin{aligned}
0\le&-\operatorname{sign}(\tilde t-t)\int_{\Omega\setminus A(t)}\nabla u(\cdot,t)\cdot\nabla\zeta\,dx\\
&-\frac{1}{|\tilde t-t|}\int_t^{\tilde t}\int_\Omega(\nabla u(\cdot,s)-\nabla u(\cdot,t))\cdot\nabla\zeta\,dxds.
\end{aligned}
$$

Passing to the limit $\tilde t\searrow t$ respectively $\tilde t\nearrow t$, we deduce by (49)

$$
-\int_{\Omega\setminus A(t)}\nabla u(\cdot,t)\cdot\nabla\zeta\,dx\ge 0
$$

respectively

$$
-\int_{\Omega\setminus A(t)}\nabla u(\cdot,t)\cdot\nabla\zeta\,dx\le 0.
$$

This implies (88). Analogously, one shows (87) which then implies our lemma. □

Proof of Lemma 10. We again drop the time dependence of all objects in the notation and proceed in three steps.

Step 1: Proof of (89). We note that due to (45) it holds $\partial^*A\cap\Omega=\partial^*A=\partial A\subset\subset\Omega$. Furthermore, Lemma 9 tells us that $\nabla\cdot\nabla u$ exists in the sense of square-integrable weak derivatives within the open sets A and $\Omega\setminus\overline{A}$, respectively, and that in fact

$$
\nabla\cdot\nabla u=0\quad\text{throughout } A \text{ and } \Omega\setminus\overline{A},\ \text{respectively.}\tag{97}
$$

In other words, $\nabla u\in Y^2_{div}(D):=\{f\in L^2(D;\mathbb{R}^2)\colon \nabla\cdot f\in L^2(D;\mathbb{R}^2)\}$ for $D\in\{A,\Omega\setminus\overline{A}\}$. It is a standard result (for instance in the context of mathematical fluid mechanics in connection with the Helmholtz decomposition; see, e.g., Simader and Sohr [26, Theorem 5.3]) that there exists a continuous linear operator $T_D\colon Y^2_{div}(D)\to W^{-\frac12,2}(\partial D)$ such that for all $f\in Y^2_{div}(D)$ and all $\zeta\in H^1(D)$

$$
\int_D\zeta(\nabla\cdot f)\,dx=-\int_D\nabla\zeta\cdot f\,dx-\langle T_D(f),\zeta\rangle,\tag{98}
$$

i.e., one may interpret $T_D f$ as $(\mathrm{n}_{\partial D}\cdot f)|_{\partial D}$ in a weak sense. Denoting now by $\eta\colon \overline{\Omega}\to[0,1]$ a C^1-cutoff such that $\eta\equiv 1$ on $\overline{A}$ and $\eta\equiv 0$ on $\partial\Omega$, we thus obtain from (97), (98) and using that ξ is compactly supported within Ω

$$
\begin{aligned}
&-\int_\Omega \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx\\
&= -\int_A \nabla u\cdot\nabla(\nabla\cdot\xi)\,dx - \int_{\Omega\setminus\overline{A}} \nabla u\cdot\nabla(\eta+(1-\eta))(\nabla\cdot\xi)\,dx\\
&= \langle T_A(\nabla u),(\nabla\cdot\xi)\rangle + \langle T_{\Omega\setminus\overline{A}}(\nabla u),\eta(\nabla\cdot\xi)\rangle + \langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)(\nabla\cdot\xi)\rangle.
\end{aligned}
\tag{99}
$$

Since $(1-\eta)\equiv 0$ on ∂A by its choice and $\nabla\cdot\xi\equiv 0$ on $\partial\Omega$, it holds $(1-\eta)(\nabla\cdot\xi)\in H^1_0(\Omega\setminus\overline{D})$ and therefore

$$
\langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)(\nabla\cdot\xi)\rangle = 0. \tag{100}
$$

Moreover, due to the boundary condition (51) and $\eta\equiv 0$ on $\partial\Omega$ by its choice, it holds $\nabla\cdot\xi = -\widetilde{w}\in W^{\frac{1}{2},2}(\partial A)$ as well as $\eta\nabla\cdot\xi = -\eta\widetilde{w}\in W^{\frac{1}{2},2}(\partial(\Omega\setminus\overline{A}))$ and therefore

$$
\begin{aligned}
&\langle T_A(\nabla u),\nabla\cdot\xi\rangle + \langle T_{\Omega\setminus\overline{A}}(\nabla u),\eta(\nabla\cdot\xi)\rangle\\
&= -\langle T_A(\nabla u),\widetilde{w}\rangle - \langle T_{\Omega\setminus\overline{A}}(\nabla u),\eta\widetilde{w}\rangle\\
&= -\langle T_A(\nabla u),\widetilde{w}\rangle - \langle T_{\Omega\setminus\overline{A}}(\nabla u),\widetilde{w}\rangle + \langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)\widetilde{w}\rangle.
\end{aligned}
$$

Based on (50) and (98), the previous display admits the upgrade

$$
\begin{aligned}
&\langle T_A(\nabla u),\nabla\cdot\xi\rangle + \langle T_{\Omega\setminus\overline{A}}(\nabla u),\eta(\nabla\cdot\xi)\rangle\\
&= \int_A \nabla u\cdot\nabla\widetilde{w}\,dx + \int_{\Omega\setminus\overline{A}} \nabla u\cdot\nabla\widetilde{w}\,dx + \langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)\widetilde{w}\rangle\\
&= \int_\Omega \nabla u\cdot\nabla\widetilde{w}\,dx + \langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)\widetilde{w}\rangle.
\end{aligned}
\tag{101}
$$

It remains to identify $\langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)\widetilde{w}\rangle$. However, comparing (98) applied to the data $(f,\zeta)=(\nabla u,(1-\eta)\widetilde{w})$, $\nabla\cdot f=0$, $D=\Omega\setminus\overline{A}$, with (88) applied to the admissible data $\zeta=(1-\eta)\widetilde{w}$ yields (recall also that $\eta\equiv 0$ along $\partial\Omega$)

$$
\langle T_{\Omega\setminus\overline{A}}(\nabla u),(1-\eta)\widetilde{w}\rangle = 0. \tag{102}
$$

The identities (99)–(102) together finally imply the claim (89).

Step 2: Proof of (90). Denote again by $\eta\colon\overline{\Omega}\to[0,1]$ a C^1-cutoff such that $\eta\equiv 1$ on $\overline{A}$ and $\eta\equiv 0$ on $\partial\Omega$. Analogous to the above arguments, we then obtain on one side

$$
-\int_{\Omega\setminus\overline{A}} \nabla\widetilde{w}\cdot\nabla\big((1-\eta)(w-\widetilde{w})\big)\,dx = \langle T_{\Omega\setminus\overline{A}}(\nabla\widetilde{w}),(1-\eta)(w-\widetilde{w})\rangle = 0.
$$

On the other side, we may exploit that the interface $\partial A=\partial^*A=\partial^*A\cap\Omega$ is $C^{1,\alpha}$ due to assumption (44). Since the boundary data in (51) is smooth, we get by standard Schauder theory that the auxiliary chemical potential is C^1 up to the interface ∂A from both sides, so that

$$
\begin{aligned}
&-\int_A \nabla\widetilde{w}\cdot\nabla(w-\widetilde{w})\,dx - \int_{\Omega\setminus\overline{A}} \nabla\widetilde{w}\cdot\nabla\big(\eta(w-\widetilde{w})\big)\,dx\\
&= -\int_\Omega \nabla\widetilde{w}\cdot\nabla\big(\eta(w-\widetilde{w})\big)\,dx = \int_{\partial^*A\cap\Omega} [\![(\mathrm{n}_{\partial^*A}\cdot\nabla)\widetilde{w}]\!](w-\widetilde{w})\,d\mathcal{H}^1.
\end{aligned}
$$

The previous two displays obviously imply (90).

Step 3: Proof of (91) *and* (92). These two follow from analogous arguments. □

5.4. **Proof of Lemma 7: Stability estimate for local terms.** Thanks to (47) and (45), we know $\partial^* A \cap \Omega \subset B_{\ell/4}(\partial\mathscr{A})$ as well as $A\Delta\mathscr{A} \subset B_{\ell/4}(\partial\mathscr{A})$. Hence, since $\frac{1}{2}|n-\xi|^2 = 1 - n\cdot\xi$ due to the first item of (13), we immediately obtain the estimates (59) and (62); recall for this also the definitions (17) and (18) of E_{rel} and E_{vol}, respectively. Furthermore, the identity (58) is a consequence of the following identities throughout $B_{\ell/4}(\partial\mathscr{A})$:

$$\begin{aligned}\xi\cdot(\partial_t\xi + (B\cdot\nabla\xi)) &= 0,\\ \partial_t\xi + (B\cdot\nabla)\xi + (\nabla B)^{\mathsf{T}}\xi &= 0.\end{aligned}$$

The first is true simply because of $|\xi| \equiv 1$ within $B_{\ell/2}(\partial\mathscr{A})$, see again the first item of (13). The second also follows from the latter by taking the spatial gradient of $\partial_t s_{\partial\mathscr{A}} + (B\cdot\nabla)s_{\partial\mathscr{A}} = 0$, which itself is true within $B_{\ell/2}(\partial\mathscr{A})$ thanks to (26). Note also in this context that the transport equation satisfied by the signed distance also directly implies (61) due to (14).

It remains to prove (60). To this end, we first obtain using (20) and (22)

$$\begin{aligned}&-\int_{\partial^* A\cap\Omega}\nabla\cdot B\,d\mathcal{H}^1\\ &\quad= -\int_{\Omega}\varrho\nabla\cdot B\,d|\mu|_{\mathbb{S}^1} \le -\int_{\overline{\Omega}\times\mathbb{S}^1}\nabla\cdot B\,d\mu + \|\nabla\cdot B\|_{L^\infty(B_{\ell/2}(\partial\mathscr{A}))}E_{rel}[A,\mu|\mathscr{A}].\end{aligned}$$

Adding zero several times, applying the compatibility condition from [14, Definition 3, item ii)] with test function $\eta = (\nabla B)^{\mathsf{T}}\xi$, and estimating based on the properties $(\xi\cdot\nabla)B = 0$ and $\frac{1}{2}|\xi - \mathrm{p}|^2 \le 1 - \xi\cdot\mathrm{p},\ \mathrm{p}\in\mathbb{S}^1$, valid throughout $B_{\ell/2}(\partial\mathscr{A})$, cf. (13) and (26), furthermore entails

$$\begin{aligned}\int_{\partial^* A\cap\Omega}\mathrm{n}_{\partial^* A}\otimes\mathrm{n}_{\partial^* A} : \nabla B\,d\mathcal{H}^1 &= \int_{\partial^* A\cap\Omega}(\mathrm{n}_{\partial^* A}-\xi)\otimes\mathrm{n}_{\partial^* A} : \nabla B\,d\mathcal{H}^1\\ &\quad+\int_{\overline{\Omega}\times\mathbb{S}^1}\xi\otimes\mathrm{p} : \nabla B\,d\mu\\ &= \int_{\partial^* A\cap\Omega}(\mathrm{n}_{\partial^* A}-\xi)\otimes(\mathrm{n}_{\partial^* A}-\xi) : \nabla B\,d\mathcal{H}^1\\ &\quad-\int_{\overline{\Omega}\times\mathbb{S}^1}(\mathrm{p}-\xi)\otimes(\mathrm{p}-\xi) : \nabla B\,d\mu\\ &\quad+\int_{\overline{\Omega}\times\mathbb{S}^1}\mathrm{p}\otimes\mathrm{p} : \nabla B\,d\mu\\ &\le \int_{\overline{\Omega}\times\mathbb{S}^1}\mathrm{p}\otimes\mathrm{p} : \nabla B\,d\mu\\ &\quad+4\|\nabla B+(\nabla B)^{\mathsf{T}}\|_{L^\infty(B_{\ell/2}(\partial\mathscr{A}))}E_{rel}[A,\mu|\mathscr{A}].\end{aligned}$$

This concludes the proof. □

5.5. **Proof of Proposition 8: Stability estimates for non-local terms.** For this whole subsection, let the assumptions and notation of Proposition 8 be in place. As before, let us drop the time dependence of all quantities in the notation. In order to streamline the proof of Proposition 8, we start by collecting an array of auxiliary constructions and results. Proofs for these are deferred until after we showed how the estimate (72) can be inferred from these. We finally conclude this subsection with the missing proofs for the estimates (73) and (74).

In order to pull off the strategy for the proof of (72) as mentioned before the statement of Proposition 8, we first introduce a C^1 diffeomorphism $\Psi^h\colon \Omega \to \Omega$ with the property

$$\Psi^h(A) = \mathscr{A}. \tag{103}$$

This can be done as follows. Let $\bar\zeta \in C^\infty_{cpt}(\mathbb{R};[0,1])$ be such that $\operatorname{supp}\bar\zeta \subset [-1/2,1/2]$, $\bar\zeta \equiv 1$ on $[-1/4,1/4]$ and $|\bar\zeta'| \leq 8$. Based on this auxiliary cutoff, define a map

$$\zeta\colon \mathbb{R}^2 \to [0,1], \quad x \mapsto \bar\zeta\Big(\frac{s_{\partial\mathscr{A}}(x)}{\ell}\Big). \tag{104}$$

To avoid cumbersome notation in the following, let us also define

$$\bar{\mathrm{n}}_{\partial\mathscr{A}}(x) := \mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x)), \tag{105}$$

$$\bar H_{\partial\mathscr{A}}(x) := (\mathrm{H}_{\partial\mathscr{A}} \cdot \mathrm{n}_{\partial\mathscr{A}})(P_{\partial\mathscr{A}}(x)), \tag{106}$$

$$\bar h(x) := h(P_{\partial\mathscr{A}}(x)) \tag{107}$$

for $x \in B_{\frac{\ell}{2}}(\partial\mathscr{A})$. We may then define the desired map Ψ^h by means of

$$\Psi^h(x) := x - \zeta(x)\bar h(x)\bar{\mathrm{n}}_{\partial\mathscr{A}}(x). \tag{108}$$

Among other things which are also needed in the sequel, we collect in the upcoming result some basic facts about Ψ^h.

Lemma 11 (Some useful identities)**.** *It holds*

$$\xi(x) = \bar{\mathrm{n}}_{\partial\mathscr{A}}(x) = \nabla s_{\partial\mathscr{A}}(x), \quad x \in B_{\frac{\ell}{2}}(\partial\mathscr{A}), \tag{109}$$

as well as

$$\nabla\xi(x) = -\frac{\bar H_{\partial\mathscr{A}}(x)}{1-\bar H_{\partial\mathscr{A}}(x)s_{\partial\mathscr{A}}(x)}\big(\mathrm{Id}-\bar{\mathrm{n}}_{\partial\mathscr{A}}(x)\otimes\bar{\mathrm{n}}_{\partial\mathscr{A}}(x)\big), \quad x \in B_{\frac{\ell}{2}}(\partial\mathscr{A}), \tag{110}$$

and thus

$$\nabla\cdot\xi(x) = -\frac{\bar H_{\partial\mathscr{A}}(x)}{1-\bar H_{\partial\mathscr{A}}(x)s_{\partial\mathscr{A}}(x)}, \quad x \in B_{\frac{\ell}{2}}(\partial\mathscr{A}). \tag{111}$$

Moreover, we have throughout Ω

$$\begin{aligned}\nabla\Psi^h = \mathrm{Id} - \zeta\frac{1}{1-\bar H_{\partial\mathscr{A}}s_{\partial\mathscr{A}}}&\bar{\mathrm{n}}_{\partial\mathscr{A}}\otimes(\nabla^{tan}h\circ P_{\partial\mathscr{A}})\\ &+\zeta\frac{\bar H_{\partial\mathscr{A}}\bar h}{1-\bar H_{\partial\mathscr{A}}s_{\partial\mathscr{A}}}\big(\mathrm{Id}-\bar{\mathrm{n}}_{\partial\mathscr{A}}\otimes\bar{\mathrm{n}}_{\partial\mathscr{A}}\big)\\ &-\big(\bar n_{\partial\mathscr{A}}\cdot\nabla\zeta\big)\bar h\,\bar{\mathrm{n}}_{\partial\mathscr{A}}\otimes\bar{\mathrm{n}}_{\partial\mathscr{A}},\end{aligned} \tag{112}$$

and thus throughout Ω

$$\det\nabla\Psi^h = \big(1-\big(\bar n_{\partial\mathscr{A}}\cdot\nabla\zeta\big)\bar h\big)\Big(1+\zeta\frac{\bar H_{\partial\mathscr{A}}\bar h}{1-\bar H_{\partial\mathscr{A}}s_{\partial\mathscr{A}}}\Big). \tag{113}$$

In particular, $\Psi^h\colon\Omega\to\Omega$ *is a* C^1 *diffeomorphism.*

Finally, it holds

$$\Psi^h(x) = x - \bar h(x)\bar{\mathrm{n}}_{\partial\mathscr{A}}(x), \qquad x \in B_{2\|h\|_{L^\infty(\partial\mathscr{A})}}(\partial\mathscr{A}), \tag{114}$$

$$(\Psi^h)^{-1}(x) = x + \bar h(x)\bar{\mathrm{n}}_{\partial\mathscr{A}}(x), \qquad x \in B_{2\|h\|_{L^\infty(\partial\mathscr{A})}}(\partial\mathscr{A}). \tag{115}$$

The next ingredient is concerned with a representation of the error functionals in terms of the graph function.

Lemma 12 (Error control in perturbative setting)**.** *For any $\delta \in (0,1)$, one may choose $C \gg_\delta 1$ from (69) such that*

$$(1-\delta)\int_{\partial\mathscr{A}} \frac{1}{2}|\nabla^{tan}h|^2\,d\mathcal{H}^1 \leq E_{rel}[A,\mu|\mathscr{A}] \leq (1+\delta)\int_{\partial\mathscr{A}} \frac{1}{2}|\nabla^{tan}h|^2\,d\mathcal{H}^1, \tag{116}$$

$$(1-\delta)\int_{\partial\mathscr{A}} \frac{1}{2}\Big(\frac{h}{\ell}\Big)^2\,d\mathcal{H}^1 \leq E_{vol}[A|\mathscr{A}] \leq (1+\delta)\int_{\partial\mathscr{A}} \frac{1}{2}\Big(\frac{h}{\ell}\Big)^2\,d\mathcal{H}^1 \tag{117}$$

and

$$(1-\delta)\mathcal{H}^1(\partial\mathscr{A}) \leq \mathcal{H}^1(\partial^*A\cap\Omega) \leq (1+\delta)\mathcal{H}^1(\partial\mathscr{A}). \tag{118}$$

In the following result, we record the PDEs satisfied by the chemical potentials after pulling back the domain A to $\mathscr{A}$.

Lemma 13 (PDEs satisfied by transformed chemical potentials)**.** *For arbitrary $v \in H^1(\Omega)$, we define $v_h := v\circ(\Psi^h)^{-1}$. We also define the uniformly elliptic and bounded coefficient field $a^h := \frac{1}{|\det\Psi^h|}(\nabla\Psi^h)^\mathsf{T}\nabla\Psi^h$. Then, $(w-\widetilde{w})_h \in H^1(\Omega)$ satisfies*

$$\Delta(w-\widetilde{w})_h = \nabla\cdot\big((\mathrm{Id}-a^h)\nabla(w-\widetilde{w})_h\big) \qquad \textit{in } \Omega\setminus\partial\mathscr{A}, \tag{119}$$

$$\mathrm{tr}_{\partial\mathscr{A}}(w-\widetilde{w})_h = \big(\mathrm{tr}_{\partial^*A}(w-\widetilde{w})\big)\circ(\Psi^h)^{-1} \qquad \textit{on } \partial\mathscr{A}, \tag{120}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)(w-\widetilde{w})_h = 0 \qquad \textit{on } \partial\Omega. \tag{121}$$

Writing $\bar{u} \in H^1(\Omega)$ for the chemical potential associated with the smoothly evolving phase $\mathscr{A}$, we also get for $\bar{u}-\widetilde{w}_h \in H^1(\Omega)$

$$\Delta(\bar{u}-\widetilde{w}_h) = \nabla\cdot\big((a^h-\mathrm{Id})\nabla\widetilde{w}_h\big) \qquad \textit{in } \Omega\setminus\partial\mathscr{A}, \tag{122}$$

$$\mathrm{tr}_{\partial\mathscr{A}}(\bar{u}-\widetilde{w}_h) = \big(H_{\partial\mathscr{A}} + (\nabla\cdot\xi)\circ(\Psi^h)^{-1}\big) \qquad \textit{on } \partial\mathscr{A}, \tag{123}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)(\bar{u}-\widetilde{w}_h) = 0 \qquad \textit{on } \partial\Omega. \tag{124}$$

In order to exploit the Hilbert space structure of $H^{1/2}_{MS}(\partial\mathscr{A})$ for an estimate of R_{dissip} and A_{dissip}, respectively, we have to smuggle in the averages of the associated chemical potentials. The following result ensures that this can be done solely at the cost of controlled quantities.

Lemma 14 (Smuggling in averages of chemical potentials)**.** *For any $\delta \in (0,1)$, one may choose $C \gg_\delta 1$ from (69) and $M \gg_\delta 1$ from (71) such that for some universal constant $\widetilde{C} > 0$*

$$\begin{aligned}&\bigg|\fint_{\partial^*A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1\bigg| + \bigg|\fint_{\partial\mathscr{A}} (w-\widetilde{w})_h\,d\mathcal{H}^1\bigg| \\ &\quad\leq \delta C_{PS}(\mathscr{A})\bigg(\int_\Omega |\nabla(w-\widetilde{w})|^2\,dx\bigg)^{\frac{1}{2}} + \frac{\widetilde{C}}{\ell}\frac{\sqrt{E[A|\mathscr{A}]}}{\sqrt{\mathcal{H}^1(\partial^*\mathscr{A}\cap\Omega)}},\end{aligned} \tag{125}$$

where $C_{PS}(\mathscr{A})$ is the constant from the Sobolev–Poincaré trace inequality: for any $v \in H^1(\mathscr{A})$,

$$\bigg(\fint_{\partial\mathscr{A}}\Big(v-\fint_{\partial\mathscr{A}} v\,d\mathcal{H}^1\Big)^2\,d\mathcal{H}^1\bigg)^{\frac{1}{2}} \leq C_{PS}(\mathscr{A})\bigg(\int_{\mathscr{A}} |\nabla v|^2\,dx\bigg)^{\frac{1}{2}}. \tag{126}$$

Furthermore,

$$\begin{aligned}\bigg|\int_{\partial^* A\cap\Omega} \mathrm{n}_{\partial^* A}\cdot\big(B-[\![\nabla\widetilde{w}]\!]\big)\,d\mathcal{H}^1\bigg| \\ \leq \widetilde{C}\big\|\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{u}]\!]\big\|_{L^\infty(\partial\mathscr{A})}\sqrt{\mathcal{H}^1(\partial\mathscr{A})}\sqrt{E_{vol}[A|\mathscr{A}]}.\end{aligned} \tag{127}$$

We also rely on a regularity estimate for the transformed chemical potential $\widetilde{w}_h$.

Lemma 15 (Schauder estimate for transformed chemical potential)**.** *There exists a constant $C_{reg}(\mathscr{A},\Lambda)\in(0,\infty)$ such that*

$$\|\widetilde{w}_h\|_{C^{1,\frac12}(\overline{\mathscr{A}\cap B_{\ell/2}(\partial\mathscr{A})})} + \|\widetilde{w}_h\|_{C^{1,\frac12}(\overline{(\Omega\setminus\mathscr{A})\cap B_{\ell/2}(\partial\mathscr{A})})} \leq C_{reg}(\mathscr{A},\Lambda). \tag{128}$$

An analogous estimate holds for $(\widetilde{w}_\vartheta)_h$.

Another useful input is given by energy estimates.

Lemma 16 (Energy estimates for transformed chemical potentials)**.** *Decompose $(w-\widetilde{w})_h = v_h^{(1)} + v_h^{(2)}$, where $v_h^{(1)}\in H^1(\Omega)$ denotes the unique solution of*

$$\Delta v_h^{(1)} = 0 \qquad \text{in } \Omega\setminus\partial\mathscr{A}, \tag{129}$$

$$\mathrm{tr}_{\partial\mathscr{A}} v_h^{(1)} = \big(\mathrm{tr}_{\partial^* A}(w-\widetilde{w})\big)\circ(\Psi^h)^{-1} \qquad \text{on } \partial\mathscr{A}, \tag{130}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)v_h^{(1)} = 0 \qquad \text{on } \partial\Omega. \tag{131}$$

For any $\delta\in(0,1)$, one may choose $C\gg_\delta 1$ from (69) such that

$$\int_\Omega |\nabla v_h^{(1)}|^2\,dx \leq (1+\delta)\int_\Omega |\nabla(w-\widetilde{w})_h|^2\,dx, \tag{132}$$

$$\int_\Omega |\nabla(w-\widetilde{w})_h|^2\,dx \leq (1+\delta)\int_\Omega |\nabla(w-\widetilde{w})|^2\,dx. \tag{133}$$

There also exists a universal constant $\widetilde{C}>0$ such that

$$\bigg|\int_\Omega \nabla v_h^{(1)}\cdot(\mathrm{Id}-a^h)\nabla\widetilde{w}_h\bigg| \leq \widetilde{C}\bigg(\int_\Omega |\nabla v_h^{(1)}|^2\,dx\bigg)^{\frac12} C_{reg}(\mathscr{A},\Lambda)\sqrt{E[A|\mathscr{A}]}, \tag{134}$$

where $C_{reg}(\mathscr{A},\Lambda)$ is the constant from Lemma 15.

The final ingredient consists of an interpolation estimate, transferring control in terms of the Hilbert space structure on $H^{1/2}_{MS}(\partial\mathscr{A})$ to control in terms of the standard H^1 norm on $\partial\mathscr{A}$ (and therefore to control in terms of our error functional).

Lemma 17 (Interpolation estimate)**.** *Fix $L\in(0,\infty)$. There exists a constant $C_{int}(\mathscr{A},\Omega,L)\in(0,\infty)$ such that for all $f\in H^2(\partial\mathscr{A})$ with $\fint_{\partial\mathscr{A}} f\,d\mathcal{H}^1 = 0$ and $\|f\|_{H^2(\partial\mathscr{A})}\leq L$ it holds*

$$\|f\|_{H^{1/2}_{MS}(\partial\mathscr{A})} \leq C_{int}(\mathscr{A},\Omega,L)\Big(\frac{1}{\ell}\|f\|_{L^2(\partial\mathscr{A})} + \|\nabla^{tan} f\|_{L^2(\partial\mathscr{A})}\Big). \tag{135}$$

We have now everything in place to proceed with the proofs.

Proof of Proposition 8, Part I: Estimate (72). Next to the decomposition $(w-\widetilde{w})_h = v_h^{(1)} + v_h^{(2)}$ as defined in Lemma 16, decompose also $\bar{u}-\widetilde{w}_h = \bar{v}_h^{(1)} + \bar{v}_h^{(2)}$, where $\bar{v}_h^{(1)}\in H^1(\Omega)$ denotes the unique solution of

$$\Delta\bar{v}_h^{(1)} = 0 \qquad \text{in } \Omega\setminus\partial\mathscr{A}, \tag{136}$$

$$\mathrm{tr}_{\partial\mathscr{A}}\bar{v}_h^{(1)} = \big(H_{\partial\mathscr{A}} + (\nabla\cdot\xi)\circ(\Psi^h)^{-1}\big) \qquad \text{on } \partial\mathscr{A}, \tag{137}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)\bar{v}_h^{(1)} = 0 \qquad \text{on } \partial\Omega. \tag{138}$$

The main claim of the proof then is that the argument for the estimate (72) boils down to an estimate of

$$Err := \bigg|\int_{\partial\mathscr{A}} \Big(v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{v}_h^{(1)}]\!]\,d\mathcal{H}^1\bigg|. \tag{139}$$

The estimate for Err itself will be a straightforward consequence of the Hilbert space structure defined on $H^{1/2}_{MS}(\partial\mathscr{A})$, the interpolation estimate (135), and that our error functional controls the H^1 norm on $\partial\mathscr{A}$ of the height function h, see (116) and (117).

Step 1: Reduction argument. Recall that we want to estimate

$$\widetilde{Err} := \bigg|\int_{\partial^*A\cap\Omega} (w-\widetilde{w})\,\mathrm{n}_{\partial^*A}\cdot\big(B + [\![\nabla\widetilde{w}]\!]\big)\,d\mathcal{H}^1\bigg|. \tag{140}$$

To this end, we first smuggle in the average $\fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1$ allowing us to deduce from (125), (127) and (118) that

$$\begin{aligned}\widetilde{Err} \leq{}& \bigg|\int_{\partial^*A\cap\Omega}\Big((w-\widetilde{w}) - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\,\mathrm{n}_{\partial^*A}\cdot\big(B+[\![\nabla\widetilde{w}]\!]\big)\,d\mathcal{H}^1\bigg| \\ &+ \frac{\delta}{8}\int_\Omega|\nabla(w-\widetilde{w})|^2\,dx \\ &+ \widetilde{C}\Big(\frac{\|B\|_{L^\infty(\partial\mathscr{A})}}{\ell} + C_{PS}^2(\mathscr{A})\|B\|^2_{L^\infty(\partial\mathscr{A})}\mathcal{H}^1(\partial\mathscr{A})\Big)E[A|\mathscr{A}].\end{aligned} \tag{141}$$

In a second step, we write by an application of the coarea formula the integral on $\partial^*A\cap\Omega$ as an integral on $\partial\mathscr{A}$. Note that the associated coarea factor on $\partial\mathscr{A}$ is simply given by $\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2} = \frac{1-H_{\partial\mathscr{A}}h}{(\mathrm{n}_{\partial^*A})_h\cdot\mathrm{n}_{\partial\mathscr{A}}}$, where $h' := \bar{t}_{\partial\mathscr{A}}\cdot\nabla^{tan}h$ with

$$J\bar{t}_{\partial\mathscr{A}} = \bar{\mathrm{n}}_{\partial\mathscr{A}} \tag{142}$$

for the counter-clockwise rotation by ninety degrees J. Indeed, this formula for the coarea factor is easily deduced from representing the interface $\partial^*A\cap\Omega$ as the image of a curve $\gamma_h(\theta) := \gamma(\theta) + h(\gamma(\theta))\mathrm{n}_{\partial\mathscr{A}}(\gamma(\theta))$, where γ is an arclength parametrization of $\partial\mathscr{A}$, and from the fact that $(\mathrm{n}_{\partial^*A})_h\cdot\mathrm{n}_{\partial\mathscr{A}} = |(\mathrm{n}_{\partial^*A})_h\cdot\mathrm{n}_{\partial\mathscr{A}}|$. Hence, decomposing $(\mathrm{n}_{\partial^*A})_h = ((\mathrm{n}_{\partial^*A})_h\cdot\bar{\mathrm{n}}_{\partial\mathscr{A}})\bar{\mathrm{n}}_{\partial\mathscr{A}} + ((\mathrm{n}_{\partial^*A})_h\cdot\bar{t}_{\partial\mathscr{A}})\bar{t}_{\partial\mathscr{A}}$ and recalling from (26) that $B = (\bar{n}_{\partial\mathscr{A}}\cdot([\![\nabla\bar{u}]\!]\circ P_{\partial\mathscr{A}}))\bar{n}_{\partial\mathscr{A}}$, we obtain from the coarea formula that

$$\begin{aligned}&\bigg|\int_{\partial^*A\cap\Omega}\Big((w-\widetilde{w}) - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\,\mathrm{n}_{\partial^*A}\cdot\big(B+[\![\nabla\widetilde{w}]\!]\big)\,d\mathcal{H}^1\bigg| \\ &\leq \bigg|\int_{\partial\mathscr{A}}(1-H_{\partial\mathscr{A}}h)\Big((w-\widetilde{w})_h - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{u}-(\nabla\widetilde{w})_h]\!]\,d\mathcal{H}^1\bigg| \\ &\quad+\bigg|\int_{\partial\mathscr{A}}\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}\big((\mathrm{n}_{\partial^*A})_h\cdot\bar{t}_{\partial\mathscr{A}}\big) \\ &\qquad\times\Big((w-\widetilde{w})_h - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\bar{t}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{u}-(\nabla\widetilde{w})_h]\!]\,d\mathcal{H}^1\bigg| \\ &=: \widetilde{Err}_1 + \widetilde{Err}_2.\end{aligned} \tag{143}$$

We proceed with an estimate for $\widetilde{Err}_2$. To this end, we first record the following elementary identities:

$$\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}\big((\mathrm{n}_{\partial^*A})_h\cdot\bar{t}_{\partial\mathscr{A}}\big)=-h' \quad \text{on } \partial\mathscr{A}, \tag{144}$$

as well as

$$\mathrm{n}_{\partial\mathscr{A}}\cdot[\![(\nabla\widetilde{w})_h]\!]=\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla(\widetilde{w}_h)]\!] \quad \text{on } \partial\mathscr{A}, \tag{145}$$

$$\bar{t}_{\partial\mathscr{A}}\cdot[\![(\nabla\widetilde{w})_h]\!]=\frac{1}{1-H_{\partial\mathscr{A}}h}\,\bar{t}_{\partial\mathscr{A}}\cdot[\![\nabla(\widetilde{w}_h)]\!] \quad \text{on } \partial\mathscr{A}. \tag{146}$$

Indeed, (144) follows from representing $\partial^*A\cap\Omega$ as the image of a curve γ_h whereas the claims (145)–(146) are direct consequences of the chain rule in the form of $(\nabla\widetilde{w})_h=(\nabla\Psi^h\circ(\Psi^h)^{-1})\nabla(\widetilde{w}_h)$ and the formulas (112) and (115). Hence, we obtain from an application of the Cauchy–Schwarz inequality, Young's inequality, the Sobolev–Poincaré trace inequality (126), the estimate (133), and the estimate (116)

$$\begin{aligned}&\widetilde{Err}_2\\ &\leq\frac{\delta}{8}\int_\Omega|\nabla(w-\widetilde{w})|^2\,dx\\ &\quad+\frac{\widetilde{C}}{\delta}\Big(C_{reg}(\mathscr{A},\Lambda)+\big\|\bar{t}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{u}]\!]\big\|_{L^\infty(\partial\mathscr{A})}\Big)^2C_{PS}^2(\mathscr{A})\mathcal{H}^1(\partial\mathscr{A})E_{rel}[A,\mu|\mathscr{A}].\end{aligned} \tag{147}$$

It remains to control $\widetilde{Err}_1$. Recalling that $|H_{\partial\mathscr{A}}|\leq 1/\ell$, it follows analogously to (147), relying this time on (117) instead of (116) and (145) instead of (146), that

$$\begin{aligned}&\widetilde{Err}_1\\ &\leq\bigg|\int_{\partial\mathscr{A}}\Big((w-\widetilde{w})_h-\fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla(\bar{u}-\widetilde{w}_h)]\!]\,d\mathcal{H}^1\bigg|\\ &\quad+\frac{\delta}{8}\int_\Omega|\nabla(w-\widetilde{w})|^2\,dx\\ &\quad+\frac{\widetilde{C}}{\delta}\Big(C_{reg}(\mathscr{A},\Lambda)+\big\|\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{u}]\!]\big\|_{L^\infty(\partial\mathscr{A})}\Big)^2C_{PS}^2(\mathscr{A})\mathcal{H}^1(\partial\mathscr{A})E_{vol}[A|\mathscr{A}].\end{aligned} \tag{148}$$

Recalling the decompositions $(w-\widetilde{w})_h=v_h^{(1)}+v_h^{(2)}$ and $\bar{u}-\widetilde{w}_h=\bar{v}_h^{(1)}+\bar{v}_h^{(2)}$, where we in addition note that $v_h^{(2)}=\bar{v}_h^{(2)}=0$ on $\partial\mathscr{A}$ by virtue of the definitions of $v_h^{(1)}$ and $\bar{v}_h^{(1)}$, we further estimate by triangle inequality and the definition (139) of Err

$$\begin{aligned}&\bigg|\int_{\partial\mathscr{A}}\Big((w-\widetilde{w})_h-\fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla(\bar{u}-\widetilde{w}_h)]\!]\,d\mathcal{H}^1\bigg|\\ &\leq Err+\bigg|\int_{\partial\mathscr{A}}\Big(v_h^{(1)}-\fint_{\partial\mathscr{A}}v_h^{(1)}\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar{v}_h^{(2)}]\!]\,d\mathcal{H}^1\bigg|\\ &=: Err+\widetilde{Err}_1'.\end{aligned} \tag{149}$$

In order to estimate the remaining term $\widetilde{Err}_1'$, we record that

$$\Delta\bar{v}_h^{(2)}=\nabla\cdot\big((a^h-\mathrm{Id})\nabla\widetilde{w}_h\big) \quad \text{in } \Omega\setminus\partial\mathscr{A}, \tag{150}$$

$$\mathrm{tr}_{\partial\mathscr{A}}\bar{v}_h^{(2)}=0 \quad \text{on } \partial\mathscr{A}, \tag{151}$$

$$(\mathrm{n}_{\partial\Omega}\cdot\nabla)\bar{v}_h^{(2)}=0 \quad \text{on } \partial\Omega. \tag{152}$$

Hence, testing in a first step the PDE satisfied by $\bar v_h^{(2)}$ with the test function $v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1$, and in a second step the PDE satisfied by $v_h^{(1)}$ with the test function $\bar v_h^{(2)}$, we get

$$
\begin{aligned}
&\int_{\partial\mathscr{A}} \Big(v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot[\![\nabla\bar v_h^{(2)}]\!]\,d\mathcal{H}^1 \\
&= \int_\Omega \nabla v_h^{(1)}\cdot\nabla\bar v_h^{(2)}\,dx - \int_\Omega \nabla v_h^{(1)}\cdot\big((a^h{-}\mathrm{Id})\nabla\widetilde w_h\big)\,dx \\
&\quad + \int_{\partial\mathscr{A}} \Big(v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot(a^h{-}\mathrm{Id})[\![\nabla\widetilde w_h]\!]\,d\mathcal{H}^1 \\
&= -\int_\Omega \nabla v_h^{(1)}\cdot\big((a^h{-}\mathrm{Id})\nabla\widetilde w_h\big)\,dx \\
&\quad + \int_{\partial\mathscr{A}} \Big((w{-}\widetilde w)_h - \fint_{\partial\mathscr{A}} (w{-}\widetilde w)_h\,d\mathcal{H}^1\Big)\mathrm{n}_{\partial\mathscr{A}}\cdot(a^h{-}\mathrm{Id})[\![\nabla\widetilde w_h]\!]\,d\mathcal{H}^1.
\end{aligned}
\tag{153}
$$

In particular, based on the same arguments leading to, e.g., (147), the definition of a^h, the formulas (112) and (113), as well as the estimates from Lemma 16, we deduce from (153) that

$$
\begin{aligned}
\widetilde{Err}_1' \leq{}& \frac{\delta}{8}\int_\Omega |\nabla(w{-}\widetilde w)|^2\,dx \\
&+ \frac{\widetilde C}{\delta} C_{reg}^2(\mathscr{A},\Lambda)\big(1{+}C_{PS}^2(\mathscr{A})\mathcal{H}^1(\partial\mathscr{A})\big)E[A|\mathscr{A}].
\end{aligned}
\tag{154}
$$

In summary, comparing our estimates (141)–(149) and (154) with the claim (72), we see that we reduced matters to an estimate of (139).

Step 2: Estimate for (139). Note that (111) and (115) directly entail

$$
H_{\partial\mathscr{A}} + (\nabla\cdot\xi)\circ(\Psi^h)^{-1} = -\Big(\frac{H_{\partial\mathscr{A}}^2}{1-H_{\partial\mathscr{A}}h}\Big)h =: f_h \quad \text{on } \partial\mathscr{A}.
$$

Furthermore, the very definition of the Hilbert space structure on $H^{1/2}_{MS}(\partial\mathscr{A})$ yields

$$
Err = \Big|\Big\langle v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1, f_h {-} \fint_{\partial\mathscr{A}} f_h\,d\mathcal{H}^1\Big\rangle_{H^{1/2}_{MS}(\partial\mathscr{A})}\Big|.
$$

Hence, from the Cauchy–Schwarz inequality, the definition of the norm on $H^{1/2}_{MS}(\partial\mathscr{A})$, i.e. $\|v_h^{(1)} - \fint_{\partial\mathscr{A}} v_h^{(1)}\,d\mathcal{H}^1\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})} = \int_\Omega |\nabla v_h^{(1)}|^2\,dx$, and the estimates (132)–(133) we infer

$$
Err \leq \frac{\delta}{8}\int_\Omega |\nabla(w{-}\widetilde w)|^2\,dx + \frac{\widetilde C}{\delta}\Big\|f_h {-} \fint_{\partial\mathscr{A}} f_h\Big\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})}.
\tag{155}
$$

Because of

$$
\begin{aligned}
\nabla^{tan} f_h = {}& -\Big(\frac{H_{\partial\mathscr{A}}^2}{1-H_{\partial\mathscr{A}}h}\Big)\nabla^{tan}h \\
&- \Big(\frac{2H_{\partial\mathscr{A}}\nabla^{tan}H_{\partial\mathscr{A}}}{1-H_{\partial\mathscr{A}}h}\Big)h \\
&- \Big(\frac{H_{\partial\mathscr{A}}^2(H_{\partial\mathscr{A}}\nabla^{tan}h + h\nabla^{tan}H_{\partial\mathscr{A}})}{(1-H_{\partial\mathscr{A}}h)^2}\Big)h
\end{aligned}
\tag{156}
$$

and $H_{\partial\mathscr{A}} \leq 1/\ell$, we obtain from the interpolation estimate (135) (for a proof of $\|f_h - \fint_{\partial\mathscr{A}} f_h \, d\mathcal{H}^1\|_{H^2(\partial\mathscr{A})} \leq C(\Omega, \mathscr{A}, \Lambda) =: L$, we refer to the proof of Lemma 15), the estimates (116)–(117) and assumption (69) that

$$
\begin{aligned}
&\Big\| f_h - \fint_{\partial\mathscr{A}} f_h \Big\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})} \\
&\leq \widetilde{C} C^2_{int}(\mathscr{A}, \Omega, \Lambda) \Big(\|\nabla^{tan} H_{\partial\mathscr{A}}\|^2_{L^\infty(\partial\mathscr{A})} + \frac{1}{\ell^4} \Big) E[A|\mathscr{A}].
\end{aligned}
\tag{157}
$$

Plugging this estimate back into (155) concludes the proof of (72). □

Proof of Lemma 11. The identity (109) simply follows from (13) and $|\nabla s_{\partial\mathscr{A}}| = 1$.

For the proof of (110) and (111), we first note that

$$
P_{\partial\mathscr{A}}(x) = x - s_{\partial\mathscr{A}}(x) \bar{\mathrm{n}}_{\partial\mathscr{A}}(x), \quad x \in B_{\frac{\ell}{2}}(\partial\mathscr{A}), \tag{158}
$$

and therefore

$$
\nabla P_{\partial\mathscr{A}} = \big(\mathrm{Id} - \bar{\mathrm{n}}_{\partial\mathscr{A}} \otimes \bar{\mathrm{n}}_{\partial\mathscr{A}}\big) - s_{\partial\mathscr{A}} \nabla \bar{\mathrm{n}}_{\partial\mathscr{A}} \quad \text{in } B_{\frac{\ell}{2}}(\partial\mathscr{A}). \tag{159}
$$

Since $(\bar{t}_{\partial\mathscr{A}} \cdot \nabla) \mathrm{n}_{\partial\mathscr{A}} = -H_{\partial\mathscr{A}} \bar{t}_{\partial\mathscr{A}}$ along $\partial\mathscr{A}$, one deduces from (159)

$$
(\bar{t}_{\partial\mathscr{A}} \cdot \nabla) P_{\partial\mathscr{A}} = \bar{t}_{\partial\mathscr{A}} + \bar{H}_{\partial\mathscr{A}} s_{\partial\mathscr{A}} (\bar{t}_{\partial\mathscr{A}} \cdot \nabla) P_{\partial\mathscr{A}} \quad \text{in } B_{\frac{\ell}{2}}(\partial\mathscr{A}). \tag{160}
$$

In other words, $(\bar{t}_{\partial\mathscr{A}} \cdot \nabla) P_{\partial\mathscr{A}} = (1 - \bar{H}_{\partial\mathscr{A}} s_{\partial\mathscr{A}})^{-1} \bar{t}_{\partial\mathscr{A}}$ in $B_{\frac{\ell}{2}}(\partial\mathscr{A})$, so that for any sufficiently regular $f \colon \partial\mathscr{A} \to \mathbb{R}$

$$
(\bar{t}_{\partial\mathscr{A}} \cdot \nabla)(f \circ P_{\partial\mathscr{A}}) = \frac{\big((\bar{t}_{\partial\mathscr{A}} \cdot \nabla) f\big) \circ P_{\partial\mathscr{A}}}{1 - \bar{H}_{\partial\mathscr{A}} s_{\partial\mathscr{A}}} \quad \text{in } B_{\frac{\ell}{2}}(\partial\mathscr{A}). \tag{161}
$$

Hence, (110) and (111) follow from (161), $(\bar{t}_{\partial\mathscr{A}} \cdot \nabla) \mathrm{n}_{\partial\mathscr{A}} = -H_{\partial\mathscr{A}} \bar{t}_{\partial\mathscr{A}}$ along $\partial\mathscr{A}$, $(\bar{\mathrm{n}}_{\partial\mathscr{A}} \cdot \nabla) \bar{\mathrm{n}}_{\partial\mathscr{A}} = 0$ (which itself is a consequence of (105)), and $(\nabla \bar{\mathrm{n}}_{\partial\mathscr{A}})^{\mathsf{T}} \bar{\mathrm{n}}_{\partial\mathscr{A}} = 0$ (which itself follows from $|\bar{\mathrm{n}}_{\partial\mathscr{A}}| = 1$).

The formula (112) is now a routine computation based on (161) and the definition (108). The formula for the Jacobian (113) follows from (112) and the general fact that for any $B = A + v \otimes w$ with invertible A, it holds $\det(B) = \det(A)(1 + w \cdot A^{-1} v)$.

Finally, the formulas (114) and (115) follow from $\operatorname{supp} \bar{\zeta} \subset [-1/2, 1/2]$, the definition (104), and (69). □

Proof of Lemma 12. The estimates (118) simply follow from coarea formula in the form of $\int_{\partial^* A \cap \Omega} 1 \, d\mathcal{H}^1 = \int_{\partial\mathscr{A}} \sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2} \, d\mathcal{H}^1$, $|H_{\partial\mathscr{A}}| \leq 1/\ell$, and (69).

For the proof of (116), we recall some basic geometric identities. Denoting by $\bar{\gamma}$ an arc-length parametrization of $\partial\mathscr{A}$, we may represent the interface $\partial^* A \cap \Omega$ as the image of the curve $\gamma_h := \bar{\gamma} + (h \circ \bar{\gamma})(\mathrm{n}_{\partial\mathscr{A}} \circ \bar{\gamma})$. A straightforward computation then reveals $\gamma_h' = \big((1 - H_{\partial\mathscr{A}} h) \bar{t}_{\partial\mathscr{A}} + h' \mathrm{n}_{\partial\mathscr{A}}\big) \circ \bar{\gamma}$, so that

$$
\mathrm{n}_{\partial^* A} \circ \gamma_h = \frac{(\gamma_h')^\perp}{|\gamma_h'|} = \Big(\frac{(1 - H_{\partial\mathscr{A}} h) \mathrm{n}_{\partial\mathscr{A}} - h' \bar{t}_{\partial\mathscr{A}}}{\sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2}} \Big) \circ \bar{\gamma}. \tag{162}
$$

Hence,

$$
\big((1 - \mathrm{n}_{\partial^* A} \cdot \xi) \circ \gamma_h\big) |\gamma_h'| = \Big(\sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2} - (1 - H_{\partial\mathscr{A}} h) \Big) \circ \bar{\gamma}. \tag{163}
$$

By a Taylor expansion and (69), the estimates (116) follow from (163).

We turn to a proof of (117). To this end, we make use of the diffeomorphism

$$
\Phi \colon B_{\ell/2}(\partial\mathscr{A}) \to \partial\mathscr{A} \times (-\ell/2, \ell/2), \quad x \mapsto (P_{\partial\mathscr{A}}(x), s_{\partial\mathscr{A}}(x)). \tag{164}
$$

Due to $(\bar{t}_{\partial\mathscr{A}}\cdot\nabla)P_{\partial\mathscr{A}} = (1-\bar{H}_{\partial\mathscr{A}}s_{\partial\mathscr{A}})^{-1}\bar{t}_{\partial\mathscr{A}}$ and $\nabla s_{\partial\mathscr{A}} = \bar{\mathrm{n}}_{\partial\mathscr{A}}$ in $B_{\frac{\ell}{2}}(\partial\mathscr{A})$, the associated Jacobian is simply given by

$$\det\nabla\Phi = \frac{1}{1-\bar{H}_{\partial\mathscr{A}}s_{\partial\mathscr{A}}} \quad \text{in } B_{\ell/2}(\partial\mathscr{A}). \tag{165}$$

Hence, by a change of variables Φ as well as the assumptions (14) and (69), we compute

$$E_{vol}[A|\mathscr{A}] = \int_{\partial\mathscr{A}}\int_0^h \frac{1}{1-H_{\partial\mathscr{A}}s}\frac{s}{\ell^2}\,\mathrm{d}s\,d\mathcal{H}^1, \tag{166}$$

so that (117) follows from (69). □

Proof of Lemma 13. These assertions are immediate consequences of the well-known transformation formulas for PDEs in distributional form. □

Proof of Lemma 14. We proceed in three steps.

Step 1: Proof of (125)*, Part I.* Adding zero, recalling (51), and exploiting the Gibbs–Thomson law (7e) with admissible test function ξ, we obtain

$$\begin{aligned}
\int_{\partial^*A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1 &= \int_{\partial^*A\cap\Omega} w+\nabla\cdot\xi\,d\mathcal{H}^1 \\
&= \int_{\partial^*A\cap\Omega} w\mathrm{n}_{\partial^*A}\cdot\xi\,d\mathcal{H}^1 + \int_{\partial^*A\cap\Omega} w(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1 \\
&\quad + \int_{\partial^*A\cap\Omega} (\mathrm{Id}-\mathrm{n}_{\partial^*A}\otimes\mathrm{n}_{\partial^*A}):\nabla\xi\,d\mathcal{H}^1 \\
&\quad + \int_{\partial^*A\cap\Omega} \mathrm{n}_{\partial^*A}\otimes\mathrm{n}_{\partial^*A}:\nabla\xi\,d\mathcal{H}^1 \\
&= \int_{\partial^*A\cap\Omega} w(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1 \\
&\quad + \int_{\partial^*A\cap\Omega} \mathrm{n}_{\partial^*A}\otimes\mathrm{n}_{\partial^*A}:\nabla\xi\,d\mathcal{H}^1.
\end{aligned} \tag{167}$$

Based on (109) and (110), we also deduce that along $\partial^*A\cap\Omega$

$$\begin{aligned}
\mathrm{n}_{\partial^*A}\otimes\mathrm{n}_{\partial^*A}:\nabla\xi &= -\frac{\bar{H}_{\partial\mathscr{A}}}{1-\bar{H}_{\partial\mathscr{A}}\bar{h}}(\mathrm{n}_{\partial^*A}\cdot\bar{t}_{\partial\mathscr{A}})^2 \\
&= -\frac{\bar{H}_{\partial\mathscr{A}}}{1-\bar{H}_{\partial\mathscr{A}}\bar{h}}(1+\mathrm{n}_{\partial^*A}\cdot\bar{\mathrm{n}}_{\partial\mathscr{A}})(1-\mathrm{n}_{\partial^*A}\cdot\xi).
\end{aligned} \tag{168}$$

Hence, inserting (168) back into (167) and adding two times another zero yields (recalling in the process also (51) and (111))

$$\begin{aligned}
\int_{\partial^*A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1 &= \int_{\partial^*A\cap\Omega} (w-\widetilde{w})(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1 \\
&\quad - \int_{\partial^*A\cap\Omega} \frac{\bar{H}_{\partial\mathscr{A}}}{1-\bar{H}_{\partial\mathscr{A}}\bar{h}}(\mathrm{n}_{\partial^*A}\cdot\bar{\mathrm{n}}_{\partial\mathscr{A}})(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1 \\
&= \Big(\fint_{\partial^*A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1\Big)E_{rel}[A,\mu|\mathscr{A}] \\
&\quad + \int_{\partial^*A\cap\Omega}\Big((w-\widetilde{w})-\fint_{\partial^*A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1\Big)(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\partial^* A\cap\Omega} \frac{\bar{H}_{\partial\mathscr{A}}}{1-\bar{H}_{\partial\mathscr{A}}\bar{h}} (\mathrm{n}_{\partial^* A}\cdot\bar{\mathrm{n}}_{\partial\mathscr{A}})(1-\mathrm{n}_{\partial^* A}\cdot\xi)\,d\mathcal{H}^1 \\
&=: I + II + III. \qquad (169)
\end{aligned}
$$

By virtue of (118) and (71), we then obtain for $M \gg 1$

$$
|I+III| \leq \frac{1}{4}\bigg|\int_{\partial^* A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1\bigg| + \frac{\widetilde{C}}{\ell}E_{rel}[A,\mu|\mathscr{A}], \qquad (170)
$$

so that we obtain the following upgrade of (169)

$$
\bigg|\int_{\partial^* A\cap\Omega} w-\widetilde{w}\,d\mathcal{H}^1\bigg| \leq \widetilde{C}|II| + \frac{\widetilde{C}}{\ell}E_{rel}[A,\mu|\mathscr{A}]. \qquad (171)
$$

It remains to bound the contribution from II, cf. (169). Because of $(1-n_{\partial^* A}\cdot\xi)^2 \leq 4(1-n_{\partial^* A}\cdot\xi)$ we get from an application of Cauchy–Schwarz inequality

$$
|II| \leq 2\bigg(\int_{\partial^* A\cap\Omega}\bigg|(w-\widetilde{w}) - \fint_{\partial^* A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1\bigg|^2 d\mathcal{H}^1\bigg)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}],
$$

so that by adding zero we obtain

$$
\begin{aligned}
&|II| \\
&\leq 2\bigg(\int_{\partial^* A\cap\Omega}\bigg|(w-\widetilde{w}) - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg|^2 d\mathcal{H}^1\bigg)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}] \\
&\quad + 2\sqrt{\mathcal{H}^1(\partial^* A\cap\Omega)}\bigg|\fint_{\partial^* A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1 - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg| E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}] \\
&\leq 2\bigg(\int_{\partial^* A\cap\Omega}\bigg|(w-\widetilde{w}) - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg|^2 d\mathcal{H}^1\bigg)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}] \\
&\quad + 2\frac{\sqrt{\mathcal{H}^1(\partial^* A\cap\Omega)}}{\mathcal{H}^1(\partial\mathscr{A})}\bigg|\int_{\partial^* A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1 - \int_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg| E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}] \\
&\quad + 2\sqrt{\mathcal{H}^1(\partial^* A\cap\Omega)}\bigg|\frac{1}{\mathcal{H}^1(\partial^* A\cap\Omega)} - \frac{1}{\mathcal{H}^1(\partial\mathscr{A})}\bigg| E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}]\bigg|\int_{\partial^* A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1\bigg| \\
&=: II' + II'' + II'''. \qquad (172)
\end{aligned}
$$

We estimate term by term. For the first term, we simply make use of the coarea formula, the Sobolev–Poincaré trace inequality (126) as well as the estimate (133) to deduce

$$
\begin{aligned}
II' &\leq \widetilde{C}\bigg(\int_{\partial\mathscr{A}}\bigg|(w-\widetilde{w})_h - \fint_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg|^2 d\mathcal{H}^1\bigg)^{\frac{1}{2}} E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}] \\
&\leq \widetilde{C}\sqrt{\mathcal{H}^1(\partial\mathscr{A})}C_{PS}(\mathscr{A})E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}]\bigg(\int_\Omega|\nabla(w-\widetilde{w})|^2\,dx\bigg)^{\frac{1}{2}}. \qquad (173)
\end{aligned}
$$

For an estimate of the second term, we first recognize through an application of coarea formula and adding zero that

$$
\begin{aligned}
&\bigg|\int_{\partial^* A\cap\Omega}(w-\widetilde{w})\,d\mathcal{H}^1 - \int_{\partial\mathscr{A}}(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg| \\
&= \bigg|\int_{\partial\mathscr{A}}\big(\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\big)(w-\widetilde{w})_h\,d\mathcal{H}^1\bigg|
\end{aligned}
$$

$$
\begin{aligned}
(174)\qquad &\leq \bigg| \int_{\partial\mathscr{A}} \big(\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\big)\Big((w-\widetilde{w})_h - \fint_{\partial\mathscr{A}} (w-\widetilde{w})_h \, d\mathcal{H}^1\Big)\, d\mathcal{H}^1 \bigg| \\
&\quad + \frac{1}{\mathcal{H}^1(\partial\mathscr{A})}\bigg(\int_{\partial\mathscr{A}} \Big|\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\Big| \, d\mathcal{H}^1\bigg) \\
&\qquad\qquad \times \bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 - \int_{\partial\mathscr{A}} (w-\widetilde{w})_h \, d\mathcal{H}^1 \bigg| \\
&\quad + \frac{1}{\mathcal{H}^1(\partial\mathscr{A})}\bigg(\int_{\partial\mathscr{A}} \Big|\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\Big| \, d\mathcal{H}^1\bigg)\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 \bigg|.
\end{aligned}
$$

Furthermore, thanks to the estimates (116)–(117), a Taylor expansion, the assumption (69), and Jensen's inequality

$$
\int_{\partial\mathscr{A}} \Big|\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\Big| \, d\mathcal{H}^1 \leq \widetilde{C}\sqrt{\mathcal{H}^1(\partial\mathscr{A})}\sqrt{E[A|\mathscr{A}]}. \tag{175}
$$

Similarly,

$$
\bigg(\int_{\partial\mathscr{A}} \Big|\sqrt{(1-H_{\partial\mathscr{A}}h)^2+(h')^2}-1\Big|^2 \, d\mathcal{H}^1\bigg)^{\frac{1}{2}} \leq \widetilde{C}\sqrt{E[A|\mathscr{A}]}. \tag{176}
$$

Hence, inserting the estimates (175) and (176) back into (174) (the second after an application of Cauchy–Schwarz inequality in the first right hand side term of (174)) and recalling the argument for (173), we may upgrade (174) to

$$
\begin{aligned}
&\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 - \int_{\partial\mathscr{A}} (w-\widetilde{w})_h \, d\mathcal{H}^1 \bigg| \\
(177)\qquad &\leq \widetilde{C}\sqrt{\mathcal{H}^1(\partial\mathscr{A})}C_{PS}(\mathscr{A})E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}]\bigg(\int_\Omega |\nabla(w-\widetilde{w})|^2 \, dx\bigg)^{\frac{1}{2}} \\
&\quad + \widetilde{C}\frac{\sqrt{E[A|\mathscr{A}]}}{\sqrt{\mathcal{H}^1(\partial\mathscr{A})}}\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 - \int_{\partial\mathscr{A}} (w-\widetilde{w})_h \, d\mathcal{H}^1 \bigg| \\
&\quad + \widetilde{C}\frac{\sqrt{E[A|\mathscr{A}]}}{\sqrt{\mathcal{H}^1(\partial\mathscr{A})}}\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 \bigg|.
\end{aligned}
$$

Overall, it follows now from (177), (118), and assumption (71) that for $M \gg 1$

$$
\begin{aligned}
(178)\qquad II'' &\leq \widetilde{C}C_{PS}(\mathscr{A})E_{rel}[A,\mu|\mathscr{A}]\bigg(\int_\Omega |\nabla(w-\widetilde{w})|^2 \, dx\bigg)^{\frac{1}{2}} \\
&\quad + \frac{1}{4}\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 \bigg|.
\end{aligned}
$$

It follows also immediately from (118), and assumption (71) that for $M \gg 1$

$$
II''' \leq \frac{1}{4}\bigg| \int_{\partial^* A\cap\Omega} (w-\widetilde{w}) \, d\mathcal{H}^1 \bigg|. \tag{179}
$$

Hence, plugging the estimates (173), (178) and (179) back into (172), we obtain as an upgrade of (172)

$$
|II| \leq \widetilde{C}\sqrt{\mathcal{H}^1(\partial\mathscr{A})}C_{PS}(\mathscr{A})E_{rel}^{\frac{1}{2}}[A,\mu|\mathscr{A}]\bigg(\int_\Omega |\nabla(w-\widetilde{w})|^2 \, dx\bigg)^{\frac{1}{2}} \tag{180}
$$

$$+ \widetilde{C} C_{PS}(\mathscr{A}) E_{rel}[A,\mu|\mathscr{A}] \bigg(\int_\Omega |\nabla(w - \widetilde{w})|^2 \, dx \bigg)^{\frac{1}{2}} + \frac{1}{2} \bigg| \int_{\partial^* A \cap \Omega} (w - \widetilde{w}) \, d\mathcal{H}^1 \bigg|.$$

In view of (171) and assumption (71), it therefore remains to choose $M \gg 1$ to conclude with (125) in terms of the average $\fint_{\partial^* A\cap\Omega} w - \widetilde{w} \, d\mathcal{H}^1$.

Step 2: Proof of (125)*, Part II.* Decomposing

$$\begin{aligned}
&\int_{\partial\mathscr{A}} (w - \widetilde{w})_h \, d\mathcal{H}^1 \\
&= \int_{\partial^* A\cap\Omega} (w - \widetilde{w}) \, d\mathcal{H}^1 \\
&\quad + \int_{\partial\mathscr{A}} \bigg((w - \widetilde{w}) - \fint_{\partial\mathscr{A}} (w - \widetilde{w}) \, d\mathcal{H}^1 \bigg) \big(1 - \sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2} \big) \, d\mathcal{H}^1 \\
&\quad + \bigg(\fint_{\partial\mathscr{A}} (w - \widetilde{w}) \, d\mathcal{H}^1 \bigg) \int_{\partial\mathscr{A}} \big(1 - \sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2} \big) \, d\mathcal{H}^1,
\end{aligned}$$

we may infer the asserted estimate (125) for $\fint_{\partial\mathscr{A}} (w - \widetilde{w})_h \, d\mathcal{H}^1$ from the corresponding estimate for $\fint_{\partial^* A\cap\Omega} (w - \widetilde{w}) \, d\mathcal{H}^1$ from Step 1 and the arguments used to derive (180).

Step 3: Proof of (127)*.* First, we simply recognize that by means of (50)–(52)

$$\int_{\partial^* A\cap\Omega} \mathrm{n}_{\partial^* A} \cdot [\![\nabla \widetilde{w}]\!] \, d\mathcal{H}^1 = 0.$$

Second, by assumption (26), the coarea formula, and the splitting $\mathrm{n}_{\partial^*\mathscr{A}} = (\mathrm{n}_{\partial^*\mathscr{A}} \cdot \bar{\mathrm{n}}_{\partial\mathscr{A}}) \bar{\mathrm{n}}_{\partial\mathscr{A}} + (\mathrm{n}_{\partial^*\mathscr{A}} \cdot \bar{t}_{\partial\mathscr{A}}) \bar{t}_{\partial\mathscr{A}}$, we obtain

$$\begin{aligned}
\int_{\partial^* A\cap\Omega} \mathrm{n}_{\partial^* A} \cdot B \, d\mathcal{H}^1 &= \int_{\partial\mathscr{A}} (1 - H_{\partial\mathscr{A}} h) \bar{\mathrm{n}}_{\partial\mathscr{A}} \cdot [\![\nabla \bar{u}]\!] \, d\mathcal{H}^1 \\
&= - \int_{\partial\mathscr{A}} H_{\partial\mathscr{A}} h \bar{\mathrm{n}}_{\partial\mathscr{A}} \cdot [\![\nabla \bar{u}]\!] \, d\mathcal{H}^1.
\end{aligned}$$

Hence,

$$\bigg| \int_{\partial^* A\cap\Omega} \mathrm{n}_{\partial^* A} \cdot B \, d\mathcal{H}^1 \bigg| \leq \widetilde{C} \big\| \mathrm{n}_{\partial\mathscr{A}} \cdot [\![\nabla \bar{u}]\!] \big\|_{L^\infty(\partial\mathscr{A})} \sqrt{\mathcal{H}^1(\partial\mathscr{A})} \sqrt{E_{vol}[A|\mathscr{A}]},$$

so that the claim (127) follows from the previous three displays. □

Proof of Lemma 15. By a change of variables Ψ^h, it follows from (50)–(52) and (111)

$$-\nabla \cdot (a^h \nabla \widetilde{w}_h) = 0 \qquad \text{in } \Omega \setminus \partial\mathscr{A}, \tag{181}$$

$$\mathrm{tr}_{\partial\mathscr{A}} \widetilde{w}_h = \frac{H_{\partial\mathscr{A}}}{1 - H_{\partial\mathscr{A}} h} \qquad \text{on } \partial\mathscr{A} \cap \Omega, \tag{182}$$

$$(\mathrm{n}_{\partial\Omega} \cdot \nabla) \widetilde{w}_h = 0 \qquad \text{on } \partial\Omega. \tag{183}$$

Hence, the regularity estimate (128) is in principle just a simple consequence of standard Schauder theory applied to the two regular open sets $\Omega \setminus \overline{\mathscr{A}}$ and $\mathscr{A}$. The corresponding estimates depend on the respective domain, the ellipticity constant of a^h, the $C^{0,1/2}$ Hölder norm of a^h on the closure of the respective domain, and finally the $C^{1,1/2}$ Hölder norm of the extended Dirichlet data $(1 - \bar{H}_{\partial\mathscr{A}} \bar{h})^{-1} \bar{h}_{\partial\mathscr{A}}$.

We therefore only need to show that this data is bounded by a constant of required form $C = C(\mathscr{A}, \Lambda)$.

Recalling the formulas (112), (113) and $a^h = \frac{1}{|\det \Psi^h|}(\nabla\Psi^h)^\mathsf{T}\nabla\Psi^h$, the claim follows once we proved that $\|h'\|_{C^{0,1/2}(\partial\mathscr{A})} \leq C(\mathscr{A}, \Lambda)$. To this end, recall that we know from Schätzle's estimate and (45)–(47) that $\|\mathrm{tr}_{\partial^* A\cap\Omega} u\|_{L^p(\partial^* A\cap\Omega)} \leq C(\mathscr{A}, \Lambda, p)$ for all $p \in [2, \infty)$. Furthermore, the Gibbs–Thomson law (7e) and the regularity (44) allow to conclude

$$\begin{aligned}\mathrm{tr}_{\partial^* A\cap\Omega} u \circ \gamma_h &= \frac{(\gamma_h')^\perp \cdot \gamma_h''}{|\gamma_h'|^3} \\ &= \bigg(\frac{h'(H_{\partial\mathscr{A}}' h + 2H_{\partial\mathscr{A}} h') + (1 - H_{\partial\mathscr{A}} h)\big(h'' + H_{\partial\mathscr{A}}(1 - H_{\partial\mathscr{A}} h)\big)}{\sqrt{(1 - H_{\partial\mathscr{A}} h)^2 + (h')^2}^3}\bigg) \circ \bar\gamma,\end{aligned}$$

where we again represented the interface $\partial^* A \cap \Omega$ as the image of the curve $\gamma_h := \bar\gamma + (h \circ \bar\gamma)(\mathrm{n}_{\partial\mathscr{A}} \circ \bar\gamma)$ with $\bar\gamma$ denoting an arc-length parametrization of $\partial\mathscr{A}$. In particular, $\|h''\|_{L^p(\partial\mathscr{A})} \leq C(\mathscr{A}, \Lambda, p)$ for all $p \in [2, \infty)$, so that we may conclude by Morrey embedding. □

Proof of Lemma 16. The estimate (133) follows from the chain rule and assumption (69). The estimate (132) in turn follows from the decomposition $(w - \widetilde{w})_h = v_h^{(1)} + v_h^{(2)}$ and the energy estimate for $v_h^{(2)}$ satisfying the PDE

$$\Delta v_h^{(2)} = \nabla \cdot \big((\mathrm{Id} - a^h)\nabla(w - \widetilde{w})_h\big) \qquad \text{in } \Omega \setminus \partial\mathscr{A}, \tag{184}$$

with boundary conditions

$$\mathrm{tr}_{\partial\mathscr{A}} v_h^{(2)} = 0 \qquad \text{on } \partial\mathscr{A}, \tag{185}$$

$$(\mathrm{n}_{\partial\Omega} \cdot \nabla) v_h^{(2)} = 0 \qquad \text{on } \partial\Omega. \tag{186}$$

For a proof of (134), we first note that by definition of a^h it holds $\mathrm{supp}(\mathrm{Id} - a^h) \subset B_{\ell/2}(\partial\mathscr{A})$. Hence, by means of the change of variables Φ from (164), we deduce (134) from Cauchy–Schwarz inequality, (128), $|\mathrm{Id} - a^h| \leq \widetilde{C}(|H_{\partial\mathscr{A}} h| + |h'|)$ in $B_{\ell/2}(\partial\mathscr{A})$, and the estimates (116)–(117). □

Proof of Lemma 17. We argue by contradiction that there exists $C = C(\mathscr{A}, \Omega, L)$ such that for all $f \in H^2(\partial\mathscr{A})$ with $\fint_{\partial\mathscr{A}} f \, d\mathcal{H}^1 = 0$ and $\|f\|_{H^2(\partial\mathscr{A})} \leq L$ it holds

$$\|f\|_{H^{1/2}_{MS}(\partial\mathscr{A})} \leq C\|f\|_{[L^2(\partial\mathscr{A}), H^1(\partial\mathscr{A})]_{1/2}}, \tag{187}$$

where $[L^2(\partial\mathscr{A}), H^1(\partial\mathscr{A})]_\theta$ denotes the interpolation space (i.e., Sobolev–Slobodeckij space) between $L^2(\partial\mathscr{A})$ and $H^1(\partial\mathscr{A})$.

For each $k \in \mathbb{N}$, assume that we may find a map $f_k \in H^2(\partial\mathscr{A})$ satisfying $\fint_{\partial\mathscr{A}} f_k \, d\mathcal{H}^1 = 0$ and $\|f_k\|_{H^2(\partial\mathscr{A})} \leq L$ such that

$$\|\nabla u_k\|_{L^2(\partial\Omega)} = \|f_k\|_{H^{1/2}_{MS}(\partial\mathscr{A})} = 1 > k\|f_k\|_{[L^2(\partial\mathscr{A}), H^1(\partial\mathscr{A})]_{1/2}}, \tag{188}$$

where $u_k \in H^1(\Omega)$ denotes the associated chemical potential, i.e.,

$$\begin{aligned} \Delta u_k &= 0 && \text{in } \Omega \setminus \partial\mathscr{A}, \\ u_k &= f_k && \text{on } \partial\mathscr{A}, \\ (\mathrm{n}_{\partial\Omega} \cdot \nabla) u_k &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since $H^2(\partial\mathscr{A})$ embeds into $[H^1(\partial\mathscr{A}),H^2(\partial\mathscr{A})]_{1/2}$, we deduce from the elliptic regularity estimate

$$\|u_k\|_{H^2(\Omega\setminus\partial\mathscr{A})} \lesssim_{\mathscr{A},\Omega} \|u_k\|_{H^1(\Omega)} + \|f_k\|_{[H^1(\partial\mathscr{A}),H^2(\partial\mathscr{A})]_{1/2}}$$

and Poincaré inequality (recall that $\fint_{\partial\mathscr{A}} u_k\,d\mathcal{H}^1 = 0$ by construction) that

$$\sup_{k\in\mathbb{N}} \|u_k\|_{H^2(\Omega\setminus\partial\mathscr{A})} \lesssim_{\mathscr{A},\Omega,L} 1.$$

Hence, modulo taking a subsequence, $(u_k)_k$ weakly converges in $H^2(\Omega\setminus\partial\mathscr{A})$ to some $u\in H^2(\Omega\setminus\partial\mathscr{A})\cap H^1(\Omega)$ such that $u_k\to u$ strongly in $H^1(\Omega)$. However, by trace theory and (188), $f_k\to 0$ strongly in $[L^2(\partial\mathscr{A}),H^1(\partial\mathscr{A})]_{1/2}$, so that u satisfies

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega\setminus\partial\mathscr{A}, \\ u &= 0 && \text{on } \partial\mathscr{A}, \\ (\mathrm{n}_{\partial\Omega}\cdot\nabla)u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Hence, $u\equiv 0$ in contradiction to $\lim_{k\to\infty}\|\nabla u_k\|_{L^2(\Omega)} = 1 = \|\nabla u\|_{L^2(\Omega)}$.

The claimed estimate (135) now immediately follows from (187) and scaling. □

Proof of Proposition 8, Part II: Estimates (73) *and* (74). The argument is naturally a very close variant of the argument in favor of (72). We leave details to the interested reader, only mentioning for a proof of (74) that

$$\Big\|\vartheta_h - \fint_{\partial\mathscr{A}} \vartheta_h\,d\mathcal{H}^1\Big\|^2_{H^{1/2}_{MS}(\partial\mathscr{A})} \leq \widetilde{C}C^2_{int}(\mathscr{A},\Omega,\Lambda)\frac{1}{\ell^4}\Big(\frac{1}{\ell^2}\|h\|^2_{L^2(\partial\mathscr{A})} + \|h'\|^2_{L^2(\partial\mathscr{A})}\Big)$$

and

$$\bigg|\int_{\partial\mathscr{A}} (\widetilde{w}_\vartheta)_h\,d\mathcal{H}^1\bigg| = \bigg|\int_{\partial\mathscr{A}} \vartheta_h\,d\mathcal{H}^1\bigg| \leq \frac{\sqrt{\mathcal{H}^1(\partial\mathscr{A})}}{\ell^2}\|h\|_{L^2(\partial\mathscr{A})}.$$

This eventually concludes the proof of Proposition 8. □

6. Proof of Proposition 5: Reduction to perturbative graph setting

6.1. **Strategy for the proof of Proposition 5.** The proof of Proposition 5 is divided into several steps which are collected and explained here. The corresponding proofs are presented in the next subsection.

First, let us fix the setting for the whole section. Fix $T'\in(0,T_*)$, $C\in(1,\infty)$, and let $\Lambda=\Lambda(\mathscr{A},A(0),T')\in(0,\infty)$ the constant from Lemma 4. We aim to find $M=M(\Omega,A(0),\mathscr{A},C,\Lambda,T')\in(1,\infty)$ such that for a.e. $t\in\mathscr{T}_{\mathrm{good}}(\Lambda,M)\cap(0,T')$, i.e.,

$$\int_\Omega \frac{1}{2}|\nabla w(\cdot,t)|^2\,dx \leq \Lambda, \tag{189}$$

$$E[A,\mu|\mathscr{A}](t) \leq \frac{\ell(t)}{M}, \tag{190}$$

the conclusions of Proposition 5 hold true, see (44)–(47). For what follows, we restrict ourselves to the subset of full measure in $(0,T')$ such that all the a.e. properties of Definition 1 (in particular, the ones of [14, Definition 3]) are satisfied, and fix $t\in(0,T')$ being an element of this subset. It will be convenient to trivially extend the oriented varifold μ_t to an element of $\mathrm{M}(\mathbb{R}^2\times\mathbb{S}^1)$ as well as the vector field $\xi(\cdot,t)$ to a map $\mathbb{R}^2\to\mathbb{R}^2$ by defining

$$|\mu_t|_{\mathbb{S}^1} := 0 \text{ and } \xi(\cdot,t) := 0 \qquad \text{in } \mathbb{R}^2\setminus\overline{\Omega}. \tag{191}$$

For readability, we suppress from now on the dependence of all quantities on t in the notation.

Heuristically, the idea for our proof of Proposition 5 is based on the competition between two effects:

- On one side, the assumptions (189)–(190) will allow us to prove that around every point $x_0 \in \operatorname{supp}\mu$, the rectifiable set $\operatorname{supp}\mu$ can be represented as a graph within a ball $B_\rho(x_0)$ (parametrized over a suitable affine subspace), where crucially the scale ρ can be chosen independently of a given point $x_0 \in \operatorname{supp}\mu$ (more precisely, $\rho \ll \ell$ uniformly on $[0,T']$). Naturally, the important building block here is Allard's regularity theory, and the assumptions (189)–(190) (in particular, the tuning of the yet to be determined constant M) are used to show that one is in the corresponding setting.
- On the other side, based on the (yet to be determined) smallness of the overall error (190), any unwanted feature of $\operatorname{supp}\mu$ contradicting the assertions of Proposition 5 (e.g., non-graphical components of $\partial^* A$ as screened from $\partial\mathscr{A}$ in normal direction, or in our case also intersections of $\partial^* A$ with $\partial\Omega$) can be shown to be of sufficiently small mass so that these in turn can be trapped inside balls of radius $\rho/2$, resulting into a contradiction with the graph property on twice this scale. For the actual implementation of this contradiction argument, it will be convenient to exploit that the reduced boundary of a set of finite perimeter in $\mathbb{R}^2$ can always be decomposed into a countable family of $\mathcal{H}^1$-rectifiable Jordan-Lipschitz curves.

For the rigorous argument, we ensure in a first step that our assumption (189) implies a curvature bound.

Lemma 18 (Curvature estimate up to the boundary)**.** *There exists* $\mathrm{H}\colon \mathbb{R}^2 \to \mathbb{R}^2 \in L^1_{loc}(\mathbb{R}^2; d|\mu|_{\mathbb{S}^1})$*,* $\operatorname{supp}\mathrm{H} \subset \overline{\Omega}$*, being the generalized mean curvature vector of* μ *with respect to tangential variations, i.e., it holds*

$$\int_{\mathbb{R}^2\times\mathbb{S}^1} (\mathrm{Id} - \mathrm{p}\otimes\mathrm{p}) : \nabla B \, d\mu(\cdot,\mathrm{p}) = -\int_{\mathbb{R}^2} \mathrm{H}\cdot B \, d|\mu|_{\mathbb{S}^1} \tag{192}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^2)$ *with* $n_{\partial\Omega}\cdot B \equiv 0$ *along* $\partial\Omega$*. Furthermore, for each* $p \in [2,\infty)$ *and* $\Lambda \in (0,\infty)$*, there exists*

$$C_\Lambda := C(\Omega, p, A(0), T, \Lambda) \in (0,\infty) \tag{193}$$

such that the assumption (189) *implies*

$$\|\mathrm{H}\|_{L^p(\mathbb{R}^2;|\mu|_{\mathbb{S}^1})} \leq \|w\|_{L^p(\overline{\Omega};|\mu|_{\mathbb{S}^1})} \leq C_\Lambda. \tag{194}$$

Based on this, our argument then relies on two results from the literature: *i)* Allard regularity theory for integer rectifiable varifolds with free boundary (cf. [27] and [10]), and *ii)* a decomposition of the reduced boundary of a set of finite perimeter in the plane into $\mathcal{H}^1$-rectifiable Jordan-Lipschitz curves (cf. [2]). The former is usually written down in the language of general varifolds and not oriented ones. However, for the oriented varifold $\mu \in \mathrm{M}(\mathbb{R}^2\times\mathbb{S}^1)$ one may always canonically associate a general varifold $\widehat{\mu} \in \mathrm{M}(\mathbb{R}^2\times\mathbf{G}(2,1))$, where $\mathbf{G}(2,1)$ denotes the space of 1-dimensional linear subspaces of $\mathbb{R}^2$, by means of

$$\int_{\mathbb{R}^2\times\mathbb{G}(2,1)} \varphi(x,\widehat{T}) \, d\widehat{\mu}(x,\widehat{T}) := \int_{\mathbb{R}^2\times\mathbb{S}^1} \varphi(x,\widehat{T}_{\mathrm{p}}) \, d\mu(x,\mathrm{p}), \quad \varphi \in C_{cpt}(\mathbb{R}^2\times\mathbf{G}(2,1)), \tag{195}$$

where for $\mathrm{p} \in \mathbb{S}^1$ we denote by $\widehat{T}_{\mathrm{p}} \in \mathbf{G}(2,1)$ the 1-dimensional linear subspace of $\mathbb{R}^2$ having s as its unit normal vector. Note that the mass measure $|\widehat{\mu}|_{\mathbf{G}(2,1)} \in \mathrm{M}(\mathbb{R}^2)$ of $\widehat{\mu}$ is simply $|\mu|_{\mathbb{S}^1}$ and that the map H from the previous lemma is the generalized mean curvature vector of $\widehat{\mu}$ in the usual sense (with respect to tangential variations). Hence, for what follows we will be cavalier about the distinction between μ and $\widehat{\mu}$.

Theorem 19 (Allard regularity theory—interior and boundary case)**.** *Fix data $\rho \in (0,\ell)$, $x_0 \in \operatorname{supp}\mu$, and $\widehat{T} \in \mathbf{G}(2,1)$.*

There exist constants $\varepsilon_{reg}, \gamma_{reg} \in (0,1)$ and $C_{reg} \in (1,\infty)$ such that:

i) If $x_0 \in \operatorname{supp}\mu \cap \Omega$ such that (192) holds for all $B \in C^1_{cpt}(B_{2\rho}(x_0);\mathbb{R}^2)$ and

$$\begin{cases} \frac{|\mu|_{\mathbb{S}^1}(B_\rho(x_0))}{\omega_1 \rho} \leq 1+\varepsilon_{reg}, \\ E^\circ_*[x_0,\rho,\widehat{T}] := \max\Big\{E_{tilt}[x_0,\rho,\widehat{T}], \frac{\rho}{\varepsilon_{reg}} \int_{B_\rho(x_0)} |\mathrm{H}|^2 \, d|\mu|_{\mathbb{S}^1}\Big\} \leq \varepsilon_{reg}, \end{cases} \tag{196}$$

where $E_{tilt}[x_0,\rho,\widehat{T}]$ is the usual tilt excess relative to the integer rectifiable varifold $\widehat{\mu}$, then there exists a $C^{1,\frac{1}{2}}$ function

$$u\colon (x_0+\widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \to \mathbb{R} \tag{197}$$

with the following properties: $u(x_0) = 0$,

$$\begin{aligned} &\operatorname{supp}\mu \cap B_{\gamma_{reg}\rho}(x_0) \\ &\quad = \big\{y+u(y)\mathrm{n}_{\widehat{T}}(y) \colon y \in (x_0+\widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0)\big\} \cap B_{\gamma_{reg}\rho}(x_0), \end{aligned} \tag{198}$$

where $\mathrm{n}_{\widehat{T}}$ is a normal vector of the affine subspace $x_0+\widehat{T}$, and

$$\begin{aligned} &\rho^{-1} \sup|u| + \sup|\nabla^{tan} u| \\ &\quad \leq C_{reg}\bigg\{E^{\frac{1}{2}}_{tilt}[x_0,\rho,\widehat{T}] + \rho^{\frac{1}{2}}\Big(\int_{B_\rho(x_0)} |\mathrm{H}|^2 \, d|\mu|_{\mathbb{S}^1}\Big)^{\frac{1}{2}}\bigg\}. \end{aligned} \tag{199}$$

ii) If $x_0 \in \operatorname{supp}\mu \cap \partial\Omega$ such that

$$\begin{cases} \frac{|\mu|_{\mathbb{S}^1}(B_\rho(x_0)) + |\mu|_{\mathbb{S}^1}(\widetilde{B}_\rho(x_0))}{\omega_1 \rho} \leq 1+\varepsilon_{reg}, \\ E_*[x_0,\rho,\mathrm{Tan}^\perp_{x_0}\partial\Omega] := \max\Big\{E^\circ_*[x_0,\rho,\mathrm{Tan}^\perp_{x_0}\partial\Omega], \frac{\rho}{\ell}\Big\} \leq \varepsilon_{reg}, \end{cases} \tag{200}$$

where $\widetilde{B}_\rho(x_0) := \{x \in \mathbb{R}^2 \colon |\widetilde{x} - x_0| < \rho$ where $\widetilde{x} := x - 2s_{\partial\Omega}(x)\mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x))\}$, then there exists a $C^{1,\frac{1}{2}}$ function

$$u\colon (x_0+\widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0) \to \mathbb{R} \tag{201}$$

with the following properties: $u(x_0) = 0$,

$$\begin{aligned} &\operatorname{supp}\mu \cap B_{\gamma_{reg}\rho}(x_0) \\ &\quad = \big\{y+u(y)\mathrm{n}_{\widehat{T}}(y) \colon y \in (x_0+\widehat{T}) \cap B_{\gamma_{reg}\rho}(x_0)\big\} \cap B_{\gamma_{reg}\rho}(x_0) \cap \overline{\Omega}, \end{aligned} \tag{202}$$

and

$$\mathrm{n}_{\partial\Omega}(x_0) \in \mathrm{Tan}_{x_0}(\operatorname{supp}\mu). \tag{203}$$

The second required result from the literature relies on the notion of Jordan curves J. Recall that these are images of continuous maps $\phi\colon [a,b] \to \mathbb{R}^2$ which are injective on $[a,b)$ and satisfy $\phi(a) = \phi(b)$. Thanks to the Jordan curve theorem, we know that any such J splits the plane into two connected components having J as their interface. Let us denote by $\mathrm{int}(J)$ the corresponding bounded component whereas the unbounded one will be denoted by $\mathrm{ext}(J)$. Finally, a Jordan curve admitting a Lipschitz parametrization is called a Jordan-Lipschitz curve.

Theorem 20 (Decomposition of planar sets of finite perimeter)**.** *Let $A \subset \mathbb{R}^2$ be a set of finite perimeter. Then there exists a family $\{J_n : n \in \mathbb{N}\}$ of $\mathcal{H}^1$-rectifiable Jordan-Lipschitz curves such that $\mathcal{H}^1(J_n \cap J_m) = 0$ for all distinct $n, m \in \mathbb{N}$ and $\partial^* A = \bigcup_{n\in\mathbb{N}} J_n$ mod $\mathcal{H}^1$.*

In a next step, we aim to define a scale $\rho_{reg} \in (0, \ell)$, uniformly on $[0, T']$, such that one may afterward choose $M \gg 1$, again uniformly on $[0, T']$, so that the assumptions (189)–(190) imply, among other things, that for all $x_0 \in \operatorname{supp}|\mu|_{\mathbb{S}^1}$, the rectifiable set $\operatorname{supp}|\mu|_{\mathbb{S}^1}$ can be locally represented around x_0 as a graph on scale $\gamma_{reg}\rho_{reg}$ (with respect to a suitably chosen subspace) in the precise sense of Theorem 19.

By locally uniform regularity of $\mathscr{A}$, one may choose

$$\varepsilon = \varepsilon(\mathscr{A}) \in \Big(0, \min\Big\{\varepsilon_{reg}, \frac{1}{8}\frac{1}{C_{reg}}\frac{1}{16C}\Big\}\Big) \quad \text{and} \quad \widetilde{C} = \widetilde{C}(\mathscr{A}) \in (1, \infty) \tag{204}$$

uniformly on $[0, T']$ such that for all

$$\rho \in (0, \rho_{reg}], \quad \text{where } \rho_{reg} := \widetilde{C}^{-1}\widetilde{\rho} \text{ and } \widetilde{\rho} := \frac{\varepsilon^2}{C_\Lambda^2}\ell, \tag{205}$$

at least the following properties are satisfied:

- for all $y \in \partial\mathscr{A}$,

$$\partial\mathscr{A} \cap B_\rho(y) \text{ can be written as a graph over } (y + \mathrm{Tan}_y\partial\mathscr{A}) \cap B_\rho(y) \tag{206}$$

  in the direction of $\mathrm{n}_{\partial\mathscr{A}}(y)$,
- for all $y_0, y \in \partial\mathscr{A}$ it holds

$$|(y - y_0) \cdot t_{\partial\mathscr{A}}(y_0)| \leq \rho \quad \Longrightarrow \quad |(y - y_0) \cdot \mathrm{n}_{\partial\mathscr{A}}(y_0)| \leq \frac{1}{4}\Big(\frac{\ell}{16C}\Big), \tag{207}$$

- given $\alpha \in (0, \frac{\pi}{2})$ defined by $2C_{reg}\varepsilon =: \tan\alpha$, it holds for all $x_0 \in \{|\xi| > \frac{1}{2}\}$ and all $x \in \partial B_\rho(x_0)$ that

$$\big|(x - x_0) \cdot t_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big| \geq \rho\cos\alpha \quad \Longrightarrow \quad |P_{\partial\mathscr{A}}(x_0) - P_{\partial\mathscr{A}}(x)| \geq \frac{1}{2}\rho\cos\alpha, \tag{208}$$

- it holds

$$x_0 \in \{|\xi| \leq 1/2\} \quad \Longrightarrow \quad B_\rho(x_0) \subset \{|\xi| \leq 3/4\}, \tag{209}$$

- for all $x_0 \in \{|\xi| > \frac{1}{2}\}$ and all $x \in B_\rho(x_0)$ it holds

$$\big|\xi(x) - \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big| \leq \frac{1}{1 + \varepsilon/2}\frac{\varepsilon^2}{4}. \tag{210}$$

With these choices in place, we then show that ρ_{reg} indeed satisfies the aforementioned goal of representing $\operatorname{supp}|\mu|_{\mathbb{S}^1}$ locally on scale $\gamma_{reg}\rho_{reg}$ around any of its points as a graph.

Lemma 21 (Applicability of Allard regularity theory)**.** *There exists $M \gg_{\mathscr{A},C_\Lambda} 1$, uniformly on $[0,T']$, such that the hypotheses (196) hold true at scale $\rho = \rho_{reg}$ for all interior points $x_0 \in \operatorname{supp}\mu \cap \Omega$ and*

$$\begin{cases} \text{any } \widehat{T} \in \mathbf{G}(2,1) & \text{if } x_0 \in \{|\xi| \leq 1/2\}, \\ \widehat{T} = \operatorname{Tan}_{P_{\partial\mathscr{A}}(x_0)}\partial\mathscr{A} & \text{if } x_0 \in \{|\xi| > 1/2\}, \end{cases} \tag{211}$$

as well as that the hypotheses (200) hold true at scale $\rho = \rho_{reg}$ for all boundary points $x_0 \in \operatorname{supp}\mu \cap \partial\Omega$ (and $\widehat{T} \in \mathbf{G}(2,1)$ fixed by $\operatorname{Tan}^{\perp}_{x_0}\partial\Omega$). Furthermore,

$$E_{tilt}^{\frac{1}{2}}[x_0, \rho_{reg}, \widehat{T}] \leq \varepsilon, \tag{212}$$

$$\rho_{reg}^{\frac{1}{2}}\Big(\int_{B_{\rho_{reg}}(x_0)} |\mathrm{H}|^2 \, d|\mu|_{\mathbb{S}^1}\Big)^{\frac{1}{2}} \leq \varepsilon \tag{213}$$

for all $x_0 \in \operatorname{supp}\mu$ with associated data $(\rho_{reg}, \widehat{T})$ as above.

By means of the previous result, we then first show that the mass measure of the varifold μ reduces to the mass measure of the reduced boundary of the finite perimeter set A and that the latter is supported sufficiently close to the interface $\partial\mathscr{A}$ of the strong solution (in particular, of positive distance to the physical boundary $\partial\Omega$).

Lemma 22 (Reduction to mass measure with no boundary intersection)**.** *One may choose $M \gg_{\mathscr{A},C_\Lambda} 1$ uniformly on $[0,T']$ such that*

$$\partial^* A \subset \{|\xi| > 1/2\} \subset B_\ell(\partial\mathscr{A}), \tag{214}$$

$$|\mu|_{\mathbb{S}^1} = \mathcal{H}^1 \llcorner \partial^* A \tag{215}$$

and

$$\begin{aligned} \int_{\partial^* A} (\mathrm{Id} - \mathrm{n}_{\partial^* A} \otimes \mathrm{n}_{\partial^* A}) : \nabla B \, d\mathcal{H}^1 &= \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\mathrm{Id} - s \otimes s) : \nabla B \, d\mu \\ &= -\int_{\mathbb{R}^2} \mathrm{H} \cdot B \, d|\mu|_{\mathbb{S}^1} \end{aligned} \tag{216}$$

for all $B \in C^1_{cpt}(\mathbb{R}^2;\mathbb{R}^2)$.

In a second step, we leverage on the previous results to show that the interface of the weak solution $\partial^* A$ is in fact "sufficiently rich" in the sense that it actually contains components admitting a graph representation relative to the interface of the strong solution $\partial\mathscr{A}$. Note that this cannot follow just from requiring smallness of the relative entropy (17)—which, by careful inspection of the proofs, is in fact up to now the only required ingredient concerning the smallness of the overall error—but also requires smallness of the volume error (18): indeed, the relative entropy provides no error control in the regime of vanishing interfacial mass $|\mu|_{\mathbb{S}^1}(\mathbb{R}^2) \downarrow 0$.

Lemma 23 (Construction of graph candidate)**.** *Denote by $(\mathscr{J}_k)_{k=1,\ldots,K}$ the finitely many connected components of $\partial\mathscr{A}$. One may choose $M \gg_{\mathscr{A},C_\Lambda} 1$ uniformly on $[0,T']$ such that for each $k \in \{1,\ldots,K\}$ there exists a Jordan-Lipschitz curve $J_k \subset \partial^* A$ such that J_k can be written as a graph over $\mathscr{J}_k$.*

We finally conclude by showing that the constructed candidate for the graph representation saturates the interface of the weak solution and satisfies all the required properties.

Lemma 24 (Graph representation and height function estimates)**.** *One may choose $M \gg_{\mathscr{A},C_\Lambda} 1$ uniformly on $[0,T']$ such that $\partial^* A = \bigcup_{k=1}^K J_k$ with the Jordan-Lipschitz curves $(J_k)_{k=1,\ldots,K}$ given by Lemma 23. Furthermore, the induced height function $h\colon \partial\mathscr{A} \to [-\ell,\ell]$ satisfies the regularity* (44) *and the estimates* (47)*.*

6.2. **Proofs.** We continue with the proofs of the several intermediate results from the previous subsection, eventually culminating into a proof of Proposition 5, which itself is the last missing ingredient for the proof of our main result, Theorem 1.

Proof of Lemma 18. Denote by $H_{|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega}\colon \operatorname{supp}(|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega) \to \mathbb{R}^d$ the map from [14, Definition 3, item iii)] and define

$$\mathrm{H} := \begin{cases} H_{|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega} & \text{in } \Omega, \\ 0 & \text{else.} \end{cases} \tag{217}$$

The identity (192) then directly follows from (217) and [14, (17h)], whereas the estimate (194) is a consequence of [14, Proposition 5, (26), Definition 3 item iv)] and (7c). ☐

Proof of Theorem 19. We naturally distinguish between two cases.

Case 1: For the case of interior points $x_0 \in \operatorname{supp}\mu \cap \Omega$, let $\bar\varepsilon, \bar\gamma \in (0,1)$ and $\bar c \in (1,\infty)$ be the constants from [27, 23.1 Theorem]. It follows from [27, 23.2 Remark, item (2)] and our hypotheses that for suitably small $\varepsilon_{reg} \in (0,\bar\varepsilon)$

$$\frac{|\mu|_{\mathbb{S}^1}(B_\sigma(x))}{\omega_1\sigma} \leq \frac{3}{2} \tag{218}$$

for all $\sigma \in (0,\frac{\varrho}{2})$ and all $x \in \operatorname{supp}\mu \cap B_{\varepsilon_{reg}\rho}(x_0)$. In particular, the 1-dimensional density of μ, denoted by $\Theta_1(\mu,\cdot)$, satisfies

$$\Theta_1(\mu,x) = 1 \qquad \text{for all } x \in \operatorname{supp}\mu \cap B_{\varepsilon_{reg}\rho}(x_0). \tag{219}$$

Furthermore, for small enough $\varepsilon_{reg} \in (0,\bar\varepsilon)$, we deduce from [27, Proof of 23.1, inequality (12)] that

$$E^\circ_*[x_0,(\varepsilon_{reg}\rho)/2,\widehat{T}] \leq \bar\varepsilon. \tag{220}$$

Hence, the hypotheses of [27, 23.1 Theorem] are fulfilled for the choices $U = B_{\varepsilon_{reg}\rho}(x_0)$ and $\rho = (\varepsilon_{reg}\rho)/2$, so that the remaining claims follow from the conclusions of [27, 23.1 Theorem] for $\gamma_{reg} := (\varepsilon_{reg}\bar\gamma)/2$ and $C_{reg} := \bar c$.

Case 2: For the case of boundary points $x_0 \in \operatorname{supp}\mu \cap \partial\Omega$, one may argue analogously, using [10, inequality between (40) and (41)], [10, inequality (51)] and [10, 4.9 Theorem] as substitutes for [27, 23.2 Remark, item (2)], [27, Proof of 23.1, inequality (12)] and [27, 23.1 Theorem], respectively. Note that the analogue of (221) is given by

$$\frac{|\mu|_{\mathbb{S}^1}(B_\sigma(x)) + |\mu|_{\mathbb{S}^1}(\widetilde{B}_\sigma(x))}{\omega_1\sigma} \leq \frac{3}{2} \tag{221}$$

for all $\sigma \in (0,\frac{\varrho}{2})$ and all $x \in \operatorname{supp}\mu \cap B_{\varepsilon_{reg}\rho}(x_0)$. In particular,

$$\begin{aligned} \Theta_1(\mu,x) &= 1 && \text{for all } x \in \operatorname{supp}\mu \cap B_{\varepsilon_{reg}\rho}(x_0) \cap \Omega, \\ \Theta_1(\mu,x) &\leq \frac{3}{4} && \text{for all } x \in \operatorname{supp}\mu \cap B_{\varepsilon_{reg}\rho}(x_0) \cap \partial\Omega; \end{aligned} \tag{222}$$

a fact which we state for future reference (cf. [10, 3.2 Corollary]). Note finally that the second statement of (222) implies $|\mu|_{\mathbb{S}^1}(B_{\varepsilon_{reg}\rho}(x_0) \cap \partial\Omega) = 0$. ☐

Proof of Theorem 20. Immediate from [2, Proposition 3 and Corollary 1]. □

Proof of Lemma 21. Using the bound (194) and the definitions (204)–(205), we get

$$\frac{\rho_{reg}}{\varepsilon_{reg}}\int_{B_{\rho_{reg}}(x_0)}|\mathrm{H}|^2\,d|\mu|_{\mathbb{S}^1}\leq\frac{\varepsilon^2}{\varepsilon_{reg}}\leq\varepsilon_{reg}. \tag{223}$$

The first inequality settles (213) whereas the second is precisely the curvature estimate required by (196) and (200). Note also that by definitions (204)–(205) it holds $\rho_{reg}/\ell\leq\varepsilon_{reg}$. It thus remains to derive the asserted estimates for the mass ratios and the tilt excess. To this end, we distinguish between the three natural cases.

Case 1: $x_0\in\operatorname{supp}\mu\cap\Omega\cap\{|\xi|\leq 1/2\}$. We start estimating by a union bound and recalling (20)–(21)

$$\begin{aligned}|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\big)\leq{}&|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\partial\Omega\big)\\&+|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\Omega\cap\{\varrho\leq 1/3\}\big)\\&+|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\Omega\cap\{\varrho=1\}\big).\end{aligned} \tag{224}$$

Recalling also (22) and from property (209) that $B_{\rho_{reg}}(x_0)\subset\{|\xi|\leq\frac{3}{4}\}$, we get

$$\begin{aligned}&|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\partial\Omega\big)+|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\Omega\cap\{\varrho\leq 1/3\}\big)\\&\leq|\mu|_{\mathbb{S}^1}(\partial\Omega)+\frac{3}{2}\int_{\Omega\cap\{\varrho\leq\frac{1}{3}\}}1-\varrho\,d|\mu|_{\mathbb{S}^1}\leq\frac{3}{2}E_{rel}[A,\mu|\mathscr{A}]\end{aligned} \tag{225}$$

and

$$\begin{aligned}&|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\cap\Omega\cap\{\varrho=1\}\big)\\&\leq\int_{\partial^*A\cap\Omega\cap\{|\xi|\leq 3/4\}}1\,d\mathcal{H}^1\\&\leq 4\int_{\partial^*A\cap\Omega}(1-\mathrm{n}_{\partial^*A}\cdot\xi)\,d\mathcal{H}^1\leq 4E_{rel}[A,\mu|\mathscr{A}].\end{aligned} \tag{226}$$

Hence, the previous three displays together with assumption (190) imply

$$\frac{|\mu|_{\mathbb{S}^1}(B_{\rho_{reg}}(x_0))}{\omega_1\rho_{reg}}\leq\frac{11}{2}\frac{\widetilde{C}\Lambda}{\omega_1\varepsilon^2}\frac{1}{M}\overset{M\gg_{\mathscr{A},C_\Lambda}1}{\leq}\varepsilon^2. \tag{227}$$

Let now $\widehat{T}\in\mathbf{G}(2,1)$ arbitrary but fixed. We may then analogously ensure by choosing $M\gg_{\mathscr{A},C_\Lambda}1$ that

$$E_{tilt}[x_0,\rho_{reg},\widehat{T}]\leq\varepsilon^2 \tag{228}$$

because of the simple observation that $E_{tilt}[x_0,\rho_{reg},\widehat{T}]\leq C_{tilt}\frac{|\mu|_{\mathbb{S}^1}(B_{\rho_{reg}}(x_0))}{\omega_1\rho_{reg}}$ for some absolute constant $C_{tilt}\in(1,\infty)$.

Case 2: $x_0\in\operatorname{supp}\mu\cap\partial\Omega$. Due to [16, Lemma 4.2], $\rho_{reg}\leq\ell/8$, (8) and (16), we know that

$$\widetilde{B}_{\rho_{reg}}(x_0)\subset B_{5\rho_{reg}}(x_0)\subset B_\ell(\partial\Omega)\subset\{|\xi|=0\}. \tag{229}$$

Hence, arguing essentially analogous to the previous case shows

$$|\mu|_{\mathbb{S}^1}(B_{\rho_{reg}}(x_0))+|\mu|_{\mathbb{S}^1}(\widetilde{B}_{\rho_{reg}}(x_0))\leq 3E_{rel}[A,\mu|\mathscr{A}], \tag{230}$$

so that a suitable choice of $M \gg_{\mathscr{A},C_\Lambda} 1$ allows to guarantee

$$\frac{|\mu|_{\mathbb{S}^1}(B_{\rho_{reg}}(x_0)) + |\mu|_{\mathbb{S}^1}(\widetilde{B}_{\rho_{reg}}(x_0))}{\omega_1\rho_{reg}} \leq \varepsilon^2 \tag{231}$$

as well as

$$E_{tilt}[x_0, \rho_{reg}, \mathrm{Tan}^{\perp}_{x_0}\partial\Omega] \leq \varepsilon^2. \tag{232}$$

Case 3: $x_0 \in \operatorname{supp}\mu \cap \Omega \cap \{|\xi| > 1/2\}$. By (224), (225) and (20) it holds

$$|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\big) \leq \frac{3}{2}E_{rel}[A,\mu|\mathscr{A}] + \int_{\partial^* A\cap B_{\rho_{reg}}(x_0)} 1\, d\mathcal{H}^1. \tag{233}$$

Note that indeed $\partial^* A \cap B_{\rho_{reg}}(x_0) = \partial^* A \cap B_{\rho_{reg}}(x_0) \cap \Omega$ since $x_0 \in \Omega \cap \{|\xi| > 1/2\}$ implies $B_{\rho_{reg}}(x_0) \subset B_\ell(\partial\mathscr{A}) \subset \Omega$ due to (16), $\rho_{reg} \leq \ell/8$ and (8).

To estimate the second term on the right hand side of (233), first introduce some additional notation. Define $T_{x_0} := P_{\partial\mathscr{A}}(x_0) + \mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0)}\partial\mathscr{A}$ and let P_{x_0} the nearest point projection onto T_{x_0}. For every $x \in \mathbb{R}^2$, we further denote by $A_{P_{x_0}(x)}$ the one-dimensional slice $A \cap \{P_{x_0}(x) + y\mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0))\colon |y| \leq \ell\}$. Finally, define the sets

$$S^{(1)}_{x_0} := \partial^* A \cap B_{\rho_{reg}}(x_0) \cap \{x \in \mathbb{R}^2\colon \mathcal{H}^0(\partial^* A_{P_{x_0}(x)}) > 1\}, \tag{234}$$

$$S^{(2)}_{x_0} := \Big\{\big(\partial^* A \cap B_{\rho_{reg}}(x_0)\big) \setminus S^{(1)}_{x_0}\colon \mathrm{n}_{\partial^* A}(x)\cdot\xi(x) \geq \frac{1{+}\varepsilon/4}{1{+}\varepsilon/2}\Big\}, \tag{235}$$

$$S^{(3)}_{x_0} := \Big\{\big(\partial^* A \cap B_{\rho_{reg}}(x_0)\big) \setminus S^{(1)}_{x_0}\colon \mathrm{n}_{\partial^* A}(x)\cdot\xi(x) < \frac{1{+}\varepsilon/4}{1{+}\varepsilon/2}\Big\}. \tag{236}$$

In particular,

$$\partial^* A \cap B_{\rho_{reg}}(x_0) = S^{(1)}_{x_0} \cup S^{(2)}_{x_0} \cup S^{(3)}_{x_0} \tag{237}$$

and, by definition of $S^{(1)}_{x_0}$ and $S^{(2)}_{x_0}$ as well as by the flatness property (210),

$$x \in S^{(2)}_{x_0} \quad\Longrightarrow\quad \mathcal{H}^0(\partial^* A_{P_{x_0}(x)}) = 1 \text{ and } \big|\mathrm{n}_{\partial^* A}(x)\cdot\mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0))\big| \geq \frac{1}{1{+}\varepsilon/2}. \tag{238}$$

Now, we estimate term by term in the decomposition (237). First, a slicing argument ensures ***Details?***

$$\mathcal{H}^1(S^{(1)}_{x_0}) \leq C_{rel}E_{rel}[A,\mu|\mathscr{A}] \tag{239}$$

for some universal constant $C_{rel} \in (1,\infty)$. Second, one immediately deduces that

$$\mathcal{H}^1(S^{(3)}_{x_0}) \leq \frac{4(1{+}\varepsilon/2)}{\varepsilon}\int_{\partial^* A\cap\Omega}(1 - \mathrm{n}_{\partial^* A}\cdot\xi)\, d\mathcal{H}^1 \leq \frac{4(1{+}\varepsilon/2)}{\varepsilon}E_{rel}[A,\mu|\mathscr{A}]. \tag{240}$$

Third, by coarea formula and (238)

$$\mathcal{H}^1(S^{(2)}_{x_0}) \leq (1{+}\varepsilon/2)\mathcal{H}^1\big(B_{\rho_{reg}}(P_{\partial\mathscr{A}}(x_0))\big) = (1{+}\varepsilon/2)\omega_1\rho_{reg}. \tag{241}$$

In summary, we may now infer from (233), (237) and (239)–(241) that

$$|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\big) \leq (1{+}\varepsilon/2)\omega_1\rho_{reg} + \Big(\frac{3}{2} + C_{rel} + \frac{4(1{+}\varepsilon/2)}{\varepsilon}\Big)E_{rel}[A,\mu|\mathscr{A}] \tag{242}$$

for some universal constant $C_{rel} \in (1,\infty)$. Hence, choosing $M \gg_{\mathscr{A},C_\Lambda} 1$ in a suitable manner entails by assumption (190)

$$|\mu|_{\mathbb{S}^1}\big(B_{\rho_{reg}}(x_0)\big) \leq (1{+}\varepsilon)\omega_1\rho_{reg}. \tag{243}$$

Having established the desired estimate on the mass ratio, we now turn to the asserted bound for the tilt excess and claim that, upon suitably choosing M,

$$E_{tilt}[x_0, \rho_{reg}, T_{x_0}] \leq \varepsilon^2. \tag{244}$$

The argument in favor of (244) is very close to the one producing (243).

First, employing the decomposition underlying (224), we obtain as the analogue of (233)

$$\begin{aligned} &E_{tilt}[x_0, \rho_{reg}, T_{x_0}] \\ &\leq C_{tilt} \frac{E_{rel}[A, \mu|\mathscr{A}]}{\rho_{reg}} + \frac{1}{\rho_{reg}} \int_{\partial^* A \cap B_{\rho_{reg}}(x_0)} \big(1 - \mathrm{n}_{\partial^* A}(x) \cdot \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big) \, d\mathcal{H}^1(x), \end{aligned} \tag{245}$$

where $C_{tilt} \in (1, \infty)$ is a universal constant. For an estimate of the second term on the right hand side of the previous display, we again make use of the decomposition (234)–(237). Hence, due to the estimates (239) and (240),

$$\begin{aligned} &\frac{1}{\rho_{reg}} \int_{\partial^* A \cap B_{\rho_{reg}}(x_0)} \big(1 - \mathrm{n}_{\partial^* A}(x) \cdot \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big) \, d\mathcal{H}^1(x) \\ &\quad \leq \frac{2}{\rho_{reg}} \big(\mathcal{H}^1(S_{x_0}^{(1)}) + \mathcal{H}^1(S_{x_0}^{(3)})\big) \\ &\qquad + \frac{1}{\rho_{reg}} \int_{S_{x_0}^{(2)}} \big(1 - \mathrm{n}_{\partial^* A}(x) \cdot \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big) \, d\mathcal{H}^1(x) \\ &\quad \leq 2\Big(C'_{tilt} + \frac{4(1{+}\varepsilon/2)}{\varepsilon}\Big) \frac{E_{rel}[A, \mu|\mathscr{A}]}{\rho_{reg}} \\ &\qquad + \frac{1}{\rho_{reg}} \int_{S_{x_0}^{(2)}} \big(1 - \mathrm{n}_{\partial^* A}(x) \cdot \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big) \, d\mathcal{H}^1(x) \end{aligned} \tag{246}$$

for some universal constant $C'_{tilt} \in (1, \infty)$. Furthermore, adding zero and estimating based on (241) and (210) yields

$$\begin{aligned} &\frac{1}{\rho_{reg}} \int_{S_{x_0}^{(2)}} \big(1 - \mathrm{n}_{\partial^* A}(x) \cdot \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big) \, d\mathcal{H}^1(x) \\ &\quad \leq \frac{E_{rel}[A, \mu|\mathscr{A}]}{\rho_{reg}} + \frac{1}{\rho_{reg}} \int_{S_{x_0}^{(2)}} \big|\xi(x) - \mathrm{n}_{\partial\mathscr{A}}\big(P_{\partial\mathscr{A}}(x_0)\big)\big| \, d\mathcal{H}^1(x) \\ &\quad \leq \frac{E_{rel}[A, \mu|\mathscr{A}]}{\rho_{reg}} + \frac{\varepsilon^2}{4}(1{+}\varepsilon/2)\omega_1. \end{aligned} \tag{247}$$

In summary, we obtain from the previous three displays that

$$E_{tilt}[x_0, \rho_{reg}, T_{x_0}] \leq 2\Big(C''_{tilt} + \frac{4(1{+}\varepsilon/2)}{\varepsilon}\Big) \frac{E_{rel}[A, \mu|\mathscr{A}]}{\rho_{reg}} + \frac{3}{4}\varepsilon^2 \tag{248}$$

for some universal constant $C''_{tilt} \in (1, \infty)$. Hence, we may infer (244) from (248) after suitably selecting $M \gg_{\mathscr{A}, C_\Lambda} 1$.

Conclusion: Since $\varepsilon \leq \varepsilon_{reg}$ by (204), the tilt excess estimates required by (212), (196) and (200) as well as the mass ratio estimates required by (196) and (200) follow from (227)–(228), (231)–(232) and (243)–(244), respectively. This eventually concludes the proof. □

Proof of Lemma 22. The proof is split into two steps.

Step 1: We first claim that

$$\operatorname{supp}\mu \cap \partial\Omega = \emptyset. \tag{249}$$

(Note that the weaker property $|\mu|_{\mathbb{S}^1}(\partial\Omega) = 0$ is immediate from Lemma 21 and the remark at the end of the proof of Theorem 19).

For a proof of (249), we argue by contradiction and assume that there exists $x_0 \in \operatorname{supp}\mu \cap \partial\Omega$. Thanks to Lemma 21, the conclusion of Theorem 19 item ii) apply, i.e., $\operatorname{supp}\mu \cap B_{\gamma_{reg}\rho_{reg}}(x_0)$ admits a graph representation with associated height function u in the precise sense of (202)–(203). It then follows from the first line of (222), [14, Definition 3, item (ii)] and Theorem 20 that there exists a Jordan-Lipschitz curve $J \subset \partial^* A$ such that $x_0 \in J$ and $\operatorname{supp}\mu \cap B_{\gamma_{reg}\rho_{reg}}(x_0) \cap \Omega = J \cap B_{\gamma_{reg}\rho_{reg}}(x_0) \cap \Omega$. By (16) and (8), we also know that $|\mu|_{\mathbb{S}^1}(J \cap B_{\gamma_{reg}\rho_{reg}}(x_0)) \leq E_{rel}[A,\mu|\mathscr{A}]$, so that assumption (190) and a suitable choice $M \gg_{\mathscr{A},C_\Lambda} 1$ imply

$$\operatorname{int}(J) \subset B_{\frac{\gamma_{reg}\rho_{reg}}{2}}(x_0). \tag{250}$$

The trapping of J in the sense of the previous display is, however, in contradiction to the above mentioned graph property of $\operatorname{supp}\mu$ within $B_{\gamma_{reg}\rho_{reg}}(x_0)$.

Step 2: By compactness of $\operatorname{supp}\mu$ and $\partial\Omega$ it now follows from the first step that $\operatorname{dist}(\operatorname{supp}\mu, \partial\Omega) > 0$. Hence, the second identity of (216) is immediate from (192), and based on that, one obtains (215) simply from (219) and [14, Definition 3, item (ii)]. Since the canonically associated general varifold $\widehat{\mu}$ of μ is 1-integer rectifiable, the first identity of (216) follows now in turn from (215) and $|\widehat{\mu}|_{\mathbf{G}(2,1)} = |\mu|_{\mathbb{S}^1}$.

It remains to verify the first inclusion of (214). To this end, one may exploit a contradiction argument being essentially analogous to the one conducted in Step 1 of this proof. Indeed, one works with the conclusions of Theorem 19 item i) instead of the ones from item ii), which are applicable due to the already established validity of (192) and Lemma 21, and exploits in addition property (209) for $\rho = \gamma_{reg}\rho_{reg}$ to ensure that $|\mu|_{\mathbb{S}^1}(B_{\gamma_{reg}\rho_{reg}}(x_0)) \leq 4E_{rel}[\mu, A|\mathscr{A}]$ for any $x_0 \in \{|\xi| \leq 1/2\}$. □

Proof of Lemma 23. We argue in two steps.

Step 1: We claim that for each $k \in \{1,\dots,K\}$ there exists a Jordan-Lipschitz curve $J_k \subset \partial^* A$ from the decomposition provided by Theorem 20 such that

$$\mathcal{H}^1\big(J_k \cap B_\ell(\mathscr{J}_k)\big) > 0. \tag{251}$$

If this is not the case, fix $\bar{k} \in \{1,\dots,K\}$ with $\mathcal{H}^1(\partial^* A \cap B_\ell(\mathscr{J}_{\bar{k}})) = 0$. Let $\mathscr{A}_{\bar{k}} \subset \mathscr{A}$ such that $\partial\mathscr{A}_{\bar{k}} = \mathscr{J}_{\bar{k}}$, so that by (214) and (16) it holds

$$\mathcal{H}^1(\partial^* A \cap \{\operatorname{dist}(\cdot, \mathscr{A}_{\bar{k}}) < \ell\}) = 0.$$

Hence, by relative isoperimetric inequality $\operatorname{vol}(A \cap \{\operatorname{dist}(\cdot, \mathscr{A}_{\bar{k}}) < \ell\}) = 0$ which in turn implies

$$\|\chi_A - \chi_{\mathscr{A}}\|_{L^1}^2 \geq \operatorname{vol}(\mathscr{A}_{\bar{k}})^2. \tag{252}$$

However, it also holds ***Details?***

$$\|\chi_A - \chi_{\mathscr{A}}\|_{L^1}^2 \leq C_{vol}\ell^3 E_{vol}[A|\mathscr{A}] \tag{253}$$

for some universal constant $C_{vol} \in (1,\infty)$. Hence, assumption (190) together with the previous two displays guarantees $\min\{\operatorname{vol}(\mathscr{A}_k)^2 \colon k = 1,\dots,K\} \leq \frac{C_{vol}}{M}\ell^4$, so that choosing $M \gg_{\mathscr{A}} 1$ sufficiently large yields a contradiction.

Step 2: We now prove that the family of Jordan-Lipschitz curves $(J_k)_{k=1,\dots,K}$ from the first step satisfies the conclusion of Lemma 23. To this end, fix $k \in \{1,\dots,K\}$ and $x_0^k \in J_k \cap B_\ell(\mathscr{J}_k) \cap \{|\xi| > 1/2\}$, with the latter set being non-empty due to the first step and (214). By (216) and Lemma 21, we may apply Theorem 19 item i) meaning there exists a $C^{1,\frac{1}{2}}$ function $u\colon (x_0^k + \mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0^k)}\partial\mathscr{A}) \cap B_{\gamma_{reg}\rho_{reg}}(x_0^k) \to \mathbb{R}$ such that J_k can be represented within $B_{\gamma_{reg}\rho_{reg}}(x_0^k)$ by means of u in the sense of (198) and that, thanks to (199) and (212)–(213), $\sup|\nabla^{tan}u| \leq 2C_{reg}\varepsilon =: \tan\alpha$. Let now $x_1^k \in J_k \cap B_\ell(\mathscr{J}_k) \cap \{|\xi| > 1/2\}$ such that

$$(x_1^k - x_0^k) \cdot t_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0^k)) > 0 \tag{254}$$

and that there exists $\widetilde{x}_0^k \in x_0^k + \mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0^k)}\partial\mathscr{A}$ such that

$$x_1^k = \widetilde{x}_0^k + u(\widetilde{x}_0^k)\mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0^k)) \in \partial B_{\frac{\gamma_{reg}\rho_{reg}}{2}}(x_0^k). \tag{255}$$

In particular, $|(x_1^k - x_0^k)\cdot t_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0^k))| \geq \frac{\gamma_{reg}\rho_{reg}}{2}\cos\alpha$, so that the flatness condition (208) applied for $\rho = \frac{\gamma_{reg}\rho_{reg}}{2}$ implies

$$\big|P_{\partial\mathscr{A}}(x_1^k) - P_{\partial\mathscr{A}}(x_0^k)\big| \geq \frac{1}{2}\Big(\frac{\gamma_{reg}\rho_{reg}}{2}\cos\alpha\Big). \tag{256}$$

Hence, one may repeat the previous reasoning with new starting point x_1^k instead of x_0^k, and therefore iterate the above process (finitely many times due to (254), (256) and the graph property (206) of $\partial\mathscr{A}$ on scale ρ_{reg}) to conclude that J_k wriggles around $\mathscr{J}_k$ in the precise sense of

$$\mathrm{int}(J_k) \supset \mathscr{A}_k \setminus B_\ell(\mathscr{J}_k), \tag{257}$$

$$\mathrm{ex}(J_k) \supset \big(\mathbb{R}^2 \setminus \overline{\mathscr{A}_k}\big) \setminus B_\ell(\mathscr{J}_k). \tag{258}$$

That J_k can be represented as a graph over $\mathscr{J}_k$ with corresponding height function h finally follows from the fact that for each $x_0 \in J_k$ one may choose—by means of the inverse function theorem—an open neighborhood $\mathscr{U}_{x_0}$ of x_0 such that $J_k \cap \mathscr{U}_{x_0} \to \mathscr{J}_k$, $x \mapsto P_{\partial\mathscr{A}}(x) = x - s_{\partial\mathscr{A}}(x)\mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x))$ is a manifold diffeomorphism. ☐

Proof of Lemma 24. We proceed in four steps.

Step 1: Graph representation. Fix $k \in \{1,\dots,K\}$, assume by contradiction that there exists a second nontrivial Jordan-Lipschitz curve $J_k' \subset \partial^*A$ from the decomposition provided by Theorem 20 such that $J_k \neq J_k'$ and $\mathcal{H}^1(J_k' \cap B_\ell(\mathscr{J}_k) \cap \{|\xi| > 1/2\}) > 0$, and fix $x_0 \in J_k' \cap B_\ell(\mathscr{J}_k) \cap \{|\xi| > 1/2\}$. Since we already know that J_k wriggles around the whole $\mathscr{J}_k$, it follows that $J_k' \cap B_{\rho_{reg}}(x_0)$ is a subset of the points within $\partial^*A \cap B_{\rho_{reg}}(x_0)$ where ∂^*A is hit more than once by slicing in normal direction starting from $\mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0)}\partial\mathscr{A}$ (or more precisely, relying on the notation introduced right before (234), $J_k' \cap B_{\rho_{reg}}(x_0) \subset S_{x_0}^{(1)}$). Hence, by a slicing argument,

$$\mathcal{H}^1(J_k' \cap B_{\rho_{reg}}(x_0)) \leq C_{rel}E_{rel}[A,\mu|\mathscr{A}] \tag{259}$$

for some universal constant $C_{rel} \in (1,\infty)$, so that one may choose $M \gg_{\mathscr{A},C_\Lambda} 1$ such that by continuity of J_k' it holds

$$\mathrm{int}(J_k') \subset B_{\frac{\gamma_{reg}\rho_{reg}}{2}}(x_0). \tag{260}$$

By (216) and Lemma 21, we may on the other side apply Theorem 19 item i) to conclude that $\mathrm{supp}\,\mu = \partial^*A$ can be represented as a graph within $B_{\gamma_{reg}\rho_{reg}}(x_0)$

in the precise sense of (198). This, however, is in contradiction with the trapping condition (260).

Step 2: Regularity (44). Fix $x_0 \in \partial^* A$ and let $y_0 := P_{\partial\mathscr{A}}(x_0)$ and $T_{x_0} := \mathrm{Tan}_{y_0}\partial\mathscr{A}$. Thanks to (216) and Lemma 21, we may use Theorem 19 item i) to write $\operatorname{supp}\mu = \partial^* A$ within $B_{\gamma_{reg}\rho_{reg}}(x_0)$ as a graph over $x_0 + T_{x_0}$ with height function u, cf. (198). Let $\mathcal{U}_{x_0} \subset (x_0{+}T_{x_0}) \cap B_{\gamma_{reg}\rho_{reg}}(x_0)$ be the open neighborhood of x_0 in $x_0{+}T_{x_0}$ defined by $\mathcal{U}_{x_0} := P_{x_0}(\partial^* A \cap B_{\gamma_{reg}\rho_{reg}}(x_0))$, where P_{x_0} represents the nearest point projection onto $x_0 + T_{x_0}$. Fix $p \in [1, \infty)$. We claim that it suffices to prove

$$u \in W^{2,p}(\mathcal{U}_{x_0}) \tag{261}$$

to infer that

$$\exists \text{ open neighborhood } \mathcal{V}_{y_0} \text{ of } y_0 \text{ in } \partial\mathscr{A} \text{ such that } h \in W^{2,p}(\mathcal{V}_{y_0}), \tag{262}$$

from which one may in turn immediately deduce (44) by compactness of $\partial\mathscr{A}$.

For a proof that "(261) $\implies$ (262)", note first that $\iota\colon \mathcal{U}_{x_0} \to \partial\mathscr{A}$ defined by $x \mapsto P_{\partial\mathscr{A}}(x{+}u(x)\mathrm{n}_{\partial\mathscr{A}}(y_0))$ induces a chart for $\partial\mathscr{A}$. Defining $\mathcal{V}_{y_0}$ as the image $\iota(\mathcal{U}_{x_0})$, the claim follows from the identity $h(\iota(x)) = s_{\partial\mathscr{A}}(x{+}u(x)\mathrm{n}_{\partial\mathscr{A}}(y_0))$, $x \in \mathcal{U}_{x_0}$, and smoothness of the signed distance function $s_{\partial\mathscr{A}}$.

Hence, it remains to establish (261). To this end, fix $\zeta \in C^\infty_{cpt}(\mathcal{U}_{x_0})$ and test the identity (216) with test function $B(x) := \zeta(P_{x_0}(x))\mathrm{n}_{\partial\mathscr{A}}(y_0)$. In coordinates induced by the height function u, the resulting identity reads (recall also (215))

$$\int_{\mathcal{U}_{x_0}} \frac{u'}{\sqrt{1{+}(u')^2}}\zeta'\,d\mathcal{H}^1 = -\int_{\mathcal{U}_{x_0}} (\mathrm{H}\cdot\mathrm{n}_{\partial^* A})\big(x{+}u(x)\mathrm{n}_{\partial\mathscr{A}}(y_0)\big)\zeta(x)\,d\mathcal{H}^1(x), \tag{263}$$

where we make use of the notation $f' := (t_{\partial\mathscr{A}}(y_0)\cdot\nabla^{tan})f$. Due to (194), we know that $x \mapsto (\mathrm{H}\cdot\mathrm{n}_{\partial^* A})\big(x{+}u(x)\mathrm{n}_{\partial\mathscr{A}}(y_0)\big) \in L^p(\mathcal{U}_{x_0})$ and thus $\frac{u'}{\sqrt{1+(u')^2}} \in W^{1,p}(\mathcal{U}_{x_0})$. Since (199) and (204) imply $\sup|u'| \le \frac{1}{64}$, we obtain (261).

Step 3: Estimate for $\sup|h|$. Assume that there exists $x_0 \in \partial^* A$ such that

$$\big|h(P_{\partial\mathscr{A}}(x_0))\big| > \frac{\ell}{16C}. \tag{264}$$

By (216) and Lemma 21, we may again apply Theorem 19 item i) to write $\operatorname{supp}\mu = \partial^* A$ within $B_{\gamma_{reg}\rho_{reg}}(x_0)$ as a graph over $x_0{+}\mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0)}\partial\mathscr{A}$ with height function u, cf. (198). Because of (199), (212)–(213), and (204)–(205)

$$\sup|u| \le \rho 2C_{reg}\varepsilon \le \frac{1}{4}\frac{\ell}{16C}, \tag{265}$$

$$\sup|\nabla^{tan}u| \le 2C_{reg}\varepsilon =: \tan\alpha,\ \alpha \in (0, \pi/2). \tag{266}$$

Together with the previous two estimates, the flatness property (207) implies that

$$\|\chi_A{-}\chi_{\mathscr{A}}\|_{L^1} \ge \big(\gamma_{reg}\rho_{reg}\cos\alpha\big)\frac{1}{2}\frac{\ell}{16C}. \tag{267}$$

In combination with the coercivity estimate (253) and assumption (190), we may therefore choose $M \gg_{\mathscr{A},C_\Lambda} 1$ to reach a contradiction.

Step 4: Estimate for $\sup|\nabla^{tan}h|$. For computational convenience, we start by representing $\partial^* A$ as the image of a curve $\gamma_h(\theta) := \gamma(\theta){+}h(\gamma(\theta))\mathrm{n}_{\partial\mathscr{A}}(\gamma(\theta))$, where γ is an arclength parametrization of $\partial\mathscr{A}$. Fix $x_0 \in \partial^* A$ and denote again by u the associated height function representing $\partial^* A$ locally around x_0 in the sense of (198). In particular, the arc $\partial^* A{\cap}B_{\gamma_{reg}\rho_{reg}}(x_0)$ is given as the image of the curve

$\gamma_u(\theta) := \gamma_{x_0}(\theta)+h(\gamma_{x_0}(\theta))\mathrm{n}_{\partial\mathscr{A}}(\gamma_{x_0}(\theta))$ with γ_{x_0} being an arclength parametrization of $B_{\gamma_{reg}\rho_{reg}}(x_0) \cap (x_0+\mathrm{Tan}_{P_{\partial\mathscr{A}}(x_0)}\partial\mathscr{A})$. Expressing the normal $\frac{\nabla\chi_A}{|\nabla\chi_A|}(x_0)$ using either of the two parametrizations, we then get two equivalent ways of expressing the scalar product $\frac{\nabla\chi_A}{|\nabla\chi_A|}(x_0)\cdot \mathrm{n}_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0))$ yielding the identity

$$\frac{1}{\sqrt{1+|\nabla^{tan}u(x_0)|^2}} = \frac{1}{\sqrt{1+\big|\frac{\nabla^{tan}h}{1-H_{\partial\mathscr{A}}h}\big|^2\big(P_{\partial\mathscr{A}}(x_0)\big)}}. \tag{268}$$

By the previous step, $|h(P_{\partial\mathscr{A}}(x_0))| \leq \frac{1}{16}\ell$, so that $|H_{\partial\mathscr{A}}(P_{\partial\mathscr{A}}(x_0))| \leq \frac{1}{\ell}$ together with the previous identity, (266) and (204) entails

$$\big|(\nabla^{tan}h)\big(P_{\partial\mathscr{A}}(x_0)\big)\big| \leq \frac{17}{16}|\nabla^{tan}u(x_0)| \leq \frac{1}{16C}. \tag{269}$$

This concludes the proof. □

Proof of Proposition 5. Immediate from Lemma 22, Lemma 23 and Lemma 24. □

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