STABILITY OF MULTIPHASE MEAN CURVATURE FLOW BEYOND CIRCULAR TOPOLOGY CHANGES

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ABSTRACT. We prove a weak-strong uniqueness principle for varifold-BV solutions to planar multiphase mean curvature flow beyond a circular topology change: Assuming that there exists a classical solution with an interface that becomes increasingly circular and shrinks to a point, any varifold-BV solution with the same initial interface must coincide with it, and any varifold-BV solution with similar initial data must undergo the same type of topology change. Our result illustrates the robustness of the relative energy method for establishing weak-strong uniqueness principles for interface evolution equations, showing that it may also be applied beyond certain topological changes.

CONTENTS

1. Introduction

For two-phase mean curvature flow, a weak solution concept with highly satisfactory properties is available in form of viscosity solutions [6, 5] to the level set formulation [21, 20]: Not only can global-in-time existence of weak solutions be shown for general initial data, but failure of uniqueness may be characterized in quite detail [3], and uniqueness of weak solutions is guaranteed as long as a classical solution exists. However, the concept of viscosity solutions crucially relies on the availability of the comparison principle, restricting it to (mostly) mean curvature flow in the context of interface evolution problems.

For interface evolution problems without comparison principle such as multiphase mean curvature flow or higher-order curvature driven flows, the question of uniqueness of weak solutions – even in the absence of topology changes – had remained open for a long time. In fact, solution concepts such as Brakke solutions for mean curvature flow [4] have even been known to admit artificial (unphysical) solutions, and attempts to develop solution concepts without these shortcomings have been made [17, 16, 23]. Recently, an approach based on relative energies has proven successful in establishing weak-strong uniqueness prior to singularities [7, 9, 8, 14], as well as in deriving sharp-interface limits of phase-field models [10, 1, 11, 15].

So far, these weak-strong uniqueness results for interface evolution problems have been limited to situations without geometric singularities and in particular without topology changes. In the present work we show that the approach of relative energies is robust and capable of handling certain controlled topology changes in the (piecewise-in-time) strong solution: We show that in the case of multiphase mean curvature flow, a weak-strong uniqueness and stability principle holds also beyond shrinking circle type singularities. More precisely, for any classical solution to planar curvature flow whose interface consists of a smooth simple curve that shrinks, becomes increasingly circular, and disappears, any weak solution with similar initial data must stay close to it and disappear in the same kind of singularity. As in [9], the weak solutions we consider are varifold-BV solutions in the sense of Stuvard-Tonegawa [23].

Recall that the classical result of Gage–Hamilton [12] and Grayson [13] asserts that any smooth, closed, and simple curve in the plane evolving by mean curvature flow (MCF) shrinks to a point in finite time, becoming increasingly circular in the process. Combining this classical result with our main result, we recover a perturbative but genuinely multiphase version of the Gage–Hamilton–Grayson theorem: If the initial interface of a varifold-BV solution to mean curvature flow is sufficiently close to a smooth, closed, simple curve (in the sense of the relative energy distance, that is, in a tilt-excess-type distance), it will over time become increasingly circular and eventually disappear in a shrinking circle type singularity. In particular, this conclusion remains valid even if initially a small amount of other phases are present in the varifold-BV solution.

2. Main result

To state our main result, we first recall the notion of relative energy of a varifold-BV solution; note that the latter consist of a time-indexed family of varifolds $\mathcal{V}_t$ and an indicator function $\chi_i(\cdot,t)$ for each phase. In [9] the relative energy of a varifold-BV solution with respect to a strong solution $(\bar\chi_i)_{1\leq i\leq P}$ was defined as

$$E_{rel}(t) := \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\int_{I_{i,j}(t)} 1-\mathrm{n}_{i,j}(\cdot,t)\cdot\xi_{i,j}(\cdot,t)\,d\mathcal{H}^1 + \int_{\mathbb{R}^2\times\mathbb{S}^1} 1-\omega(x,t)\,d\mathcal{V}_t(x,s) \tag{1}$$

where $\xi_{i,j}(\cdot,t)$ denotes a suitable extension of the unit normal vector field of the interface between phases i and j in the strong solution, $I_{i,j}(t) := \partial^*\{\chi_i(\cdot,t) = 1\}\cap\partial^*\{\chi_j(\cdot,t)=1\}$ denotes the interface between phases i and j in the varifold-BV solution, $\mathrm{n}_{i,j}$ is its corresponding unit normal vector, and $\omega(\cdot,t)\in[0,1]$ is the local ratio between the surface measure $\frac{1}{2}\sum_{i=1}^{P}|\nabla\chi_i|(\cdot,t)$ and the weight measure $\mu_t(\cdot) := \int_{\mathbb{S}^1} d\mathcal{V}_t(\cdot,s)$. For our present results the relative energy will share the same structure as in (1), except that as in [9] we also add a lower-order term for coercivity; it is merely the vector fields $\xi_{i,j}$ that will be modified suitably (in a way that corresponds to simply shifting the strong solution $\bar\chi$ in space and time).

Observe that the relative energy (1) measures the mismatch between the classical solution and the varifold-BV solution in a tilt-excess-type way; furthermore, the second term on the right-hand side of (1) measures the mismatch in multiplicity between the varifold $\mathcal{V}_t$ and the surface measure $\frac{1}{2}\sum_{i=1}^{P}|\nabla\chi_i|(\cdot,t)$.

Recall that the general goal of weak-strong uniqueness proofs via relative energy methods is to establish a Gronwall-type estimate $\frac{d}{dt}E_{rel}\leq CE_{rel}$, which enables one to conclude. However, previous weak-strong stability results of this form (e.g., [9] and [14]) have been limited to time horizons before the first topology change of the strong solution: The reason is that for typical topology changes such as the circular topology change considered in the present work, a naive estimation of the terms on the right-hand side of the relative energy inequality would lead to a Gronwall estimate of the form $\frac{d}{dt}E_{rel}\leq C(t)E_{rel}$ with $C(t)\sim\frac{1}{|T-t|}$. Note that the time-dependent constant $C(t)$ is borderline non-integrable, leading to a loss of any assertion on stability past the topology change. This suggests that in order to deal with topology changes, a more refined estimation is needed to control the right-hand side of the relative energy estimate, e. g., by combining the relative entropy approach with a linearized stability analysis.

A linearized stability analysis for the relative energy (1) beyond a circular topology change however reveals the presence of two unstable modes and one borderline stable mode. It turns out that the unstable modes correspond to translational degrees of freedom, while the borderline stable mode corresponds to a shift in time (see Figure 1 and Figure 2 for a more detailed explanation). We overcome this issue of unstable modes by developing a weak-strong stability theory for circular topology change up to dynamic shift, which amounts to dynamically adapting the strong solution to the weak solution to a degree which takes care of the leading-order non-integrable contributions in the Gronwall estimate.

The precise statement of our main result reads as follows.

Theorem 1 (Weak-strong stability up to shift for circular topology change)**.** *Let $d = 2$ and $P \geq 2$. Consider a global-in-time varifold-BV solution $(\mathcal{V}, \chi)$ with $\chi = (\chi_1, \dots, \chi_P)$ (or a BV solution $\chi = (\chi_1, \dots, \chi_P)$) to multiphase MCF in the sense of Definition 3 (or Definition 2). Consider also a smoothly evolving two-phase strong solution to MCF $\bar\chi = (\bar\chi_1, \dots, \bar\chi_P \equiv 1-\bar\chi_1)$ with extinction time $T_{ext} =: \frac{1}{2} r_0^2 > 0$. Fix $\alpha \in (1, 5)$.*

There exists $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ such that if for all $t \in (0, T_{ext})$ the interior of the phase $\{\bar\chi_1(\cdot, t){=}1\} \subset \mathbb{R}^2$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext}{-}t)}$ in the sense of Definition 4, the evolution of $\bar\chi$ is unique and stable until the extinction time T_{ext} modulo shift *in the following sense:*

There exists $\delta \ll 1$ as well as an error functional $E[\mathcal{V}_0, \chi_0|\bar\chi_0] \in [0, \infty)$ for the initial data $(\mathcal{V}_0, \chi_0)$ and $\bar\chi_0$ of $(\mathcal{V}, \chi)$ and $\bar\chi$, respectively, such that if

$$E[\mathcal{V}_0, \chi_0|\bar\chi_0] < \delta r_0, \tag{2}$$

one may then choose

$$\begin{aligned} &\text{a time horizon } t_\chi > 0, \\ &\text{a path of translations } z \in W^{1,\infty}((0, t_\chi); \mathbb{R}^2), \text{ and} \\ &\text{a strictly increasing bijection } T \in W^{1,\infty}((0, t_\chi); (0, T_{ext})) \end{aligned}$$

with the properties $(z(0), T(0)) = (0, 0)$,

$$\frac{1}{r_0} \|z\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0} E[\mathcal{V}_0, \chi_0|\bar\chi_0]}, \tag{3}$$

$$\frac{1}{T_{ext}} \|T - \mathrm{id}\|_{L^\infty_t(0,t_\chi)} \leq \sqrt{\frac{1}{r_0} E[\mathcal{V}_0, \chi_0|\bar\chi_0]}, \tag{4}$$

such that for a.e. $t \in (0, t_\chi)$ it holds

$$E[\mathcal{V}, \chi|\bar\chi^{z,T}](t) \leq E[\mathcal{V}_0, \chi_0|\bar\chi_0] \Big(\frac{r_T(t)}{r_0}\Big)^\alpha \tag{5}$$

where $\bar\chi^{z,T}(x, t) := \bar\chi(x{-}z(t), T(t))$, $(x, t) \in \mathbb{R}^2{\times}[0, t_\chi)$, denotes the shifted strong solution*, $r_T(t) := r(T(t))$ for $t \in [0, t_\chi)$, and $E[\mathcal{V}, \chi|\bar\chi^{z,T}](t)$ is an error functional satisfying*

$$E[\mathcal{V}, \chi|\bar\chi^{z,T}](t) = 0 \quad \iff \quad \begin{cases} \chi(\cdot, t) = \bar\chi^{z,T}(\cdot, t) & \mathcal{H}^2\text{-a.e. in } \mathbb{R}^2, \\ \mu_t = \frac{1}{2} \sum_{i=1}^P |\nabla\bar\chi(\cdot, t)| & \mathcal{H}^1\text{-a.e. in } \mathbb{R}^2. \end{cases} \tag{6}$$

In particular, under the assumption of (2), the varifold-BV solution $(\mathcal{V}, \chi)$ goes extinct and the associated time horizon t_χ provides an upper bound for the extinction time.

We phrased our main result in a form emphasizing the main contribution of this work, i.e., stability of the evolution for times close to a circular topology change (formalized above by means of the notion of quantitative closeness to a shrinking circle, see Definition 4 for details). One may also derive a corresponding stability estimate starting from initial data not entailing an approximately self-similarly evolving solution at early times.

Remark 1. Consider a smoothly evolving two-phase solution to mean curvature flow $\bar\chi = (\bar\chi_1, \dots, \bar\chi_P \equiv 1{-}\bar\chi_1)$ with initial data $\bar\chi_{0,1} = \chi_{\mathcal{A}_0}$ for some smooth, bounded, open and simply connected initial set $\mathcal{A}_0 \subset \mathbb{R}^2$. By the Gage–Hamilton–Grayson theorem ([12],[13]), the solution goes extinct at time $T_{ext} = \frac{\mathrm{vol}(\mathcal{A}_0)}{\pi}$, and for any given $\delta_{\mathrm{asymp}} \in (0,1)$, there exists a time $t_0 = t_0(\mathcal{A}_0, \delta_{\mathrm{asymp}}) < T_{ext}$ such that for all $t \in [t_0, T_{ext})$ it holds that the interior of $\{\bar\chi_1(\cdot,t) = 1\}$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext} - t)}$ in the sense of Definition 4.

In particular, from some time onwards one is in the asymptotic regime close to the extinction time for which the conclusions of Theorem 1 apply, at least if at time $t_0 = t_0(\mathcal{A}_0, \delta_{\mathrm{asymp}})$ the assumption (2) on the smallness of the initial error is satisfied (i.e., with respect to $r(t_0)$). Based on the weak-strong stability estimate prior to topology changes from [9], this requirement can be translated into a condition at the initial time $t = 0$: there exists a constant $\mu_0 = \mu_0(t_0, \mathcal{A}_0) > 0$ such that if $E[\chi_0|\bar\chi_0] < \frac{1}{\mu_0}\delta r_0$ then $E[\chi|\bar\chi](t_0) < \delta r(t_0)$.

In summary, for general initial data $\mathcal{A}_0$ as considered in this remark, one first has, thanks to the main result of [9], at least stability in the sense of $\frac{d}{dt}E[\chi|\bar\chi](t) \leq C(t)E[\chi|\bar\chi](t)$ for times $t \in (0, t_0)$ where $C(t) \sim (2(T_{ext} - t))^{-1} = r(t)^{-2}$. Then, if $E[\chi_0|\bar\chi_0] < \frac{1}{\mu_0}\delta r_0$, in addition the decay estimate (5) from Theorem 1 holds true for all times in the asymptotic regime (t_0, T_{ext}). $\diamondsuit$

Before we recall the precise definitions of the two weak solution concepts to which our main result applies, we provide two comments on the latter.

- First, note that the decay exponent $\alpha < 5$ in our stability estimate (5) is optimal in the sense that it is consistent with the results obtained by Gage–Hamilton [12]. More precisely, from [12, Corollary 5.7.2] one can read off that for a smooth, closed and simple curve $\partial\mathcal{A}(t)$ shrinking by MCF to the origin $x = 0$, it holds asymptotically as $t \uparrow T_{ext}$ that $\sup_{\partial\mathcal{A}(t)} |\nabla^{\tan} H_{\partial\mathcal{A}(t)}| \lesssim r(t)^{-\tilde\alpha}$, for any $0 < \tilde\alpha \ll 1$. Since $H_{\partial\mathcal{A}} := -\nabla^{\tan} \cdot \mathrm{n}_{\partial\mathcal{A}}$, by dimensional analysis, one then expects from the fact that our error functional behaves like a tilt excess that one gets decay for any exponent $\alpha = 5 - \tilde\alpha$, $0 < \tilde\alpha \ll 1$.
- Second, note that the error bounds (3)–(4) on the space-time shift (z, T) are optimal in terms of the scaling $\sqrt{\frac{1}{r_0}E[\mathcal{V}_0, \chi_0|\bar\chi_0]}$. For example, let $0 < \delta \ll 1$ and consider, next to a shrinking circle with initial radius r_0, a shrinking circle with initial radius $(1 + \sqrt{\delta})r_0$ (both centered at the origin). Note that the initial error between the two solutions indeed satisfies our assumption (2), cf. (83)–(84). Since the relative error between the two extinction times is $\sim \sqrt{\delta}$, this shows the claim for (4). Shifting instead a shrinking circle with initial radius r_0 initially by $\sqrt{\delta}r_0 v$, $v \in \mathbb{S}^1$, in turn illustrates the claim for (3).

Our arguments to prove weak-strong stability up to dynamic shift for circular topology change work for both BV solutions in the sense of Laux-Otto-Simon ([17], [18], [19]) and for more general varifold-BV solutions, recently introduced by Stuvard and Tonegawa in [23]. Here we recall the definitions of both of these weak solution concepts (cf. [9, Definition 12] and [9, Definition 18]).

Definition 2 (BV solution to multiphase MCF)**.** Let $d = 2$ and $P \geq 2$. A measurable map

$$\chi = (\chi_1, \ldots, \chi_P) \colon \mathbb{R}^2 \times [0, \infty) \to \{0,1\}^P$$

(or the corresponding tuple of sets $\Omega_i(t) := \{\chi_i(t) = 1\}$ for $i = 1, \ldots, P$) is called a *global-in-time BV solution to multiphase MCF* with initial data $\chi_0 =$

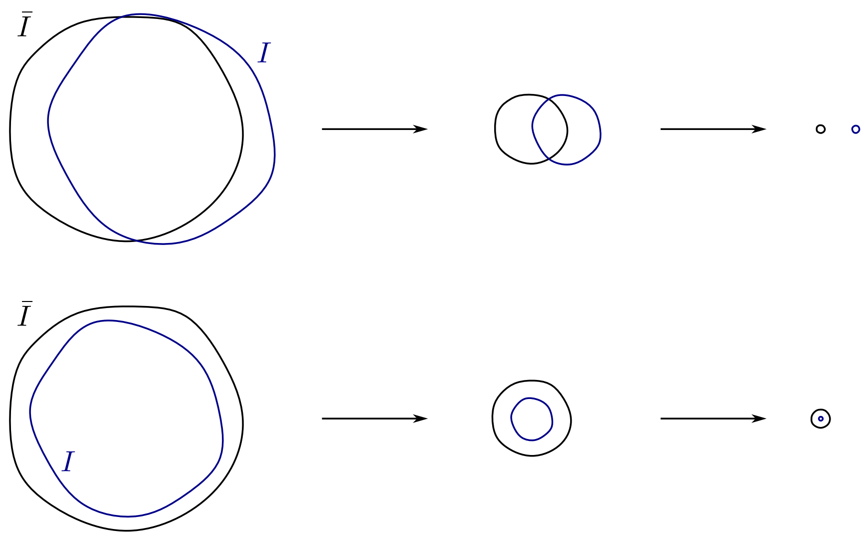

FIGURE 1. First case - Motivation for space shift z: $\bar{I} = \partial\{\bar{\chi}_1 = 1\}$ and $I = \partial^*\{\chi_1 = 1\}$ simultaneously shrink to two distinct points which are shifted by z. Second case - Motivation for time shift T: $\bar{I} = \partial\{\bar{\chi}_1 = 1\}$ and $I = \partial^*\{\chi_1 = 1\}$ shrink to the same point but at distinct times.

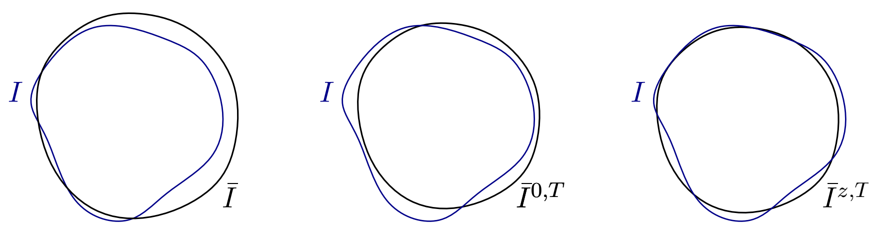

FIGURE 2. The interface $\bar{I} = \partial\{\bar{\chi}_1 = 1\}$ is dynamically adapted to $I = \partial^*\{\chi_1 = 1\}$ by means of the space-time shift (z, T), namely $\bar{I}^{z,T} = \partial\{\bar{\chi}_1^{z,T} = 1\}$.

$(\chi_{0,1}, \ldots, \chi_{0,P})\colon\ \mathbb{R}^2 \to \{0,1\}^P$ (of finite interface energy in the sense of [9, Definition 12]) if the following conditions are satisfied:

i) For any $T_{\mathrm{BV}} \in (0, \infty)$, χ is a BV solution to multiphase MCF on $[0, T_{\mathrm{BV}})$ with initial data χ_0 in the sense of [9, Definition 13] (with trivial surface tension matrix $\sigma = \mathrm{diag}(1, \ldots, 1) \in \mathbb{R}^{P\times P}$) such that
   i.a) (Partition with finite interface energy) For almost every $T \in [0, T_{\mathrm{BV}})$, $\chi(T)$ is a partition of $\mathbb{R}^2$ with interface energy

$$E[\chi] := \frac{1}{2} \sum_{i,j=1, i\neq j}^{P} \mathcal{H}^1(I_{i,j}(t)) \tag{7}$$

such that

$$\operatorname*{ess\,sup}_{T\in[0,T_{\mathrm{BV}})} E[\chi(\cdot,T)] < \infty, \tag{8}$$

where $I_{i,j}(t) = \partial^*\{\chi_i(\cdot,t) = 1\} \cap \partial^*\{\chi_j(\cdot,t) = 1\}$ for $i \neq j$ is the interface between the i-th and the j-th phase. We also define $\mathrm{n}_{i,j}(\cdot,t) := -\frac{\nabla\chi_i(\cdot,t)}{|\nabla\chi_i(\cdot,t)|} = \frac{\nabla\chi_j(\cdot,t)}{|\nabla\chi_j(\cdot,t)|}$ abeing the unit normal vector field along $I_{i,j}(t)$ pointing from the i-th to the j-th phase.

i.b) (Evolution equation) For all $i \in \{1,\dots,P\}$, there exist normal velocities $V_i \in L^2(\mathbb{R}^2\times[0,T_{\mathrm{BV}}), |\nabla\chi_i|\otimes\mathcal{L}^1)$ in the sense that each χ_i satisfies the evolution equation

$$\begin{aligned}&\int_{\mathbb{R}^2}\chi_i(\cdot,T)\varphi(\cdot,T)\,dx - \int_{\mathbb{R}^2}\chi_{0,i}\varphi(\cdot,0)\,dx\\ &= \int_0^T\int_{\mathbb{R}^2} V_i\varphi\,\mathrm{d}|\nabla\chi_i|dt + \int_0^T\int_{\mathbb{R}^2}\chi_i\partial_t\varphi\,dxdt\end{aligned} \tag{9}$$

for almost every $T \in [0,T_{\mathrm{BV}})$ and all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^2\times[0,T_{\mathrm{BV}}])$. Moreover, the (reflection) symmetry condition $V_i\frac{\nabla\chi_i}{|\nabla\chi_i|} = V_j\frac{\nabla\chi_j}{|\nabla\chi_j|}$ shall hold $\mathcal{H}^1$-almost everywhere on the interfaces $I_{i,j}$ for $i \neq j$.

i.c) (BV formulation of mean curvature) The normal velocities satisfy the equation

$$\begin{aligned}&\sum_{i,j=1,i\neq j}^{P}\int_0^{T_{\mathrm{BV}}}\int_{I_{i,j}(t)} V_i\frac{\nabla\chi_i}{|\nabla\chi_i|}\cdot\mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}dt\\ &= \sum_{i,j=1,i\neq j}^{P}\int_0^{T_{\mathrm{BV}}}\int_{I_{i,j}(t)}\Big(\mathrm{Id}-\frac{\nabla\chi_i}{|\nabla\chi_i|}\otimes\frac{\nabla\chi_i}{|\nabla\chi_i|}\Big):\nabla\mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}dt\end{aligned} \tag{10}$$

for all $B \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^2\times[0,T_{\mathrm{BV}}];\mathbb{R}^d)$.

ii) For all $[s,\tau] \subset [0,\infty)$, the energy dissipation inequality

$$E[\chi(\cdot,\tau)] + \int_s^\tau \sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}(t)}\frac{1}{2}|V_i|^2\,d\mathcal{H}^1dt \leq E[\chi(\cdot,s)] \tag{11}$$

holds true, and in addition more generally the corresponding Brakke inequality in the BV-setting [18, Definition 2.1]. ◊

Definition 3 (Varifold-BV solution to multiphase MCF, cf. [23])**.** Let $\mathcal{V} = (\mathcal{V}_t)_{t\in(0,\infty)}$ be a measurable family of integral and rectifiable $(d-1)$-varifolds; denote by $(\mu_t)_{t\in(0,\infty)}$ the associated family of weight measures. Let $(\chi_1,...,\chi_P)\colon\mathbb{R}^d\times[0,\infty)\to\{0,1\}^P$ denote a family of indicator functions of sets with bounded perimeter subject to the properties in item i.a) of Definition 2.

The tuple $(\mathcal{V},\chi)$ is called a *global-in-time varifold-BV solution to multiphase MCF* with initial data $(\mathcal{V}_0,\chi_0)$, where $\mathcal{V}_0$ is an integral and rectifiable $(d-1)$-varifold and χ_0 is as in Definition 2, if the following conditions are satisfied:

i) For a.e. $t \in [0,\infty)$, there exists a generalized mean curvature vector $H_\mu(\cdot,t) \in L^2(\mathbb{R}^d,\mu_t)$ of $\mathcal{V}_t$ in the sense that

$$-\int_{\mathbb{R}^d} H_\mu\cdot B\,\mathrm{d}\mu_t = \int_{\mathbb{R}^d\times\mathbf{G}(d,d-1)}\mathrm{Id}_{\mathbf{G}(d,d-1)}:\nabla B\,\mathrm{d}\mathcal{V}_t \tag{12}$$

for all $B \in C^\infty_{cpt}(\mathbb{R}^d;\mathbb{R}^d)$, where $\mathbf{G}(d,d-1)$ denotes the space of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$.

ii) The family of varifolds $\mathcal{V}$ is a Brakke solution to multiphase mean curvature flow (cf. [23, Definition 2.1]). Furthermore, for all $[s,\tau] \subset [0,\infty)$ the global energy dissipation estimate

$$\mu_\tau(\mathbb{R}^d) + \int_s^\tau \int_{\mathbb{R}^d} |H_\mu|^2 \,\mathrm{d}\mu_t \leq \mu_s(\mathbb{R}^d) \tag{13}$$

holds true.

iii) For a.e. $t \in (0,\infty)$, the varifold $\mathcal{V}_t$ describes the interfaces $\partial^*\{\chi_i(\cdot,t)=1\}$ in the sense that

$$\frac{1}{2}\sum_{i=1}^P |\nabla\chi_i(\cdot,t)| \leq \mu_t. \tag{14}$$

iv) The indicator functions χ_i evolve according to the mean curvature of $\mathcal{V}$ in the sense that

$$\partial_t\chi_i + H_\mu \cdot \nabla\chi_i = 0 \tag{15}$$

holds distributionally for all $i \in \{1,...,P\}$. ♢

We finally formalize the notion of being quantitatively close to a circle.

Definition 4 (Quantitative closeness to circle)**.** Let $\mathcal{A} \subset \mathbb{R}^2$ be a bounded, open and simply connected set with C^∞ boundary $\partial\mathcal{A}$. Fix two constants $\delta_{\mathrm{asymp}} \in (0,\frac{1}{2})$ and $r > 0$. We refer to $\mathcal{A}$ as δ_{asymp}-*close to a circle with radius* r if there exists an arc-length parametrization $\gamma\colon [0,L) \to \mathbb{R}^2$ of $\partial\mathcal{A}$ such that $\frac{1}{2}r$ is a tubular neighborhood width of $\partial\mathcal{A}$ and

$$\frac{1}{2\pi r}|L - 2\pi r| \leq \delta_{\mathrm{asymp}}, \tag{16}$$

$$\sup_{\theta\in[0,L)} \big|\mathrm{n}_{\partial\mathcal{A}}\big(\gamma(\theta)\big) - (-e^{2\pi i\frac{\theta}{L}})\big| \leq \delta_{\mathrm{asymp}}, \tag{17}$$

$$\sup_{\theta\in[0,L)} r\Big|H_{\partial\mathcal{A}}\big(\gamma(\theta)\big) - \frac{1}{r}\Big| \leq \delta_{\mathrm{asymp}}, \tag{18}$$

$$\sup_{\theta\in[0,L)} r^2\big|(\nabla^{\tan}H_{\partial\mathcal{A}})\big(\gamma(\theta)\big)\big| \leq \delta_{\mathrm{asymp}}, \tag{19}$$

where $\mathrm{n}_{\partial\mathcal{A}}$ denotes the unit normal vector field along $\partial\mathcal{A}$ pointing inside $\mathcal{A}$ and $H_{\partial\mathcal{A}} := -\nabla^{\tan}\cdot\mathrm{n}_{\partial\mathcal{A}}$ is the associated scalar mean curvature of $\partial\mathcal{A}$. ♢

Notation and some elementary differential geometry. For the smoothly evolving $\bar\chi$, we write $\mathrm{n}_{\bar I}(\cdot,t)$ for the unit normal vector field of $\bar I(\cdot,t) := \partial\{\bar\chi_1(\cdot,t)=1\}$ pointing inside $\{\bar\chi_1(\cdot,t)=1\}$, and also define a tangent vector field through $\tau_{\bar I}(\cdot,t) := J^{-1}\mathrm{n}_{\bar I}(\cdot,t)$ with $J \in \mathbb{R}^{2\times 2}$ being counter-clockwise rotation by 90°, $t \in (0,T)$. Curvature is defined by $H_{\bar I}(\cdot,t) := -\nabla^{\tan}\cdot\mathrm{n}_{\bar I}(\cdot,t)$ for $t \in (0,T_{ext})$. In particular, it holds

$$\nabla^{\tan}\mathrm{n}_{\bar I} = -H_{\bar I}\tau_{\bar I}\otimes\tau_{\bar I}, \qquad \nabla^{\tan}\tau_{\bar I} = H_{\bar I}\mathrm{n}_{\bar I}\otimes\tau_{\bar I}. \tag{20}$$

Within the tubular neighborhood $\{x \in \mathbb{R}^2\colon \mathrm{dist}(x,\bar I(t)) < r(t)/2\}$, the nearest-point projection onto $\partial\{\bar\chi(\cdot,t)=1\}$ is denoted by $P_{\bar I}(\cdot,t)$, whereas we write $\mathrm{sdist}_{\bar I}(\cdot,t)$ for the signed distance function, with orientation fixed through the requirement $\nabla\,\mathrm{sdist}_{\bar I}(\cdot,t)|_{\bar I} = \mathrm{n}_{\bar I}(\cdot,t)$, $t \in (0,T_{ext})$.

Given a map $f\colon \mathbb{R}^2 \times [0,T_{ext}) \to \mathbb{R}^m$ (or $f\colon \bigcup_{t\in[0,T_{ext})} \bar{I}(t)\times\{t\} \to \mathbb{R}^m$), we will use the notation $f^{z,T}$ to refer to the space-time shifted function $\mathbb{R}^2 \times (0,t_\chi) \ni (x,t) \mapsto f(x-z(t),T(t)) \in \mathbb{R}^m$ (or in the other case $\bigcup_{t\in[0,t_\chi)}(z(t)+\bar{I}(T(t))\times\{t\} \ni (x,t) \mapsto f(x-z(t),T(t)) \in \mathbb{R}^m$) for any $t_\chi \in (0,\infty)$, $z\colon [0,t_\chi) \to \mathbb{R}^2$ and $T\colon [0,t_\chi) \to [0,T_{ext})$. We also define $\bar{I} := \bigcup_{t\in[0,T_{ext})} \bar{I}(t)\times\{t\}$.

The shifted geometry itself will be abbreviated by $\bar{I}^{z,T}(t) := z(t) + \bar{I}(T(t))$, $t \in (0,t_\chi)$, and analogously for an associated arc-length parametrization $\bar{\gamma}(\cdot,t)$ of $\bar{I}(\cdot,t)$: $\bar{\gamma}^{z,T}(\cdot,t) := z(t)+\bar{\gamma}(\cdot,T(t))$, $t \in (0,t_\chi)$. We also write $\bar{I}^{z,T} := \bigcup_{t\in[0,t_\chi)} \bar{I}^{z,T}(t)\times\{t\}$. Note then that

$$\operatorname{sdist}_{\bar{I}}^{z,T}(\cdot,t) = \operatorname{sdist}_{\bar{I}^{z,T}}(\cdot,t), \tag{21}$$

and thus as a direct consequence

$$\mathrm{n}_{\bar{I}}^{z,T}(\cdot,t) = \mathrm{n}_{\bar{I}^{z,T}}(\cdot,t). \tag{22}$$

Indeed, the former simply follows from

$$\operatorname{sdist}_{\bar{I}}(\cdot,t) = \operatorname{dist}\big(\cdot,\mathbb{R}^2 \setminus \{\bar{\chi}_1(\cdot,t)=1\}\big) - \operatorname{dist}\big(\cdot,\{\bar{\chi}_1(\cdot,t)=1\}\big).$$

Furthermore, within the tubular neighborhood $\{x \in \mathbb{R}^2\colon \operatorname{dist}(x-z(t),\bar{I}(T(t))) < r(T(t))/2\} = \{x \in \mathbb{R}^2\colon \operatorname{dist}(x,\bar{I}^{z,T}(t)) < r(T(t))/2\}$ it holds

$$P_{\bar{I}}^{z,T}(\cdot,t) = -z(t) + P_{\bar{I}^{z,T}}(\cdot,t). \tag{23}$$

Finally, for simplicity, we will denote $\frac{d}{\mathrm{d}t}f$ by $\dot{f}$.

3. Overview of the strategy

For the rest of the paper, we consider the more general framework of varifold-BV solutions. In particular, it follows that all the results hold also for BV solutions.

We fix a global-in-time varifold-BV solution $(\mathcal{V},\chi = (\chi_1,\ldots,\chi_P))$ to (planar) multiphase MCF in the sense of Definition 3 as well as a smoothly evolving two-phase solution to MCF $\bar{\chi} = (\bar{\chi}_1,\ldots,\bar{\chi}_P \equiv 1-\bar{\chi}_1)$ with extinction time $T_{ext} =: \frac{1}{2}r_0^2 > 0$. We also assume that for all $t \in (0,T_{ext})$ the interior of the phase $\{\bar{\chi}_1(\cdot,t)=1\} \subset \mathbb{R}^2$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext}-t)}$ in the sense of Definition 4. Consistent with the claim of Theorem 1, we will choose a suitable value of the constant δ_{asymp} in the course of the proof.

3.1. Heuristics: Leading-order behaviour near extinction time.

The aim of this subsection is to compute heuristically the time evolution of our linearized error functional in the simplified case of a centered self-similarly shrinking circle. As a result, our analysis reveals the instability of our linearized error functional near the extinction time.

Consider a centered circle self-similarly shrinking by mean curvature flow: $t \mapsto \partial B_{r(t)} = \operatorname{im}\bar{\gamma}(t) \subset \mathbb{R}^2$, where $\bar{\gamma}(t)\colon [0,2\pi r(t)) \to \partial B_{r(t)}$, $\theta \mapsto r(t)e^{i\frac{\theta}{r(t)}}$, is an arc-length parametrization of $\partial B_{r(t)}$. In particular, $\dot{r} = -\frac{1}{r}$ in the interval $(0,\frac{1}{2}r_0^2 = T_{ext})$ for $r_0 := r(0) > 0$, i.e., $r(t) = \sqrt{2(T_{ext}-t)}$.

Apart from the shrinking circle, let us consider a second solution to mean curvature flow, for which we in addition assume that it can be written as a smooth graph over the self-similarly shrinking circle. More precisely, there exists a smooth time-dependent height function $h(\cdot,t)\colon \partial B_{r(t)} \to \mathbb{R}$ with $|h(\cdot,t)| \ll r(t)$ and $|h'(\cdot,t)| \ll 1$

such that this second solution is represented as the image of the curve

$$\gamma_h(\cdot,t) := \big(\mathrm{id} + h(\cdot,t)\mathrm{n}_{\partial B_{r(t)}}\big)\circ\bar\gamma(\cdot,t) \quad \text{on } [0,2\pi r(t)), \tag{24}$$

where $\mathrm{n}_{\partial B_{r(t)}}$ denotes the inward-pointing unit normal along $\partial B_{r(t)}$ and by slight abuse of notation $h'(\cdot,t) := (\tau_{\partial B_{r(t)}}\cdot\nabla^{\tan})h(\cdot,t)$ for the choice of tangent vector field $\tau_{\partial B_{r(t)}}\big(\bar\gamma(\theta,t)\big) = ie^{i\frac{\theta}{r(t)}}$. As we will show in Lemma 5, our error functional in this perturbative setting corresponds to leading order to

$$E_h(t) := \int_{\partial B_{r(t)}} \frac{1}{2}\frac{h^2(\cdot,t)}{r^2(t)} + \frac{1}{2}(h')^2(\cdot,t)\,d\mathcal{H}^1. \tag{25}$$

For the current purposes, we content ourselves with studying the stability of $E_h(t)$ near the extinction time.

To this end, we have to derive the PDE satisfied by the height function h (and its derivative). Dropping from now on for ease of notation the time dependence of all involved quantities, we first note that by definition in case of self-similarly shrinking circle

$$\partial_t\bar\gamma = \Big(\frac{1}{r}\mathrm{n}_{\partial B_r} + \lambda\tau_{\partial B_r}\Big)\circ\bar\gamma \quad \text{on } [0,2\pi r), \tag{26}$$

where λ denotes the tangential velocity. Second, we may then, on one side, directly compute based on the definition (24)

$$\partial_t\gamma_h = \bigg(\Big(\frac{1}{r} + \partial_t h + \lambda h'\Big)\mathrm{n}_{\partial B_r}\bigg)\circ\bar\gamma + \bigg(\lambda\Big(1-\frac{h}{r}\Big)\tau_{\partial B_r}\bigg)\circ\bar\gamma. \tag{27}$$

On the other side, since γ_h is assumed to evolve by mean curvature flow, it holds

$$H_{\gamma_h} = \partial_t\gamma_h\cdot\mathrm{n}_{\gamma_h} \quad \text{on } [0,2\pi r), \tag{28}$$

where the normal n_{γ_h} and mean curvature H_{γ_h} of the curve γ_h are given by the elementary formulas (with J denoting the counter-clockwise rotation by 90°)

$$\mathrm{n}_{\gamma_h} = J\frac{\partial_\theta\gamma_h}{|\partial_\theta\gamma_h|} = \left(\frac{\big(1-\frac{h}{r}\big)\mathrm{n}_{\partial B_r} - h'\tau_{\partial B_r}}{\sqrt{\big(1-\frac{h}{r}\big)^2 + (h')^2}}\right)\circ\bar\gamma \tag{29}$$

and

$$H_{\gamma_h} = \frac{\partial_{\theta\theta}\gamma_h}{|\partial_\theta\gamma_h|^2}\cdot\mathrm{n}_{\gamma_h} = \left(\frac{\big(1-\frac{h}{r}\big)\big(\frac{1}{r}+h''-\frac{h}{r^2}\big) + 2\frac{(h')^2}{r}}{\Big(\big(1-\frac{h}{r}\big)^2+(h')^2\Big)^{\frac{3}{2}}}\right)\circ\bar\gamma. \tag{30}$$

From (27)–(30), one may now deduce the non-linear PDE satisfied by the height function h. However, because in what follows we are only interested in identifying the leading-order behavior, we suppose from now on that the height function h instead satisfies the corresponding linearized equation:

$$\partial_t h = h'' + \frac{h}{r^2} \quad \text{on } \partial B_r. \tag{31}$$

From this, using $(\partial_t h)' = \partial_t h' + \frac{h}{r^2}$, we in particular deduce

$$\partial_t h' = h''' + 2\frac{h'}{r^2}. \tag{32}$$

Indeed, this follows easily from (31) and exploiting the change of variables $\widetilde{h}(\theta) := h(re^{i\theta})$ as a useful computational device.

Recalling (25), we thus get from the transport theorem as well as (31)–(32)

$$\begin{aligned}\frac{d}{dt}E_h &= \int_{\partial B_r} \partial_t\Big(\frac{1}{2}\frac{h^2}{r^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 - \int_{\partial B_r} H^2_{\partial B_r}\Big(\frac{1}{2}\frac{h^2}{r^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1 \\ &= \int_{\partial B_r}\frac{h}{r}\Big(2\frac{h}{r^3}+\frac{h''}{r}\Big)\,d\mathcal{H}^1 + \int_{\partial B_r} h'\Big(h'''+2\frac{h'}{r^2}\Big)\,d\mathcal{H}^1 \\ &\quad - \int_{\partial B_r}\frac{1}{r^2}\Big(\frac{1}{2}\frac{h^2}{r^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1.\end{aligned} \tag{33}$$

Integrating by parts and collecting similar terms therefore yields

$$\frac{d}{dt}E_h + \int_{\partial B_r}(h'')^2\,d\mathcal{H}^1 = \int_{\partial B_r}\frac{3}{2}\frac{h^2}{r^4}+\frac{1}{2}\frac{(h')^2}{r^2}\,d\mathcal{H}^1. \tag{34}$$

Fourier decomposing

$$[0,2\pi)\ni\theta\mapsto\widetilde{h}(\theta)=h(re^{i\theta})=a_0\frac{1}{\sqrt{2\pi}}\chi_{[0,2\pi]}+\sum_{k=1}^{\infty}a_k\frac{\cos(k\theta)}{\sqrt{\pi}}+b_k\frac{\sin(k\theta)}{\sqrt{\pi}}, \tag{35}$$

where we also recall the formulas for the associated Fourier coefficients

$$a_0=\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\widetilde{h}(\theta)\,d\theta,\quad a_k=\int_0^{2\pi}\widetilde{h}(\theta)\frac{\cos(k\theta)}{\sqrt{\pi}}\,d\theta,\quad b_k=\int_0^{2\pi}\frac{\sin(k\theta)}{\sqrt{\pi}}\,d\theta,$$

then rearranges (34) as

$$\frac{d}{dt}E_h+\frac{1}{r^3}\sum_{k=1}^{\infty}k^4\big(a_k^2+b_k^2\big)=\frac{1}{r^3}\frac{3}{2}a_0^2+\frac{1}{r^3}\sum_{k=1}^{\infty}\Big(\frac{3}{2}+\frac{1}{2}k^2\Big)\big(a_k^2+b_k^2\big). \tag{36}$$

Since $k^4-\frac{3}{2}-\frac{1}{2}k^2>0$ for $k\geq 2$, we infer that only the modes (a_0,a_1,b_1) are unstable near the extinction time (in the sense that these are precisely those inducing the borderline non-integrable singularity r^{-2} in the Gronwall estimate of E_h).

3.2. **Heuristics: Decay estimate.** Geometrically, the unstable modes correspond to time dilations and spatial translations. The basic idea of the present work is to correct these by dynamically adapting the smoothly evolving strong solution. In the simplified context of a self-similarly shrinking circle, this works heuristically as follows.

Consider $t_h>0$ (to be interpreted as an upper bound for the perturbed solution) as well as a smooth path $z\colon(0,t_h)\to\mathbb{R}^2$ of translations together with a smooth time diffeomorphism $T\colon(0,t_h)\to(0,\frac{1}{2}r_0^2)$, the latter to be thought of as a perturbation of the identity: $T(t)=:t+\mathfrak{T}(t)$ for $t\in(0,t_h)$. Based on this input, we then introduce the dynamically adapted solution

$$\bar{\gamma}^{z,T}(\theta,t):=\bar{\gamma}(\theta,T(t))+z(t),\quad \theta\in[0,2\pi r_T(t)),\ t\in(0,t_h), \tag{37}$$

where $r_T(t):=r(T(t))$, and assume that the perturbed solution γ_h is given by

$$\gamma_h(\cdot,t)=\big(\mathrm{id}+h(\cdot,t)\mathrm{n}_{\partial B_{r_T(t)}(z(t))}\big)\circ\bar{\gamma}^{z,T}(\cdot,t),\quad t\in(0,t_h), \tag{38}$$

where $|h(\cdot,t)|\ll r_T(t)$ and $|h'(\cdot,t)|\ll 1$. We are again interested in the stability properties of

$$E_h^{z,T}(t):=\int_{\partial B_{r_T(t)}(z(t))}\frac{1}{2}\frac{h^2(\cdot,t)}{r^2(t)}+\frac{1}{2}(h')^2(\cdot,t)\,d\mathcal{H}^1,\quad t\in(0,t_h). \tag{39}$$

In fact, we actually aim to identify ODEs for z and $\mathfrak{T}$ such that $E_h^{z,T}$ satisfies a quantitative decay estimate on $(0,t_h)$. One of course already expects the ODE for $\mathfrak{T}$ to involve the mode a_0, whereas the ODE for z is expected to be encoded in terms of (a_1,b_1). From now on, we again make use of the notational convention of suppressing the time dependence of all involved quantities. To this end, it will be convenient to associate to any map $f(\cdot,t)\colon \partial B_{r(t)}\to\mathbb{R}$ its time-rescaled version $f_T(\cdot,t)\colon \partial B_{r_T(t)}\to\mathbb{R}$ defined by $f_T(\cdot,t):=f(\cdot,T(t))$.

We start by computing the normal speed of $\partial B_{r_T}(z)$. By definition (37),

$$\partial_t\bar{\gamma}^{z,T} = (\partial_t\bar{\gamma})_T\big(1+\dot{\mathfrak{T}}\big)+\dot{z}. \tag{40}$$

Hence, the normal speed of $\partial B_{r_T}(z)$ in the direction of $\mathrm{n}_{\partial B_{r_T}(z)}$ is given by

$$V_{\partial B_{r_T}(z)} = \frac{1}{r_T}\big(1+\dot{\mathfrak{T}}\big)+\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot{z}. \tag{41}$$

The tangential speed in the direction of $\tau_{\partial B_{r_T}(z)}$ is furthermore given by

$$\lambda_{\partial B_{r_T}(z)} = \lambda_T\big(1+\dot{\mathfrak{T}}\big)+\tau_{\partial B_{r_T}(z)}\cdot\dot{z}, \tag{42}$$

where λ is the tangential velocity from (26). In particular, we may now compute

$$\begin{aligned}\partial_t\gamma_h &= \Big(\Big(V_{\partial B_{r_T}(z)}+\partial_t h+\lambda_{\partial B_{r_T}(z)}h'\Big)\mathrm{n}_{\partial B_{r_T}(z)}\Big)\circ\bar{\gamma}^{z,T}\\ &\quad+\Big(\Big(\lambda_{\partial B_{r_T}(z)}-\lambda_T\big(1+\dot{\mathfrak{T}}\big)\frac{h}{r_T}\Big)\tau_{\partial B_{r_T}(z)}\Big)\circ\bar{\gamma}^{z,T}.\end{aligned} \tag{43}$$

Furthermore, the analogous versions of the formulas (29)–(30) hold true:

$$\mathrm{n}_{\gamma_h} = \Bigg(\frac{\big(1-\frac{h}{r_T}\big)\mathrm{n}_{\partial B_{r_T}(z)}-h'\tau_{\partial B_{r_T}(z)}}{\sqrt{\big(1-\frac{h}{r_T}\big)^2+(h')^2}}\Bigg)\circ\bar{\gamma}^{z,T} \tag{44}$$

and

$$H_{\gamma_h} = \Bigg(\frac{\big(1-\frac{h}{r_T}\big)\big(\frac{1}{r_T}+h''-\frac{h}{r_T^2}\big)+2\frac{(h')^2}{r_T}}{\sqrt{\big(1-\frac{h}{r_T}\big)^2+(h')^2}^{\,3}}\Bigg)\circ\bar{\gamma}^{z,T}. \tag{45}$$

Combining the information provided by (41)–(44), we deduce

$$\begin{aligned}\partial_t\gamma_h\cdot\mathrm{n}_{\gamma_h} &= \Big(1-\frac{h}{r_T}\Big)\Big(V_{\partial B_{r_T}(z)}+\partial_t h+\lambda_{\partial B_{r_T}(z)}h'\Big)\\ &\quad- h'\Big(\Big(1-\frac{h}{r_T}\Big)\lambda_T\big(1+\dot{\mathfrak{T}}\big)+\tau_{\partial B_{r_T}(z)}\cdot\dot{z}\Big)\\ &= \Big(1-\frac{h}{r_T}\Big)\Big(\frac{1}{r_T}\big(1+\dot{\mathfrak{T}}\big)+\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot{z}+\partial_t h\Big)-\frac{h}{r_T}h'\tau_{\partial B_{r_T}(z)}\cdot\dot{z}.\end{aligned}$$

Turning as in Section 3.1 to the linearized PDE satisfied by the height function, we therefore obtain

$$\partial_t h = h''+\frac{h}{r_T^2}-\frac{\dot{\mathfrak{T}}}{r_T}-\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot{z} \tag{46}$$

as well as

$$(\partial_t h') = h'''+(2+\dot{\mathfrak{T}})\frac{h'}{r_T^2}+\frac{1}{r_T}\tau_{\partial B_{r_T}(z)}\cdot\dot{z}. \tag{47}$$

We may now finally compute based on the transport theorem, the definition (39), and the formulas (41) as well as (46)–(47)

$$\begin{aligned}\frac{d}{dt}E_h^{z,T} &= \int_{\partial B_{r_T}(z)} \partial_t\Big(\frac{1}{2}\frac{h^2}{r_T^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1\\&\quad-\int_{\partial B_{r_T}(z)} H_{\partial B_{r_T}(z)}V_{\partial B_{r_T}(z)}\Big(\frac{1}{2}\frac{h^2}{r_T^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1\\&=\int_{\partial B_{r_T}(z)} \frac{h}{r_T}\Big(\frac{1+\dot{\mathfrak{T}}}{r_T^3}h+\frac{1}{r_T}\Big(h''+\frac{h}{r_T^2}-\frac{\dot{\mathfrak{T}}}{r_T}-\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot z\Big)\Big)\,d\mathcal{H}^1\\&\quad+\int_{\partial B_{r_T}(z)} h'\Big(h'''+(2+\dot{\mathfrak{T}})\frac{h'}{r_T^2}+\frac{1}{r_T}\tau_{\partial B_{r_T}(z)}\cdot\dot z\Big)\,d\mathcal{H}^1\\&\quad-\int_{\partial B_{r_T}(z)}\frac{1}{r_T^2}\Big(\frac{1}{2}\frac{h^2}{r_T^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1\\&\quad-\int_{\partial B_{r_T}(z)} H_{\partial B_{r_T}(z)}\big(V_{\partial B_{r_T}(z)}-H_{\partial B_{r_T}(z)}\big)\Big(\frac{1}{2}\frac{h^2}{r_T^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1.\end{aligned}$$

Hence, integrating by parts and collecting again similar terms yields

$$\begin{aligned}\frac{d}{dt}E_h^{z,T} &= \int_{\partial B_{r_T}(z)}\frac{3}{2}\frac{h^2}{r_T^4}\,d\mathcal{H}^1-\int_{\partial B_{r_T}(z)}\frac{h}{r_T^3}\dot{\mathfrak{T}}\,d\mathcal{H}^1\\&\quad+\int_{\partial B_{r_T}(z)}\frac{1}{2}\frac{(h')^2}{r_T^2}\,d\mathcal{H}^1-\int_{\partial B_{r_T}(z)}2\frac{h}{r_T^2}\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot z\,d\mathcal{H}^1\\&\quad-\int_{\partial B_{r_T}(z)}(h'')^2\,d\mathcal{H}^1\\&\quad+R_{h.o.t.},\end{aligned}\tag{48}$$

where

$$R_{h.o.t.} := \int_{\partial B_{r_T}(z)}\frac{1}{r_T}\Big(\frac{\dot{\mathfrak{T}}}{r_T}-\mathrm{n}_{\partial B_{r_T}(z)}\cdot\dot z\Big)\Big(\frac{1}{2}\frac{h^2}{r_T^2}+\frac{1}{2}(h')^2\Big)\,d\mathcal{H}^1.\tag{49}$$

Based on the Fourier decomposition (35), the identity (48) now motivates to define

$$\dot{\mathfrak{T}} = \frac{c_T}{r_T}\fint_0^{2\pi}\widetilde h\,d\theta,\quad \dot z = \frac{c_z}{r_T^2}\fint_0^{2\pi}\widetilde h(-e^{i\theta})\,d\theta,\tag{50}$$

where the constants (c_T, c_z) are yet to be chosen. Indeed, with these choices we get

$$\begin{aligned}\frac{d}{dt}E_h^{z,T}+\frac{(c_T-3/2)}{r_T^2}\frac{a_0^2}{r_T}+\frac{(c_z-1)}{r_T^2}\frac{a_1^2+b_1^2}{r_T}&\\+\frac{1}{r_T^2}\sum_{k=2}^{\infty}\Big(k^4-\frac{3}{2}-\frac{1}{2}k^2\Big)\frac{a_k^2+b_k^2}{r_T}&=R_{h.o.t.},\end{aligned}\tag{51}$$

where, due to $|\dot{\mathfrak{T}}|\le c_T\frac{1}{r_T}\|h\|_{L^\infty(\partial B_{r_T}(z))}$ and $|\dot z|\le c_z\frac{1}{r_T^2}\|h\|_{L^\infty(\partial B_{r_T}(z))}$, one has an estimate for the remainder term in the form of

$$\Big|R_{h.o.t.}\Big|\le\Big(c_T+c_z\Big)\frac{\|h\|_{L^\infty(\partial B_{r_T}(z))}}{r_T}\frac{1}{r_T^3}\Big(\frac{1}{2}a_0^2+\sum_{k=1}^{\infty}\frac{1}{2}(1+k^2)(a_k^2+b_k^2)\Big).\tag{52}$$

Hence, for given $\widetilde{\delta} \in (0,1)$, if $|h| \ll_{\widetilde{\delta},c_T,c_z} r_T$, one gets an upgrade of (48) in the form of

$$\begin{aligned}\frac{d}{dt}E_h^{z,T} + \frac{c_T - 3/2(1+\widetilde{\delta})}{r_T^2}\frac{a_0^2}{r_T} + \frac{c_z - (1+\widetilde{\delta})}{r_T^2}\frac{a_1^2+b_1^2}{r_T} \\ + \frac{1}{r_T^2}\sum_{k=2}^{\infty}\Big(k^4 - (1+\widetilde{\delta})\Big(\frac{3}{2}+\frac{1}{2}k^2\Big)\Big)\frac{a_k^2+b_k^2}{r_T} \leq 0.\end{aligned} \tag{53}$$

Because of

$$E_h^{z,T} = \frac{1}{2}\frac{a_0^2}{r_T} + \sum_{k=1}^{\infty}\frac{1}{2}(1+k^2)\frac{a_k^2+b_k^2}{r_T}, \tag{54}$$

we deduce that for any constant $\alpha > 1$ satisfying

$$\alpha \leq \min\{2c_T - 3(1+\widetilde{\delta}), c_z - (1+\widetilde{\delta})\}, \tag{55}$$

$$\alpha\frac{1}{2}(1+k^2) \leq k^4 - (1+\widetilde{\delta})\Big(\frac{3}{2}+\frac{1}{2}k^2\Big), \qquad k \geq 2, \tag{56}$$

it holds

$$\frac{d}{dt}E_h^{z,T} + \frac{\alpha}{r_T^2}E_h^{z,T} \leq 0. \tag{57}$$

Choosing $c_T := 4$ and $c_z := 6$, optimizing shows that for any desired exponent $\alpha \in (1,5)$ there exists a choice of the constant $\widetilde{\delta}$ such that (57) holds true (in the perturbative regime $|h| \ll_{c_T,c_z,\widetilde{\delta}} 1$ with linearized evolution law (46)). Indeed, the function $f\colon [2,\infty) \to [0,\infty)$, $x \mapsto \big(\frac{1}{2}(1+x^2)\big)^{-1}\big(x^4 - (\frac{3}{2}+\frac{1}{2}x^2)\big)$ is monotonically increasing and satisfies $f(2) = 5$.

Finally, since $|\dot{\mathfrak{T}}| \leq c_T\frac{1}{r_T}\|h\|_{L^\infty(\partial B_{r_T}(z))}$, one may choose for any $\alpha \in (1,5)$ the constant $\widetilde{\delta}$ such that (in the perturbative regime $|h| \ll_{c_T,c_z,\widetilde{\delta}} 1$ with linearized evolution law (46)) it even holds $\frac{d}{dt}E_h^{z,T} + (1+\dot{\mathfrak{T}})\frac{\alpha}{r_T^2}E_h^{z,T} \leq 0$, so that $\frac{d}{dt}r_T^\alpha = -\alpha r_T^{\alpha-1}\frac{1}{r_T} = -(1+\dot{\mathfrak{T}})\frac{\alpha}{r_T^2}r_T^\alpha$ implies

$$E_h^{z,T}(t) \leq E_h^{z,T}(0)\Big(\frac{r_T(t)}{r_0}\Big)^\alpha, \quad t \in (0,t_h). \tag{58}$$

This is precisely the type of decay estimate (or, weak-strong stability estimate up to shift) claimed in our main result, Theorem 1.

Before we turn in the upcoming subsections to a description of the key ingredients and steps for our proof of Theorem 1 (with the above considerations, of course, being their main motivation), let us provide some final remarks on the main assumptions behind the derivation of the decay estimate (58).

First, one may derive a version of (48) also in the case where the time-evolving curve $\bar{\gamma}$ is not parametrizing a perfect circle. The main difference in this case is that the coefficients are not anymore simply constant along $\bar{\gamma}$ (i.e., not proportional to inverse powers of r_T). It is precisely at this stage where we exploit our notion of quantitative closeness of the strong solution to a circular solution, cf. Definition 4, allowing us to effectively reduce the situation to the constant-coefficient computation (48) (i.e., in PDE jargon, we perform nothing else than a global freezing of coefficients).

A second simplifying assumption was the usage of the linearized evolution law (46) for the height function h as well as that we only considered the stability of the leading order contribution E_h to our actual error functional. It will turn out that these linearization errors are harmless and only impact the final stability estimate qualitatively in the same manner as the term $R_{h.o.t.}$ from (48).

Needless to say, in the general setting of Theorem 1 where we aim for quantitative stability beyond circular topology change even for the broader class of weak (i.e., varifold-BV) solutions, we can not rely on the above considerations (e.g., transport theorem, derivation of the (linearized) evolution law (46)) in order to rigorously derive the evolution of the error functional. In order to still unravel the structure of the right hand side of (48), we instead make use of the recently introduced notions of gradient flow calibrations and relative entropies for multiphase mean curvature flow from [9], serving as a robust replacement of the above considerations to the weak setting.

Last but not least, one of course also needs an independent argument ensuring that one can reduce the whole estimation strategy to a perturbative graph setting as above. This, however, is precisely one of the key points of the upcoming subsections.

3.3. **A general stability estimate for multiphase MCF.** Starting point of our strategy is a stability estimate, see Lemma 2 below, which one may essentially directly infer from the combination of [9, Proposition 17] and [9, Lemma 20] (or more precisely, their proofs), together with the following compatibility properties of the varifold $\mathcal{V}_t$ and the indicator functions χ_i. From Definition 3 one may infer that, for each $i \in \{1, ..., P\}$ and a.e. $t \in (0, \infty)$, the Radon–Nikodým derivatives

$$\omega_i(\cdot,t) := \frac{\mathrm{d}|\nabla\chi_i(\cdot,t)|}{\mathrm{d}\mu_t} \in [0,1], \quad \omega(\cdot,t) := \frac{\mathrm{d}\frac{1}{2}\sum_{i=1}^{P}|\nabla\chi_i(\cdot,t)|}{\mathrm{d}\mu_t} \in [0,1] \tag{59}$$

exist. Note that $\omega = \frac{1}{2}\sum_{i=1}^{P}\omega_i$. Since $\mathcal{V}$ a family of integral varifolds, we have that $\omega \in \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and

$$\mu_t \llcorner \Big\{\frac{1}{2}\sum_{i=1}^{P}\omega_i(\cdot,t) = 1\Big\} = \mathcal{H}^{d-1} \llcorner \Big(\Big\{\frac{1}{2}\sum_{i=1}^{P}\omega_i(\cdot,t) = 1\Big\} \cap \bigcup_{i\neq j} I_{i,j}(t)\Big)$$

and, since $\mathcal{V}_t$ is rectifiable for a.e. $t \in (0, \infty)$, it follows that

$$\begin{aligned} &\mathcal{V}_t \llcorner \Big\{\frac{1}{2}\sum_{i=1}^{P}\omega_i(\cdot,t) = 1\Big\} \\ &= \frac{1}{2}\sum_{i=1}^{P}\Big(\operatorname{supp}|\nabla\chi_i(\cdot,t)| \llcorner \Big\{\frac{1}{2}\sum_{i=1}^{P}\omega_i(\cdot,t) = 1\Big\} \otimes \big(\delta_{\mathrm{Tan}_x^{d-1}(\operatorname{supp}|\nabla\chi_i(\cdot,t)|)}\big)_{x\in\operatorname{supp}|\nabla\chi_i|}\Big) \end{aligned}$$

for a.e. $t \in (0, \infty)$. In particular, from Brakke's perpendicularity theorem it follows that for a.e. $t \in (0, \infty)$

$$H_\mu(\cdot,t) = \Big(H_\mu(\cdot,t)\cdot\frac{\nabla\chi_i(\cdot,t)}{|\nabla\chi_i(\cdot,t)|}\Big)\frac{\nabla\chi_i(\cdot,t)}{|\nabla\chi_i(\cdot,t)|}$$

$\mathcal{H}^{d-1}$-a.e. on $\{\frac{1}{2}\sum_{i=1}^{P}\omega_i(\cdot,t) = 1\} \cap \operatorname{supp}|\nabla\chi_i(\cdot,t)|$ for all $i \in \{1, ..., P\}$. We define

$$V_i := -H_\mu(\cdot,t)\cdot\frac{\nabla\chi_i(\cdot,t)}{|\nabla\chi_i(\cdot,t)|}$$

and $V_{i,j} := V_i = -V_j$ for all $i, j \in \{1, ..., P\}$, $i \neq j$.

Lemma 2 (Preliminary stability estimate)**.** *Let $((\xi_i)_{i=1,\ldots,P}, (\vartheta_i)_{i=1,\ldots,P-1}, B)$—to be thought of as being constructed from $\bar\chi$—such that*

$$\xi_i \in W^{1,\infty}_{loc}\big([0,T_{ext}); W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)\big) \cap L^\infty_{loc}\big([0,T_{ext}); W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)\big),$$
$$\vartheta_i \in W^{1,1}_{loc}\big([0,T_{ext}); L^1(\mathbb{R}^2)\big) \cap L^1_{loc}\big([0,T_{ext}); (W^{1,1}\cap W^{1,\infty})(\mathbb{R}^2)\big),$$
$$B \in L^\infty_{loc}\big([0,T_{ext}); W^{2,\infty}(\mathbb{R}^2;\mathbb{R}^2)\big),$$

where $(\vartheta_i)_{i=1,\ldots,P-1}$ is supposed to satisfy, for all $t \in (0,T_{ext})$,

$$\begin{aligned} &\vartheta_1(\cdot,t) < 0 && \text{in the interior of } \{\bar\chi_1(\cdot,t)=1\},\\ &\vartheta_1(\cdot,t) > 0 && \text{in the exterior of } \overline{\{\bar\chi_1(\cdot,t)=1\}},\\ &\vartheta_i(\cdot,t) = 1 && \text{throughout } \mathbb{R}^2 \text{ for } i \in \{1,\ldots,P-1\}. \end{aligned}$$

Define $\xi_{i,j} := \xi_i - \xi_j$ for all distinct $i,j \in \{1,\ldots,P\}$. Consider in addition a triple (t_χ, z, T) so that $t_\chi \in (0,\infty)$, $z \in W^{1,\infty}_{loc}((0,t_\chi);\mathbb{R}^2)$ and $T \in W^{1,\infty}((0,t_\chi);(0,T_{ext}))$, and define for all $t \in (0,t_\chi)$

$$E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) := \mu_t(\mathbb{R}^d) - \sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\int_{I_{i,j}(t)} \mathrm{n}_{i,j}(\cdot,t)\cdot\xi^{z,T}_{i,j}(\cdot,t)\,d\mathcal{H}^1, \tag{60}$$

$$E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](t) := \sum_{i=1}^{P-1}\int_{\mathbb{R}^2} |\chi_i(\cdot,t) - \bar\chi^{z,T}_i(\cdot,t)||\vartheta^{z,T}_i(\cdot,t)|\,dx, \tag{61}$$

$$E[\mathcal{V},\chi|\bar\chi^{z,T}](t) := E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) + E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](t). \tag{62}$$

Then, for all $[s,\tau] \subset [0,t_\chi)$ and all $\psi \in C^1_{cpt}([0,t_\chi);[0,\infty))$, it holds

$$\begin{aligned} &\psi(\tau)E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](\tau) + \int_s^\tau \psi(t)\sum_{i,j=1,i\neq j}^{P}\frac{1}{2}\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t)\,dt\\ &\quad\le \psi(s)E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](s) + \int_s^\tau \psi(t) RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t)\,dt\\ &\qquad + \int_s^\tau \psi(t)\sum_{i,j=1,i\neq j}^{P}\frac{1}{2}RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\,dt\\ &\qquad + \int_s^\tau \Big(\frac{d}{dt}\psi(t)\Big)E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](t)\,dt, \end{aligned} \tag{63}$$

as well as

$$\begin{aligned} &\psi(\tau)E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](\tau)\\ &\quad = \psi(s)E_{\mathrm{bulk}}[\chi|\bar\chi^{z,T}](s) + \int_s^\tau \psi(t)\sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}](t)\,dt\\ &\qquad + \int_s^\tau \Big(\frac{d}{dt}\psi(t)\Big)E_{\mathrm{bulk}}[\mathcal{V},\chi|\bar\chi^{z,T}](t)\,dt, \end{aligned} \tag{64}$$

where the individual terms are given by

$$\begin{aligned} \mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) := &\int_{I_{i,j}(t)} \frac{1}{2}\big|V_{i,j} + \nabla\cdot\xi^{z,T}_{i,j}\big|^2(\cdot,t)\,d\mathcal{H}^1\\ &+ \int_{I_{i,j}(t)} \frac{1}{2}\big|V_{i,j}\mathrm{n}_{i,j} - (B^{z,T}\cdot\xi^{z,T}_{i,j})\xi^{z,T}_{i,j}\big|^2(\cdot,t)\,d\mathcal{H}^1, \end{aligned}$$

$$
\begin{aligned}
&RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar{\chi}^{z,T}](t)\\
&:= -\int_{\mathbb{R}^d}|H_\mu|^2\Big(1-\frac{1}{2}\sum_{i=1}^P\rho_i\Big)\,d\mu_t+\int_{\mathbb{R}^d}H_\mu\cdot B\Big(1-\frac{1}{2}\sum_{i=1}^P\rho_i\Big)\,d\mu_t\\
&\quad-\sum_{i,j=1,i\neq j}^P\frac{1}{2}\int_{I_{i,j}}\big(\operatorname{Id}-\mathrm{n}_{i,j}\otimes\mathrm{n}_{i,j}\big):\nabla B\,d\mathcal{H}^1+\int_{\mathbb{R}^d\times\mathbf{G}(d,d-1)}\operatorname{Id}_{\mathbf{G}(d,d-1)}:\nabla B\,d\mathcal{V}_t,
\end{aligned}
$$

and

$$
\begin{aligned}
&RHS^{\text{int}}_{i,j}[\chi|\bar{\chi}^{z,T}](t)\\
&:= -\int_{I_{i,j}(t)}\big(\partial_t\xi^{z,T}_{i,j}+(B^{z,T}\cdot\nabla)\xi^{z,T}_{i,j}+(\nabla B^{z,T})^{\mathsf{T}}\xi^{z,T}_{i,j}\big)(\cdot,t)\cdot(\mathrm{n}_{i,j}-\xi^{z,T}_{i,j})(\cdot,t)\,d\mathcal{H}^1\\
&\quad-\int_{I_{i,j}(t)}\big(\partial_t\xi^{z,T}_{i,j}+(B^{z,T}\cdot\nabla)\xi^{z,T}_{i,j}\big)(\cdot,t)\cdot\xi^{z,T}_{i,j}(\cdot,t)\,d\mathcal{H}^1\\
&\quad+\int_{I_{i,j}(t)}\frac{1}{2}\big|\nabla\cdot\xi^{z,T}_{i,j}+B^{z,T}\cdot\xi^{z,T}_{i,j}\big|^2(\cdot,t)\,d\mathcal{H}^1\\
&\quad-\int_{I_{i,j}(t)}\frac{1}{2}\big|B^{z,T}\cdot\xi^{z,T}_{i,j}\big|(\cdot,t)\big(1-|\xi^{z,T}_{i,j}|^2\big)(\cdot,t)\,d\mathcal{H}^1\\
&\quad-\int_{I_{i,j}(t)}(1-\mathrm{n}_{i,j}\cdot\xi^{z,T}_{i,j})(\cdot,t)\nabla\cdot\xi^{z,T}_{i,j}(\cdot,t)(B^{z,T}\cdot\xi^{z,T}_{i,j})(\cdot,t)\,d\mathcal{H}^1\\
&\quad+\int_{I_{i,j}(t)}\big((\operatorname{Id}-\xi^{z,T}_{i,j}\otimes\xi^{z,T}_{i,j})B^{z,T}\big)(\cdot,t)\cdot\big((V_{i,j}+\nabla\cdot\xi^{z,T}_{i,j})\mathrm{n}_{i,j}\big)(\cdot,t)\,d\mathcal{H}^1\\
&\quad+\int_{I_{i,j}(t)}(1-\mathrm{n}_{i,j}\cdot\xi^{z,T}_{i,j})(\cdot,t)\nabla\cdot B^{z,T}(\cdot,t)\,d\mathcal{H}^1\\
&\quad-\int_{I_{i,j}(t)}(\mathrm{n}_{i,j}-\xi^{z,T}_{i,j})(\cdot,t)\otimes(\mathrm{n}_{i,j}-\xi^{z,T}_{i,j})(\cdot,t):\nabla B^{z,T}(\cdot,t)\,d\mathcal{H}^1,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&RHS^{\text{bulk}}_{i}[\chi|\bar{\chi}^{z,T}](t)\\
&:= -\sum_{j=1,j\neq i}^P\int_{I_{i,j}(t)}\vartheta^{z,T}_i(\cdot,t)(B^{z,T}\cdot\xi^{z,T}_{i,j}-V_{i,j})(\cdot,t)\,d\mathcal{H}^1\\
&\quad-\sum_{j=1,j\neq i}^P\int_{I_{i,j}(t)}\vartheta^{z,T}_i(\cdot,t)B^{z,T}(\cdot,t)\cdot(\mathrm{n}_{i,j}-\xi^{z,T}_{i,j})(\cdot,t)\,d\mathcal{H}^1\\
&\quad+\int_{\mathbb{R}^2}(\chi_i-\bar{\chi}^{z,T}_i)(\cdot,t)\vartheta^{z,T}_i(\cdot,t)\nabla\cdot B^{z,T}(\cdot,t)\,dx\\
&\quad+\int_{\mathbb{R}^2}(\chi_i-\bar{\chi}^{z,T}_i)(\cdot,t)\big(\partial_t\vartheta^{z,T}_i+(B^{z,T}\cdot\nabla)\vartheta^{z,T}_i\big)(\cdot,t)\,dx.
\end{aligned}
$$

One may rewrite (60) as (cf. [9, Section 4.4])

$$
E_{\text{int}}[\mathcal{V},\chi|\bar{\chi}^{z,T}](t)=\int_{\mathbb{R}^d}1-\frac{1}{2}\sum_{i=1}^P\omega_i(\cdot,t)\,\mathrm{d}\mu_t+E_{\text{int}}[\chi|\bar{\chi}^{z,T}](t),\tag{65}
$$

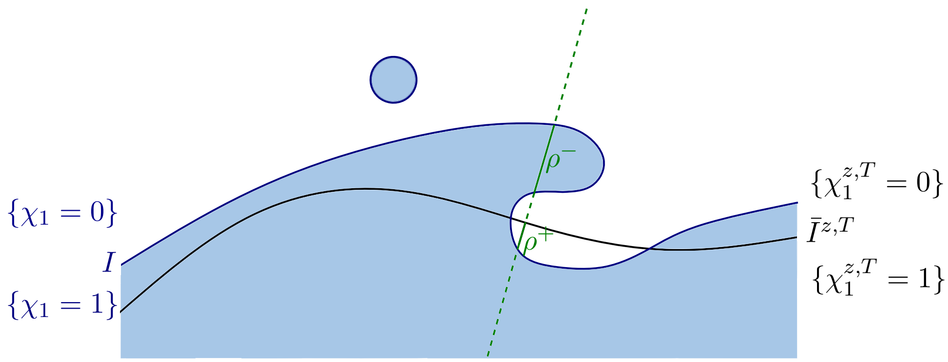

FIGURE 3. Interface error heights

where $E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}]$ is the interface error for BV solutions in the sense of Definition 2, namely

$$E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](t) = \sum_{i,j=1, i\neq j}^{P} \frac{1}{2}\int_{I_{i,j}(t)} 1 - \mathrm{n}_{i,j}(\cdot,t)\cdot \xi_{i,j}^{z,T}(\cdot,t)\, d\mathcal{H}^1.$$

Observe that the first term on the right hand side of (65) is nonnegative by (14) and provides control of the multiplicity of the varifold whenever it exceeds the multiplicity of the BV interface $\frac{1}{2}\sum_{i=1}^{P}|\nabla\chi_i(\cdot,t)|$. In particular, the varifold-BV interface error (60) controls the interface error for BV solutions.

The next two steps of our strategy are concerned with the construction of the input data for Lemma 2: first (t_χ, z, T) and second $((\xi_i)_{i=1,\ldots,P}, (\vartheta_i)_{i=1,\ldots,P-1}, B)$.

3.4. **Construction of dynamic shifts.** In case of a single closed curve, a characteristic length scale associated with the evolution of $\bar{\chi}$ is given by $r(t) := \sqrt{\frac{\mathrm{vol}(\{\bar{\chi}_1(\cdot,t)=1\})}{\pi}}$, $t\in[0,T_{ext})$. Since

$$\frac{d}{dt}\mathrm{vol}(\{\bar{\chi}(\cdot,t){=}1\}) = -2\pi$$

we infer that

$$\begin{cases} \dot{r}(t) = -\frac{1}{r(t)}, & t\in(0,T_{ext}),\\ r(0) = r_0 := \sqrt{\frac{\mathrm{vol}(\{\bar{\chi}_1(\cdot,0)=1\})}{\pi}}, \end{cases} \tag{66}$$

Hence, $r(t) = \sqrt{2(T_{ext}-t)}$ and $T_{ext} = \frac{1}{2}r_0^2$.

In Subsection 3.2, we already derived the defining ODEs for $(z, T = \mathrm{id}{+}\mathfrak{T})$, at least in a regime where the weak solution is represented as a sufficiently regular graph over the smooth solution, cf. (50). Of course, there is no guarantee to be in that regime for all times, so that the general construction needs a robust version of (50). To this end, it is convenient to work with the notion of interface error heights (see Figure 3).

Construction 1 (Interface error heights)**.** *Consider a triple (t_χ, z, T) so that $t_\chi\in(0,\infty)$, $z\in W^{1,\infty}_{loc}((0,t_\chi);\mathbb{R}^2)$ and $T\in W^{1,\infty}((0,t_\chi);(0,T_{ext}))$. Let $\zeta\colon\mathbb{R}\to[0,1]$ be a smooth cutoff function such that $\zeta(s)=1$ for $|s|\le 1/(16C_\zeta)$ and $\zeta(s)=0$*

for $|s| > 1/(8C_\zeta)$*, where* $C_\zeta \in [1,\infty)$ *is a given constant. We then define* interface error heights

$$\rho(\cdot,\cdot;z,T), \rho_\pm(\cdot,\cdot;z,T) \colon \bar{I}^{z,T} \to \mathbb{R}$$

through a slicing construction (recall that $r_T(t) := r(T(t))$, $t \in (0,t_\chi)$*):*

$$\rho_+(x,t;z,T) := \int_0^{\frac{1}{2}r_T(t)} (\bar{\chi}_1^{z,T} - \chi_1)\big(x+\ell \mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\big)\zeta\Big(\frac{\ell}{r_T(t)}\Big)\,d\ell, \tag{67}$$

$$\rho_-(x,t;z,T) := \int_{-\frac{1}{2}r_T(t)}^{0} (\chi_1 - \bar{\chi}_1^{z,T})\big(x+\ell \mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\big)\zeta\Big(\frac{\ell}{r_T(t)}\Big)\,d\ell, \tag{68}$$

$$\rho(x,t;z,T) := \rho_+(x,t;z,T) - \rho_-(x,t;z,T). \tag{69}$$

We have everything in place to construct the dynamic shifts.

Lemma 3 (Existence of space-time shifts)**.** *There exists a unique choice of*

- *a time horizon* $t_\chi > 0$*,*
- *a path of translations* $z \in W^{1,\infty}_{loc}((0,t_\chi);\mathbb{R}^2)$*, and*
- *a strictly increasing bijection* $T \in W^{1,\infty}((0,t_\chi);(0,T_{ext}))$*,*

which in addition satisfy $(z(0),T(0)) = (0,0)$ *as well as, by defining* $\mathfrak{T} := T - \mathrm{id}$*,*

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathfrak{T}}(t) \end{bmatrix} = \begin{bmatrix} \frac{6}{r_T^2(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\,d\mathcal{H}^1 \\ \frac{4}{r_T(t)} \fint_{\bar{I}^{z,T}(t)} \rho(\cdot,t;z,T)\,d\mathcal{H}^1 \end{bmatrix}, \quad t \in (0,t_\chi). \tag{70}$$

Moreover, for given $\delta_{\mathrm{err}} \in (0,\frac{1}{2})$ *one may choose the constant* $C_\zeta \gg_{\delta_{\mathrm{err}}} 1$ *from Construction 1 such that*

$$|\dot{z}(t)| \leq \delta_{\mathrm{err}} \frac{1}{r_T(t)}, \quad |\dot{\mathfrak{T}}(t)| \leq \delta_{\mathrm{err}}, \quad t \in (0,t_\chi). \tag{71}$$

The proof of Lemma 3 is given in Section 6.1.

3.5. **Construction of gradient flow calibrations.** In contrast to [9], in the present work the smoothly evolving solution $\bar{\chi}$ stems from a simple two-phase geometry instead of a more complicated multiphase geometry with branching interfaces. As a consequence, the construction of a gradient flow calibration (cf. [9, Definition 2 and Definition 4]) is particularly simple and can be given directly as follows.

Construction 2 (Gradient flow calibration up to extinction time)**.** *Consider a smooth cutoff function* $\eta \colon \mathbb{R} \to [0,1]$ *such that* $\eta(s) = 1$ *for* $|s| \leq 1/8$*,* $\eta(s) = 0$ *for* $|s| \geq 1/4$ *and* $\|\eta'\|_{L^\infty(\mathbb{R})} \leq 16$*. We then define an extension* $\xi \colon \mathbb{R}^2{\times}[0,T_{ext}) \to \mathbb{R}^2$ *of the unit vector field* $\mathrm{n}_{\bar{I}}$ *by means of*

$$\xi(x,t) = \eta\Big(\frac{\mathrm{sdist}_{\bar{I}}(x,t)}{r(t)}\Big)\mathrm{n}_{\bar{I}}\big(P_{\bar{I}}(x,t),t\big), \quad (x,t) \in \mathbb{R}^2{\times}[0,T_{ext}). \tag{72}$$

Based on this auxiliary construction, we may now introduce families of vector fields $(\xi_i)_{i=1,\dots,P}$ *and* $(\xi_{i,j})_{i,j\in\{1,\dots,P\},i\neq j}$ *(defined as maps* $\mathbb{R}^2{\times}[0,T_{ext}) \to \mathbb{R}^2$*) by the following simple procedure:*

- $\xi_{i,j} := \xi_i - \xi_j$ *for any* $i,j \in \{1,\dots,P\}$*,* $i \neq j$*.*
- $\xi_i \equiv 0$ *for all* $i \notin \{1,P\}$*.*
- $\xi_1 := -\frac{1}{2}\xi$ *and* $\xi_P := \frac{1}{2}\xi$*.*

Furthermore, we define an extension $B\colon \mathbb{R}^2\times[0,T_{ext}) \to \mathbb{R}^2$ *of the normal velocity field* $H_{\bar I}\mathrm{n}_{\bar I}$ *of* $\bar\chi$ *through*

$$B(x,t) := \eta\Big(\frac{\mathrm{sdist}_{\bar I}(x,t)}{r(t)}\Big)(H_{\bar I}\mathrm{n}_{\bar I})\big(P_{\bar I}(x,t),t\big), \quad (x,t)\in\mathbb{R}^2\times[0,T_{ext}). \tag{73}$$

Finally, for the construction of the family $(\vartheta_i)_{i=1,\ldots,P-1}$ *(defined as functions mapping* $\mathbb{R}^2\times[0,T_{ext}) \to [-1,1]$*), we proceed as follows. Let* $\bar\vartheta\colon\mathbb{R}\to[-1,1]$ *be a smooth function such that* $\bar\vartheta(s) = -s$ *for* $|s|\le 1/4$, $\bar\vartheta(s) = -1$ *for* $s\ge 1/2$, $\bar\vartheta(s)=1$ *for* $s\le -1/2$, *and* $\|\bar\vartheta'\|_{L^\infty(\mathbb{R})}\le 4$. *We then define*

$$\vartheta(x,t) := \frac{1}{r(t)}\bar\vartheta\Big(\frac{\mathrm{sdist}_{\bar I}(x,t)}{r(t)}\Big), \quad (x,t)\in\mathbb{R}^2\times[0,T_{ext}), \tag{74}$$

and, at last,

- $\vartheta_1 := \vartheta$,
- $\vartheta_i := 1$ *for all* $i\in\{2,\ldots,P-1\}$.

Note that the gradient flow calibration $((\xi_i)_{i=1,\ldots,P},(\vartheta_i)_{i=1,\ldots,P-1},B)$ from Construction 2 is an admissible input for Lemma 2. From now on, whenever we refer to an admissible element from either (t_χ,z,T) or $((\xi_i)_{i=1,\ldots,P},(\vartheta_i)_{i=1,\ldots,P-1},B)$, we always mean their specific realizations provided by Lemma 3 or Construction 2, respectively.

3.6. **Time splitting: Regular vs. non-regular times.** With the input for Lemma 2 being constructed, the main remaining major task is to upgrade the preliminary stability estimates (63) and (64) to the decay estimate (5) for the overall error. The main idea here is to reduce the whole estimation strategy to a regime where the weak solution χ is only a small perturbation of $\bar\chi$, for which we in turn already formally identified the leading-order contributions to the stability estimates in Subsections 3.1–3.2.

Definition 5 (Regular and non-regular times)**.** Fix $\Lambda > 0$. We then define a disjoint decomposition

$$(0,t_\chi) = \mathcal{T}_{\text{non-reg}}(\Lambda)\cup\mathcal{T}_{\text{reg}}(\Lambda)$$

where

$$\mathcal{T}_{\text{non-reg}}(\Lambda) := \left\{t\in(0,t_\chi) : \int_{\mathbb{R}^d}|H_\mu(\cdot,t)|^2\,d\mu_t \ge \Lambda\frac{2\pi}{r_T(t)}\right\} \tag{75}$$

and $H_\mu(\cdot,t)$ is the generalized mean curvature vector of $\mathcal{V}_t$, see (12).

Observe that in the framework of BV solutions in the sense of Definition 2, the defining inquality in (75) reduces to

$$\sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}(t)}\frac12|V_{i,j}(\cdot,t)|^2\,d\mathcal{H}^1 \ge \Lambda\frac{2\pi}{r_T(t)}.$$

The motivation behind the previous definition is as follows. On one side, for non-regular times, the right hand sides of the preliminary stability estimates (63) and (64) turn out to be easily estimated thanks to the defining condition of disproportionally large dissipation of the weak solution, cf. (75). On the other side, the opposite of (75) together with a smallness assumption on the overall error (consistent with the decay (5)) imply for regular times the desired perturbative setting. The latter is formalized in the following result.

Proposition 4 (Perturbative regime at regular times)**.** *Fix* $\Lambda > 0$ *and let* $t \in \mathcal{T}_{\mathrm{reg}}(\Lambda)$, *i.e.,* $t \in (0, t_\chi)$ *such that*

$$\int_{\mathbb{R}^d} |H_\mu(\cdot,t)|^2 \, d\mu_t < \Lambda \frac{2\pi}{r_T(t)}. \tag{76}$$

Given $C_\zeta \geq 1$ *from Construction 1 and given any* $C, C' \geq 1$, *there exists a constant* $\delta \ll_{\Lambda,C,C',C_\zeta} \frac{1}{2}$ *such that*

$$E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t) \tag{77}$$

implies:

- $\chi_i(\cdot,t) \equiv 0$ *for all* $i \notin \{1,P\}$ *and*
$$\mathcal{V}_t = \big(\mathcal{H}^1 \llcorner \operatorname{supp} |\nabla\chi_1(\cdot,t)|\big) \otimes \big(\delta_{\mathrm{Tan}_x(\operatorname{supp}|\nabla\chi_1(\cdot,t)|)}\big)_{x\in\operatorname{supp}|\nabla\chi_1(\cdot,t)|}.$$
- *There exists a* height function
$$h(\cdot,t) \in H^2(\bar{I}^{z,T}(t)) \tag{78}$$
such that the only remaining interface is given by
$$I_{1,P}(t) = \big\{x \in \bar{I}^{z,T}(t) \colon x + h(x,t)\mathrm{n}_{\bar{I}^{z,T}}(x,t)\big\}. \tag{79}$$
- *Finally, it holds*

$$\|h(\cdot,t)\|_{L^\infty(\bar{I}^{z,T}(t))} \leq \frac{r_T(t)}{16\max\{C, C_\zeta\}}, \tag{80}$$

$$\|h'(\cdot,t)\|_{L^\infty(\bar{I}^{z,T}(t))} \leq \frac{1}{C'}. \tag{81}$$

In particular, the height function $h(\cdot,t)$ *coincides with the interface error height* $\rho(\cdot,t;z,T)$ *from Construction 1 and (70) simply reads*

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathfrak{T}}(t) \end{bmatrix} = \begin{bmatrix} \frac{6}{r_T^2(t)} \fint_{\bar{I}^{z,T}(t)} h(\cdot,t)\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t)\, d\mathcal{H}^1 \\ \frac{4}{r_T(t)} \fint_{\bar{I}^{z,T}(t)} h(\cdot,t)\, d\mathcal{H}^1 \end{bmatrix}. \tag{82}$$

In the perturbative regime of Proposition 4, our error functionals take the following form.

Lemma 5 (Error functionals in perturbative regime)**.** *Fix* $t \in (0, t_\chi)$ *and assume that the conclusions of Proposition 4 hold true. Given* $\delta_{\mathrm{err}} \in (0,1)$, *one may select* $C, C' \gg_{\delta_{\mathrm{err}}} 1$ *from (80)–(81) such that*

$$\begin{aligned}(1-\delta_{\mathrm{err}}) &\int_{\bar{I}^{z,T}(t)} \frac{1}{2}\Big(\frac{h(\cdot,t)}{r_T(t)}\Big)^2 d\mathcal{H}^1 \\ &\leq E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t) \leq (1+\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}\Big(\frac{h(\cdot,t)}{r_T(t)}\Big)^2 d\mathcal{H}^1\end{aligned} \tag{83}$$

as well as

$$\begin{aligned}(1-\delta_{\mathrm{err}}) &\int_{\bar{I}^{z,T}(t)} \frac{1}{2}|h'(\cdot,t)|^2 \, d\mathcal{H}^1 \\ &\leq E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}](t) \leq (1+\delta_{\mathrm{err}}) \int_{\bar{I}^{z,T}(t)} \frac{1}{2}|h'(\cdot,t)|^2 \, d\mathcal{H}^1.\end{aligned} \tag{84}$$

The proofs of Proposition 4 and of Lemma 5 are given in Section 7.1 and in Section 7.3, respectively.

3.7. **Stability estimates at non-regular times.** In a next step, we take care of the estimation of the right hand sides of (63) and (64) in the case of disproportionally large dissipation.

Lemma 6. *There exist $\delta, \delta_{\mathrm{asymp}} \ll \frac{1}{2}$ as well as $\Lambda \gg_{\delta,\delta_{\mathrm{asymp}}} 1$ such that for every $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ satisfying (16) and $E[\mathcal{V},\chi|\bar\chi^{z,T}](t) \leq \delta r_T(t)$ it holds*

$$
\begin{aligned}
(85)\qquad &\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(-\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\Big)\\
&\quad + RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}](t)\\
&\quad \leq -\frac{1}{2}\int_{\mathbb{R}^d} |H_\mu(\cdot,t)|^2\,d\mu_t.
\end{aligned}
$$

We may easily post-process the estimate (85) to an estimate in terms of our error functional consistent with the final decay estimate (5).

Corollary 7. *There exist $\delta, \delta_{\mathrm{asymp}} \ll \frac{1}{2}$ as well as $\Lambda \gg_{\delta,\delta_{\mathrm{asymp}}} 1$ such that for every $t \in \mathcal{T}_{\text{non-reg}}(\Lambda)$ satisfying (16) and $E[\mathcal{V},\chi|\bar\chi^{z,T}](t) \leq \delta r_T(t)$ it holds*

$$
\begin{aligned}
(86)\qquad &\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(-\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\Big)\\
&\quad + RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}](t)\\
&\quad \leq -\frac{5}{r_T^2(t)}E[\mathcal{V},\chi|\bar\chi^{z,T}](t).
\end{aligned}
$$

In fact, there is nothing particular about having 5 as a factor on the right hand side of (86), and it can indeed be replaced by any constant C. The proofs of Lemma 6 and Corollary 7 can be found in Section 5.2.

3.8. **Stability estimates for perturbative regime.** We proceed with the estimation of the right hand sides of (63) and (64) in the perturbative regime described by Proposition 4. We first derive the version of the stability estimate (48) without making use of the assumption on $\bar\chi$ being quantitatively close to a shrinking circle. The derivation of these estimates, namely the proofs of the following lemmas, are contained in Sections 5.3-5.4-5.5.

Lemma 8 (Stability estimate in perturbative setting: variable coefficients)**.** *Fix $t \in (0, t_\chi)$ and assume that the conclusions of Proposition 4 hold true. Given*

$\delta_{\mathrm{err}} \in (0,1)$, *one may choose the constants* $C, C' \gg_{\delta_{\mathrm{err}}} 1$ *from* (80)–(81) *such that*

$$\begin{aligned}
&\sum_{i,j=1, i\neq j}^{P} \frac{1}{2}\Big(-\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\Big) \\
&\quad + RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_{i}[\chi|\bar\chi^{z,T}](t) \\
&\quad \leq R_{l.o.t.} + R_{h.o.t.},
\end{aligned} \tag{87}$$

where the leading order terms are given by

$$\begin{aligned}
R_{l.o.t.} := &- \int_{\bar I^{z,T}(t)} (h'')^2(\cdot,t)\, d\mathcal{H}^1 \\
&+ \int_{\bar I^{z,T}(t)} \Big(\frac{3}{2} H^2_{\bar I^{z,T}}(\cdot,t) - \frac{1}{r_T^2(t)}\Big)(h')^2(\cdot,t)\, d\mathcal{H}^1 \\
&+ \int_{\bar I^{z,T}(t)} \frac{1}{r_T^2(t)}\Big(\frac{1}{2} H^2_{\bar I^{z,T}}(\cdot,t) + \frac{1}{r_T^2(t)}\Big) h^2(\cdot,t)\, d\mathcal{H}^1 \\
&- \int_{\bar I^{z,T}(t)} \Big(\frac{1}{r_T^2(t)} + H^2_{\bar I^{z,T}}(\cdot,t)\Big) h(\cdot,t) \mathrm{n}_{\bar I^{z,T}}(\cdot,t)\cdot \dot z(t)\, d\mathcal{H}^1 \\
&- \int_{\bar I^{z,T}(t)} \frac{1}{r_T^2(t)} H_{\bar I^{z,T}}(\cdot,t) h(\cdot,t) \dot{\mathfrak{T}}(t)\; d\mathcal{H}^1 \\
&- \int_{\bar I^{z,T}(t)} H'_{\bar I^{z,T}}(\cdot,t)\big(\tau_{\bar I^{z,T}}(\cdot,t)\cdot \dot z(t)\big) h(\cdot,t)\, d\mathcal{H}^1 \\
&- \int_{\bar I^{z,T}(t)} H'_{\bar I^{z,T}}(\cdot,t)\dot{\mathfrak{T}}(t) h'(\cdot,t)\, d\mathcal{H}^1 \\
&+ \int_{\bar I^{z,T}(t)} 2 H_{\bar I^{z,T}}(\cdot,t) H'_{\bar I^{z,T}}(\cdot,t) h(\cdot,t) h'(\cdot,t)\, d\mathcal{H}^1
\end{aligned}$$

and the higher order terms are given by

$$\begin{aligned}
R_{h.o.t.} := &\, \delta_{\mathrm{err}} \int_{\bar I^{z,T}(t)} (h'')^2(\cdot,t)\, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar I^{z,T}(t)} \Big(\frac{1}{r_T^2(t)} + \big|H'_{\bar I^{z,T}}\big|(\cdot,t)\Big)(h')^2(\cdot,t)\, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar I^{z,T}(t)} \Big(\frac{1}{r_T^4(t)} + \big(H'_{\bar I^{z,T}}\big)^2(\cdot,t)\Big) h^2(\cdot,t)\, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar I^{z,T}(t)} \frac{1}{r_T(t)}\big|h'(\cdot,t)\tau_{\bar I^{z,T}}\cdot \dot z\big| + \frac{1}{r_T^2(t)}\big|h(\cdot,t)\mathrm{n}_{\bar I^{z,T}}\cdot \dot z\big|\, d\mathcal{H}^1 \\
&+ \delta_{\mathrm{err}} \int_{\bar I^{z,T}(t)} \frac{1}{r_T^3(t)}\big|h(\cdot,t)\dot{\mathfrak{T}}\big| + \big|H'_{\bar I^{z,T}}(\cdot,t) h'(\cdot,t)\dot{\mathfrak{T}}\big|\, d\mathcal{H}^1.
\end{aligned}$$

Note that in case $\bar\chi_1$ describes a circle shrinking by mean curvature flow, then the leading order terms on the right hand side of (87) are indeed precisely those captured by (48).

In a second step, we post-process the previous estimate (87) to the constant-coefficient estimate (48). In PDE jargon, this amounts to nothing else than a freezing of coefficients, only exploiting the estimates from Definition 4.

Lemma 9 (Stability estimate in perturbative setting: frozen coefficients)**.** *Fix $t \in (0, t_\chi)$, assume that the conclusions of Proposition 4 hold true, and define $\widetilde{h}(\cdot,t)\colon [0,2\pi) \to \mathbb{R}$, $\theta \mapsto h\big(\bar\gamma^{z,T}\big(\frac{L_{\bar\gamma^{z,T}}}{2\pi}\theta, t\big), t\big)$, where, for each $\widetilde{t} \in (0, T_{ext})$, $\bar\gamma(\cdot,\widetilde{t})$ denotes an arc-length parametrization of $\bar I(\widetilde{t})$. Given $\delta_{\mathrm{err}} \in (0,1)$, one may choose the constants $C, C' \gg_{\delta_{\mathrm{err}}} 1$ from (80)–(81) as well as the constant $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} \frac{1}{2}$ from Definition 4 such that*

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) + RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\Big) \\
&\qquad + RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}](t) \\
&\qquad \leq \widetilde{R}_{l.o.t.} + \widetilde{R}_{h.o.t.},
\end{aligned}
\tag{88}
$$

where the leading order terms are given by

$$
\begin{aligned}
\widetilde{R}_{l.o.t.} :={}& -\frac{1}{r_T^3(t)} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2(\cdot,t) - \frac{1}{2}(\partial_\theta \widetilde{h})^2(\cdot,t) - \frac{3}{2}\widetilde{h}^2(\cdot,t)\,d\theta \\
&- 4\frac{1}{r_T^3(t)}\bigg(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \widetilde{h}(\cdot,t)\,d\theta\bigg)^2 - 6\frac{1}{r_T^3(t)}\bigg|\frac{1}{\sqrt{\pi}}\int_0^{2\pi} \widetilde{h}(\cdot,t)e^{i\theta}\,d\theta\bigg|^2
\end{aligned}
$$

and the higher order term is simply given by

$$
\widetilde{R}_{h.o.t.} := \delta_{\mathrm{err}} \frac{1}{r_T^3(t)} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2(\cdot,t) + \frac{1}{2}(\partial_\theta \widetilde{h})^2(\cdot,t) + \frac{3}{2}\widetilde{h}^2(\cdot,t)\,d\theta.
$$

Since we reduced matters to the constant coefficient case, we may now in a third step employ Fourier methods to obtain in the perturbative regime a stability estimate consistent with the decay estimate (5).

Lemma 10 (Final stability estimate in perturbative setting)**.** *Fix $t \in (0, t_\chi)$ and assume that the conclusions of Proposition 4 hold true. Given $\alpha \in (1,5)$, one may choose the constants $C, C' \gg_\alpha 1$ from (80)–(81), the constant $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ from Definition 4, and the constant $C_\zeta \gg_\alpha 1$ from Construction 1 such that*

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(- \mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}](t) + \frac{1}{2}RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}](t)\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}](t) \\
&\quad \leq -\frac{\alpha}{r_T^2(t)}(1+\dot{\mathfrak{T}})E[\chi|\bar\chi^{z,T}](t).
\end{aligned}
\tag{89}
$$

3.9. **A priori stability estimate up to extinction time.** The penultimate step of our strategy is simply a summary of our estimates from Subsections 3.6–3.8 (see Section 5.6 for the proof).

Theorem 11. *Fix a decay exponent $\alpha \in (1,5)$ and a time $\widetilde{t}_\chi \in (0, t_\chi)$. One may choose the constant $C_\zeta \gg_\alpha 1$ from Construction 1 as well as constants $\delta_{\mathrm{asymp}} \ll_\alpha \frac{1}{2}$ and $\delta \ll_{\alpha, C_\zeta} \frac{1}{2}$ (all independent of $\widetilde{t}_\chi$) such that if for all $t \in (0, T_{ext})$ the interior of $\{\bar\chi_1(\cdot,t){=}1\} \subset \mathbb{R}^2$ is δ_{asymp}-close to a circle with radius $r(t) := \sqrt{2(T_{ext}{-}t)}$ in the sense of Definition 4 and*

$$
E[\mathcal{V},\chi|\bar\chi^{z,T}](t) \leq \delta r_T(t) \text{ for all } t \in [0, \widetilde{t}_\chi), \tag{90}
$$

then it holds for all $[s,\tau]\subset[0,\widetilde{t}_\chi)$ *and all* $\psi\in C^1_{cpt}([0,t_\chi);[0,\infty))$

$$\begin{aligned}\psi(\tau)E[\mathcal{V},\chi|\bar\chi^{z,T}](\tau)+\int_s^\tau\psi(t)\frac{\alpha}{r_T^2(t)}\big(1+\dot{\mathfrak{T}}(t)\big)E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\,dt\\ \leq\psi(s)E[\mathcal{V},\chi|\bar\chi^{z,T}](s)+\int_s^\tau\Big(\frac{d}{dt}\psi(t)\Big)E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\,dt.\end{aligned}\tag{91}$$

The unconditional decay estimate (5) from Theorem 1 now follows by means of a simple ODE argument (cf. Section 4). The asserted estimates (3)–(4) on the space-time shifts are in turn the content of the following result (see Subsection 6.2 for a proof) .

Lemma 12. *In the setting of Theorem 11, one may choose the constants such that assumption* (90) *implies*

$$\frac{1}{r_0}\|z\|_{L^\infty_t(0,t_\chi)}\leq\sqrt{\frac{1}{r_0}E[\mathcal{V},\chi_0|\bar\chi_0]},\tag{92}$$

$$\frac{1}{T_{ext}}\|T-\mathrm{id}\|_{L^\infty_t(0,t_\chi)}\leq\sqrt{\frac{1}{r_0}E[\mathcal{V},\chi_0|\bar\chi_0]}.\tag{93}$$

4. Proof of the main theorem

We proceed in two steps. For the whole proof, fix $\alpha\in(1,5)$, and choose $C_\zeta\gg_\alpha 1$, $\delta_{\mathrm{asymp}}\ll_\alpha\frac12$ and $\delta\ll_{\alpha,C_\zeta}\frac12$ such that Theorem 11 applies. We then also fix an auxiliary constant $\kappa\in(0,\delta r_0)$.

Step 1: Post-processed a priori stability estimate. Let the conclusion of Theorem 11 hold true for some $\widetilde{t}_\chi\in(0,t_\chi)$. We then claim that for a.e. $t\in(0,\widetilde{t}_\chi)$

$$E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\leq\big(E[\mathcal{V}_0,\chi_0|\bar\chi_0]+\kappa\big)\Big(\frac{r_T(t)}{r_0}\Big)^\alpha=:e(t)\quad\text{for any }\kappa>0.\tag{94}$$

For a proof of (94), we first note that $(0,t_\chi)\ni t\mapsto E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\in(0,\infty)$ is of bounded variation in $(0,t_\chi)$ due to the conditions satisfied by χ being a varifold-BV solution for multiphase mean curvature flow in the sense of Definition 3, and the identity

$$\begin{aligned}E[\mathcal{V},\chi|\bar\chi^{z,T}](t)=E[\mathcal{V},\chi(\cdot,t)]-\sum_{i=1}^P\int_{\mathbb{R}^2}\chi_i(\cdot,t)(\nabla\cdot\xi^{z,T})(\cdot,t)\,dx\\ +\sum_{i=1}^{P-1}\int_{\mathbb{R}^2}(\chi_i-\bar\chi_i^{z,T})(\cdot,t)\vartheta_i^{z,T}(\cdot,t)\,dx.\end{aligned}\tag{95}$$

By slight abuse of notation, we denote the associated distributional derivative by $\frac{d}{dt}E[\mathcal{V},\chi|\bar\chi^{z,T}]$. It then follows from (91) that

$$\frac{d}{dt}E[\mathcal{V},\chi|\bar\chi^{z,T}]\leq-\frac{\alpha}{r_T^2}\big(1+\dot{\mathfrak{T}}\big)E[\mathcal{V},\chi|\bar\chi^{z,T}]\quad\text{in distributional sense.}\tag{96}$$

Since $t\mapsto e(t)$ is a smooth function, one may infer from the product rule of distributional derivatives that $\frac{d}{dt}\frac{E[\mathcal{V},\chi|\bar\chi^{z,T}]}{e}=\frac{(\frac{d}{dt}E[\mathcal{V},\chi|\bar\chi^{z,T}])e-E[\mathcal{V},\chi|\bar\chi^{z,T}]\frac{d}{dt}e}{e^2}$, so that (96) and $\frac{d}{dt}e=-\frac{\alpha}{r_T^2}\big(1+\dot{\mathfrak{T}}\big)e$ imply

$$\frac{d}{dt}\frac{E[\mathcal{V},\chi|\bar\chi^{z,T}]}{e}\leq 0\quad\text{in distributional sense.}\tag{97}$$

Testing (97) by standard bump functions, we obtain for a.e. $t \in (0, \widetilde{t}_\chi)$ and a.e. $s \in (0,t)$ that $\frac{E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t)}{e(t)} \leq \frac{E[\mathcal{V},\chi|\bar{\chi}^{z,T}](s)}{e(s)}$. Since (91) furthermore implies that $[0,\widetilde{t}_\chi) \ni t \mapsto E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t)$ is non-increasing, we obtain from taking $(0,t) \ni s \downarrow 0$ that (94) indeed holds true.

Step 2: Proof of (5) *under assumption* (2). We define

$$\mathcal{T} := \big\{ t \in (0, t_\chi) \colon E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t) > e(t) \big\}, \tag{98}$$

and we argue in favor of (5) by contradiction. Hence, we assume $\mathcal{T} \neq \emptyset$ and define $\widetilde{t}_\chi := \inf \mathcal{T} \in [0, t_\chi)$. Since $E[\mathcal{V}_0, \chi_0|\bar{\chi}_0^{z,T}] < e(0)$, it is not hard to show that $\widetilde{t}_\chi \neq 0$. Then, by construction and hypothesis (2), we observe that assumption (90) is in place for all $t \in [0, \widetilde{t}_\chi)$. In other words, the estimate (94) applies on $(0, \widetilde{t}_\chi)$. However, on the other side $E[\mathcal{V},\chi|\bar{\chi}^{z,T}](\widetilde{t}_\chi) \leq \lim_{t \uparrow \widetilde{t}_\chi} E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t)$ by virtue of the energy $E[\mathcal{V},\chi]$ being non-increasing and the remaining constituents of $E[\mathcal{V},\chi|\bar{\chi}^{z,T}]$ from (95) being absolutely continuous. In other words, $\frac{E[\mathcal{V},\chi|\bar{\chi}^{z,T}](\widetilde{t}_\chi)}{e(\widetilde{t}_\chi)} \leq 1$ contradicting our assumption $\mathcal{T} \neq \emptyset$. Hence, $\mathcal{T} = \emptyset$ and taking the limit $\kappa \downarrow 0$ implies the decay estimate (5). Finally, the bounds (3) and (4) follow from Lemma 12. ☐

5. Weak-strong stability estimates

5.1. **Proof of Lemma 2: Preliminary stability estimate.** Without the additional non-negative test function in time ψ (or more precisely, in case $\psi \equiv 1$ on $[s,\tau]$), the proof of (63) and (64) directly follows from the arguments used in the proofs of [9, Proposition 17] and [9, Lemma 20], respectively, and the arguments of [9, Section 4.4]. The extension to general $\psi \in C^1_{cpt}([0,t_\chi);[0,\infty))$ in turn follows along the same lines, with the only additional ingredient being that one has to rely on the general form of Brakke's inequality (see [23, Definition 2.1 (d)]) instead of just the energy dissipation inequality (11). ☐

5.2. **Proof of Lemma 6 and Corollary 7: Stability at non-regular times.** Before we turn to the proofs of Lemma 6 and Corollary 7, respectively, we start with two useful auxiliary results. The first is concerned with bounds for our gradient flow calibration.

Lemma 13. *Consider the gradient flow calibration* $((\xi_i)_{i=1,\ldots,P}, (\vartheta_i)_{i=1,\ldots,P-1}, B)$ *from Construction 2 and recall that* $\xi_{i,j} := \xi_i - \xi_j$ *for all distinct* $i,j \in \{1,\ldots,P\}$. *There exists a universal constant* $\widetilde{C} \in [1,\infty)$ *such that for all* $t \in [0,t_\chi)$ *and all* $i,j \in \{1,\ldots,P\}$ *with* $i \neq j$, *it holds*

$$\big\| \big(\partial_t \xi^{z,T}_{i,j}\big)(\cdot,t) \big\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\widetilde{C}}{r_T^2(t)}, \tag{99}$$

$$\big\| \big(\nabla \cdot \xi^{z,T}_{i,j}\big)(\cdot,t) \big\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\widetilde{C}}{r_T(t)}, \tag{100}$$

$$\big\| \big(\partial_t \vartheta^{z,T}_{i}\big)(\cdot,t) \big\|_{L^\infty(\mathbb{R}^2)} \leq \frac{\widetilde{C}}{r_T^3(t)}. \tag{101}$$

Proof of Lemma 13. Fix $t \in [0,t_\chi)$ and $i,j \in \{1,\ldots,P\}$ with $i \neq j$. Recalling Construction 2, we observe that $\xi_{i,j} \in \{\pm\xi, \pm\frac{1}{2}\xi, 0\}$, so that it suffices to estimate in terms of the vector field ξ. Recalling its definition (72) it follows that

$$\begin{aligned}\xi^{z,T}(\cdot,t) &= \eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\\ &= \eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\big(\nabla\,\mathrm{sdist}_{\bar I^{z,T}}\big)(\cdot,t),\end{aligned}\tag{102}$$

we directly compute

$$\begin{aligned}\big(\nabla\cdot\xi^{z,T}\big)(\cdot,t) &= \frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\\ &\quad - \eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)}{1-H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)},\end{aligned}$$

so that (100) follows from $|\eta'|\le 16$, $|H_{\bar I}(\cdot,t)|\le 2/r(t)$ due to Definition 4, and $|\,\mathrm{sdist}_{\bar I}(\cdot,t)|\le r(t)/4$ on $\operatorname{supp}\xi(\cdot,t)$, $t\in[0,T_{ext})$.

Furthermore, since

$$\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t) = -H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)(1+\dot{\mathfrak{T}}) - \mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\cdot\dot z,\tag{103}$$

which itself one may either directly read off from the obvious generalization of (41) or alternatively from (21), we also get

$$\begin{aligned}\big(\partial_t\xi^{z,T}\big)(\cdot,t) &= \eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\big(\nabla\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\big)(\cdot,t)\\ &\quad+\frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\big(\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\big)(\cdot,t)\mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\\ &\quad+\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T^3(t)}(1+\dot{\mathfrak{T}})\mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\\ &= -\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H'_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)(1+\dot{\mathfrak{T}})}{1-H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}\\ &\quad+\eta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\tau_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\cdot\dot z}{1-H_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}\\ &\quad+\frac{1}{r_T(t)}\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\big(\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\big)(\cdot,t)\mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big)\\ &\quad+\eta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T^3(t)}(1+\dot{\mathfrak{T}})\mathrm{n}_{\bar I^{z,T}}\big(P_{\bar I^{z,T}}(\cdot,t),t\big).\end{aligned}\tag{104}$$

Hence, (99) follows from (104), (103), (19), (71), and the estimates used for the derivation of (100).

Recalling Construction 2, we observe that $\vartheta_i\in\{\vartheta,1\}$, so that it suffices to estimate in terms of the function ϑ. Recalling its definition (72) in the form of

$$\vartheta^{z,T}(\cdot,t) = \frac{1}{r_T(t)}\bar\vartheta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big),\tag{105}$$

we obtain

$$
\begin{aligned}
\big(\partial_t\vartheta^{z,T}\big)(\cdot,t) &= \frac{(1{+}\dot{\mathfrak{T}})}{r_T^3(t)}\bar\vartheta\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big) \\
&\quad + \frac{1}{r_T(t)}\bar\vartheta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\big(\partial_t\,\mathrm{sdist}_{\bar I^{z,T}}\,\big)(\cdot,t)}{r_T(t)} \\
&\quad + \frac{1}{r_T(t)}\bar\vartheta'\Big(\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T(t)}\Big)\frac{\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)}{r_T^3(t)}(1{+}\dot{\mathfrak{T}}),
\end{aligned} \tag{106}
$$

so that, based on the previous ingredients, we have $|\bar\vartheta'|\leq 2$ and $|\,\mathrm{sdist}_{\bar I}(\cdot,t)|\leq r(t)/4$ on $\mathrm{supp}\,\bar\vartheta'\big(\,\mathrm{sdist}_{\bar I}(\cdot,t)/(r(t)/4)\big)$, $t\in[0,T_{ext})$, and we may deduce (101). □

The second result is concerned with a crude upper bound for the mass of the varifold $\mathcal{V}_t$ in terms of the length scale r of the strong solution.

Lemma 14. *Fix $t\in(0,t_\chi)$. The condition $E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\leq\delta r_T(t)$ together with the requirements of Definition 4 implies*

$$
\frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\mathcal{H}^1(I_{i,j}(t))\leq\mu_t(\mathbb{R}^2)\leq\widetilde{C}r_T(t), \tag{107}
$$

where $\widetilde{C}\in[0,\infty)$ is a universal constant.

Proof. One can compute

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P}\frac{1}{2}\int_{I_{i,j}(t)}1\,d\mathcal{H}^{d-1} \\
&\leq\mu_t(\mathbb{R}^2)=E_{\mathrm{int}}[\mathcal{V},\chi|\bar\chi^{z,T}](t)+\sum_{i=1}^{P}\int_{\mathbb{R}^2}\chi_i(\cdot,t)\nabla\cdot\xi_i^{z,T}(\cdot,t)\,dx.
\end{aligned}
$$

Due to the computation below (102), the estimate (100), the properties of the cutoff function η from Construction 2, and Definition 4, it follows

$$
\sum_{i=1}^{P}\int_{\mathbb{R}^d}\chi_i(\cdot,t)\nabla\cdot\xi_i^{z,T}(\cdot,t)\,dx\leq\int_{\{|\,\mathrm{sdist}_{\bar I^{z,T}}(\cdot,t)|\leq r_T(t)\}}|\nabla\cdot\xi^{z,T}|\,dx\lesssim r_T(t).
$$

Whence, we can deduce (107) from the previous two displays and the assumption $E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\leq\delta r_T(t)$. □

Proof of Lemma 6. Fix $t\in\mathcal{T}_{\text{non-reg}}(\Lambda)$ for yet to be chosen $\Lambda\gg1$. For notational simplicity, let us in the sequel drop the dependence on t of all quantities. Since the definition of the error functionals is independent of the actual choice of the vector field B, we may interpret the right hand sides of (63) and (64) with $B\equiv0$ and therefore obtain for all $i,j\in\{1,\dots,P\}$ with $i\neq j$

$$
\begin{aligned}
&-\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}]+RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}] \\
&=-\int_{I_{i,j}}|V_{i,j}|^2\,d\mathcal{H}^1-\int_{I_{i,j}}V_{i,j}\nabla\cdot\xi_{i,j}^{z,T}\,d\mathcal{H}^1-\int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}^{z,T}\,d\mathcal{H}^1,
\end{aligned} \tag{108}
$$

$$
RHS^{\text{var-BV}}[\mathcal{V},\chi|\bar\chi^{z,T}](t)=-\int_{\mathbb{R}^d}|H_\mu|^2\Big(1-\frac{1}{2}\sum_{i=1}^{P}\rho_i\Big)\,d\mu_t\leq0, \tag{109}
$$

and for all $i \in \{1,\ldots,P-1\}$

$$RHS_i^{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}] = \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i^{z,T})\partial_t\vartheta_i^{z,T}\,dx + \sum_{j=1,\, i\neq j}^{P} \int_{I_{i,j}} V_{i,j}\vartheta_i^{z,T}\,d\mathcal{H}^1. \tag{110}$$

Note that, since $\frac{1}{2}\sum_{i=1}^P \rho_i \leq 1$, the right hand side of (109) is nonpositive. Before we start estimating the right hand sides of (108) and (110), we fix $\delta, \delta_{\mathrm{asymp}} \ll 1$ such that the conclusion of Lemma 14 applies for the choice $\delta_{\mathrm{err}} = \frac{1}{2}$.

From Hölder's inequality, (107) and (100), we then directly infer

$$\bigg|\int_{I_{i,j}} V_{i,j}\nabla\cdot\xi_{i,j}^{z,T}\,d\mathcal{H}^1\bigg| \lesssim \frac{1}{\sqrt{r_T}}\bigg(\int_{I_{i,j}} |V_{i,j}|^2\,d\mathcal{H}^1\bigg)^{\frac{1}{2}}.$$

Similarly, we may estimate due to (99)

$$\bigg|\int_{I_{i,j}} \mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}^{z,T}\,d\mathcal{H}^1\bigg| \lesssim \frac{1}{r_T}$$

and, since $|\vartheta_i^{z,T}| \leq 1/r_T$, also

$$\bigg|\int_{I_{i,j}} V_{i,j}\vartheta_i^{z,T}\,d\mathcal{H}^1\bigg| \lesssim \frac{1}{\sqrt{r_T}}\bigg(\int_{I_{i,j}} |V_{i,j}|^2\,d\mathcal{H}^1\bigg)^{\frac{1}{2}}.$$

Finally, the estimates (107), (16) and (101) together with the isoperimetric inequality imply

$$\bigg|\int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i^{z,T})\partial_t\vartheta_i^{z,T}\,dx\bigg| \lesssim \frac{1}{r_T}.$$

Plugging these estimates back into (108) and (110), we may infer by an absorption argument the claim (85) from employing the defining condition (75) of non-regular times for $\Lambda \gg 1$. □

Proof of Corollary 7. Denote by $\widetilde{\Lambda}$ and $(\widetilde{\delta}, \widetilde{\delta}_{\mathrm{asymp}})$ the constants from Lemma 6. The choices $\Lambda := \max\{\widetilde{\Lambda}, 10\}$ and $(\delta, \delta_{\mathrm{asymp}}) := (\widetilde{\delta}, \widetilde{\delta}_{\mathrm{asymp}})$ then imply the claim. Indeed, for $t \in \mathcal{T}_{\mathrm{non\text{-}reg}}(\Lambda)$ satisfying the assumption $E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t) \leq \delta r_T(t)$, it follows from the defining condition (75)

$$\frac{5}{r_T^2(t)}E[\mathcal{V},\chi|\bar{\chi}^{z,T}](t) \leq \frac{5}{r_T(t)} \leq \frac{1}{2}\frac{\Lambda}{r_T(t)} \leq \int_{\mathbb{R}^d} |H_\mu(\cdot,t)|^2\,d\mu_t,$$

so that the validity of (85) implies (86). □

5.3. **Proof of Lemma 8: Stability estimate in perturbative setting I.** The asserted bound (87) follows directly from the estimates (154)–(167) established in Lemma 22 in Section 8 and the fact that $RHS^{\mathrm{var\text{-}BV}}[\mathcal{V},\chi|\bar{\chi}^{z,T}](t) = 0$ due to the conclusions of Proposition 4. □

5.4. **Proof of Lemma 9: Stability estimate in perturbative setting II.** For notational simplicity, we again neglect the dependence on t of all quantities. Our proof of the estimate (88) proceeds in several steps.

Step 1: Leading order terms involving $H'_{\bar{I}^{z,T}}$. We start by providing a preliminary estimate for the last three right hand side terms of $R_{l.o.t.}$ from Lemma 8. To this end, for each of the three terms we make use of Definition 4 in the form of $|H'_{\bar{I}^{z,T}}| \leq \delta_{\mathrm{asymp}}/r_T^2$. Hence, by Young's inequality and $|H_{\bar{I}^{z,T}}| \leq 2/r_T$

$$\int_{\bar{I}^{z,T}} 2H_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} hh' \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1. \tag{111}$$

Furthermore, by the defining ODE for the space-time shift in the form of (82), Jensen's inequality and (16), we obtain

$$-\int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} \big(\tau_{\bar{I}^{z,T}} \cdot \dot{z}\big) h \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 \tag{112}$$

as well as

$$-\int_{\bar{I}^{z,T}} H'_{\bar{I}^{z,T}} \dot{\mathfrak{T}} h' \, d\mathcal{H}^1 \lesssim \delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \tag{113}$$

where for the latter we also used Young's inequality.

Step 2: Freezing of coefficients in leading order quadratic terms. As a simple consequence of (18), $|H_{\bar{I}^{z,T}}| \leq 2/r_T$ and $a^2 - b^2 = (a-b)(a+b)$, it holds

$$\begin{aligned} &\int_{\bar{I}^{z,T}} \Big(\frac{3}{2} H^2_{\bar{I}^{z,T}} - \frac{1}{r_T^2}\Big)(h')^2 \, d\mathcal{H}^1 + \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}\Big(\frac{1}{2} H^2_{\bar{I}^{z,T}} + \frac{1}{r_T^2}\Big) h^2 \, d\mathcal{H}^1 \\ &\leq \int_{\bar{I}^{z,T}} \frac{1}{2}\frac{1}{r_T^2}(h')^2 + \frac{3}{2}\frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 + \frac{9}{2}\delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^2}(h')^2 + \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1. \end{aligned} \tag{114}$$

Step 3: Freezing of coefficients in leading order correction terms. By the arguments from the previous two steps, we may estimate

$$\begin{aligned} &-\int_{\bar{I}^{z,T}} \Big(\frac{1}{r_T^2} + H^2_{\bar{I}^{z,T}}\Big) h \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z} \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^2} H_{\bar{I}^{z,T}} h \dot{\mathfrak{T}} \, d\mathcal{H}^1 \\ &\leq -\int_{\bar{I}^{z,T}} \frac{2}{r_T^2} h \mathrm{n}_{\bar{I}^{z,T}} \cdot \dot{z} \, d\mathcal{H}^1 - \int_{\bar{I}^{z,T}} \frac{1}{r_T^3} h \dot{\mathfrak{T}} \, d\mathcal{H}^1 \\ &\quad + \widetilde{C}\delta_{\mathrm{asymp}} \int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1, \end{aligned} \tag{115}$$

where $\widetilde{C} > 0$ is some universal constant.

Step 4: Change of variables in quadratic terms. Recalling the definition $[0, 2\pi) \ni \theta \mapsto h\big(\bar{\gamma}^{z,T}(\frac{L_{\bar{\gamma}^{z,T}}}{2\pi}\theta)\big)$, a simple change of variables together with condition (16) entails

$$\frac{1}{(1+\delta_{\mathrm{asymp}})^3} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2 \, d\theta \leq \int_{\bar{I}^{z,T}} (h'')^2 \, d\mathcal{H}^1, \tag{116}$$

$$\int_{\bar{I}^{z,T}} (h'')^2 \, d\mathcal{H}^1 \leq \frac{1}{(1-\delta_{\mathrm{asymp}})^3} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta^2 \widetilde{h})^2 \, d\theta, \tag{117}$$

$$\int_{\bar{I}^{z,T}} \frac{1}{r_T^2} (h')^2 \, d\mathcal{H}^1 \leq \frac{1}{(1-\delta_{\mathrm{asymp}})} \frac{1}{r_T^3} \int_0^{2\pi} (\partial_\theta \widetilde{h})^2 \, d\theta, \tag{118}$$

$$\int_{\bar{I}^{z,T}} \frac{1}{r_T^4} h^2 \, d\mathcal{H}^1 \leq (1+\delta_{\mathrm{asymp}}) \frac{1}{r_T^3} \int_0^{2\pi} \widetilde{h}^2 \, d\theta. \tag{119}$$

Step 5: Change of variables in correction terms. We claim that

$$\begin{aligned}
&\bigg| -\int_{\bar I^{z,T}} \frac{2}{r_T^2} h \mathrm{n}_{\bar I^{z,T}} \cdot \dot z \, d\mathcal{H}^1 - \bigg(-6\frac{1}{r_T^3(t)}\Big|\frac{1}{\sqrt{\pi}}\int_0^{2\pi} \widetilde h(\cdot,t)e^{i\theta}\,d\theta\Big|^2\bigg)\bigg| \\
&\quad + \bigg| -\int_{\bar I^{z,T}} \frac{1}{r_T^2} H_{\bar I^{z,T}} h\dot{\mathfrak{T}} \, d\mathcal{H}^1 - \bigg(-4\frac{1}{r_T^3(t)}\Big(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \widetilde h(\cdot,t)\,d\theta\Big)^2\bigg)\bigg| \\
&\qquad \leq \widetilde C \delta_{\mathrm{asymp}} \frac{1}{r_T^3}\int_0^{2\pi} \widetilde h^2\,d\theta,
\end{aligned} \tag{120}$$

where $\widetilde C > 0$ is some universal constant. Indeed, this follows similarly to the previous steps, exploiting in the process the two conditions (16) and (17) as well as the defining ODE of the space-time shift (82).

Step 6: Conclusion. Based on the previous steps, we infer that, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that the leading order contribution $R_{l.o.t.}$ from Lemma 8 is estimated by $\widetilde R_{l.o.t.} + \frac{1}{2}\widetilde R_{h.o.t.}$. Since the higher order contribution $R_{h.o.t.}$ from Lemma 8 can be easily estimated in terms of $\frac{1}{2}\widetilde R_{h.o.t.}$ for a suitable choice of $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ by means of the previous arguments, this concludes the proof of Lemma 9. ☐

5.5. **Proof of Lemma 10: Final stability estimate in perturbative setting.** First, we observe that by Lemma 5 and the estimates (118)–(119) from the previous proof that, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $C, C' \gg_{\delta_{\mathrm{err}}} 1$ and $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that

$$E[\chi|\bar\chi^{z,T}] \leq (1+\delta_{\mathrm{err}})\frac{1}{r_T}\frac{1}{2}\big\|\widetilde h\big\|^2_{H^1(0,2\pi)} =: RHS. \tag{121}$$

Second, thanks to Lemma 9, for given $\delta_{\mathrm{err}} \in (0,1)$, one may choose $C, C' \gg_{\delta_{\mathrm{err}}} 1$ and $\delta_{\mathrm{asymp}} \ll_{\delta_{\mathrm{err}}} 1$ such that

$$\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \frac{1}{2}\Big(-\mathcal{D}_{i,j}[\chi|\bar\chi^{z,T}] + \frac{1}{2} RHS^{\mathrm{int}}_{i,j}[\chi|\bar\chi^{z,T}]\Big) + \sum_{i=1}^{P-1} RHS^{\mathrm{bulk}}_i[\chi|\bar\chi^{z,T}] \\
&\leq -\frac{1}{r_T^3}\int_0^{2\pi} (1-\delta_{\mathrm{err}})(\partial_\theta^2 \widetilde h)^2 - (1+\delta_{\mathrm{err}})\frac{1}{2}(\partial_\theta \widetilde h)^2 - (1+\delta_{\mathrm{err}})\frac{3}{2}\widetilde h^2\,d\theta \\
&\quad - 4\frac{1}{r_T^3}\Big(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\widetilde h\,d\theta\Big)^2 - 6\frac{1}{r_T^3}\Big|\frac{1}{\sqrt{\pi}}\int_0^{2\pi}\widetilde h e^{i\theta}\,d\theta\Big|^2 \\
&=: LHS.
\end{aligned} \tag{122}$$

Now, fix $\alpha \in (1,5)$. We claim that there exist $\delta_{\mathrm{err}} \ll_\alpha 1$ as well as a choice of the constant $C_\zeta \gg_\alpha 1$ from Construction 1 such that

$$LHS \leq -\frac{\alpha}{r_T^2}(1+\dot{\mathfrak{T}})RHS, \tag{123}$$

so that the claim (89) follows from (121)–(123). Fourier decomposing both sides of the asserted inequality (123), we may indeed derive the validity of (123) for suitably chosen $\delta_{\mathrm{err}} \ll_\alpha 1$ and $C_\zeta \gg_\alpha 1$ analogously to our analysis towards the end of Subsection 3.2 (cf. (53)–(56)), exploiting in the process also the bound (71). ☐

5.6. **Proof of Theorem 11: Overall a priori stability estimate.** The stability estimate (91) follows directly from combining all results from Subsections 3.3–3.8, in particular Lemma 2, Corollary 7, and Lemma 10. □

Note in this context that assumption (90) implies $E[\mathcal{V},\chi|\bar\chi^{z,T}](t) \leq \delta r_T(t)$ for all $t\in(0,t_\chi)$.

6. Construction and properties of space-time shifts

6.1. **Proof of Lemma 3: Existence of space-time shifts.** Our aim is to prove the existence of a time horizon $t_\chi\in(0,\infty)$, a locally Lipschitz map $z\colon[0,t_\chi)\to\mathbb{R}^d$ and a strictly increasing Lipschitz map $T=:\mathrm{id}+\mathfrak{T}\colon[0,t_\chi)\to[0,\infty)$ such that $(z(0),T(0))=(0,0)$ and

$$t_\chi = \sup\Big\{t : T(t) < \frac{1}{2}r_0^2 = T_{ext}\Big\}, \tag{124}$$

$$\begin{bmatrix}\dot z(t)\\ \dot{\mathfrak{T}}(t)\end{bmatrix} = F(z(t),T(t),t),\quad t\in(0,t_\chi), \tag{125}$$

where

$$F(z,T,t) := \begin{bmatrix}\frac{6}{r_T^2(t)}\fint_{\bar I^{z,T}(t)}\rho(\cdot,t;z,T)\mathrm{n}_{\bar I^{z,T}}(\cdot,t)\,d\mathcal{H}^1\\ \frac{4}{r_T(t)}\fint_{\bar I^{z,T}(t)}\rho(\cdot,t;z,T)\,d\mathcal{H}^1\end{bmatrix}. \tag{126}$$

Note that the asserted Lipschitz bounds (71) are then immediate consequences of integrating (125) and $|\rho(x,t;z,T)|\leq r_T(t)/(8C_\zeta)$, cf. Construction 1 .

The proof of existence of the solution is obtained by successive approximations and an application of the Picard–Lindelöf argument. We have to resort to an approximation argument to circumvent blowing up constants (originating from negative powers of $r_T(t)$ for $t\to t_\chi$) preventing the use of the Picard–Lindelöf argument. To this end, we introduce an auxiliary version of our problem labeled by integers $k\geq1$, which reads as

$$\begin{bmatrix}\dot z_k(t)\\ \dot{\mathfrak{T}}_k(t)\end{bmatrix} = F_k(z_k(t),T_k(t),t),\quad (z_k(0),T_k(0))=(0,0), \tag{127}$$

where the right hand side $F_k\colon\mathbb{R}^2\times[0,\infty)\times[0,\infty)\to\mathbb{R}^3$ is defined by truncation:

$$F_k(z,T,t) = F\big(z,\min\big\{T,\tfrac12 r_0^2(1-\tfrac1k)\big\},t\big),\quad t\in[0,\infty). \tag{128}$$

We will show below that the fixed point equation obtained from integrating (127) admits a unique solution $z_k\in C_b([0,\infty);\mathbb{R}^2)$ and $T_k\in C_b([0,\infty);[0,\infty))$, where $t\mapsto T_k(t)$ is strictly increasing such that

$$\frac12 t\leq T_k(t)\leq\frac32 t. \tag{129}$$

(The latter two properties are consequences of $|\dot{\mathfrak{T}}_k|\leq\frac12$ due to the estimate $|\rho(x,t;z_k,\min\{T_k,\frac12 r_0^2(1-\frac1k)\})|\leq r_T(t)/(8C_\zeta)$ and $C_\zeta\geq1$.)

Taking the existence of such a sequence of solutions $(z_k,T_k)_{k\geq1}$ for granted for the moment, we then define $t_0:=0$ and for $k\geq1$

$$t_k := \sup\big\{t\colon T_k(t)<T_{ext}(1-\tfrac1k)\big\}. \tag{130}$$

By the properties of T_k, the uniqueness of solutions to (127), as well as the definitions (128) and (130), the sequence $(t_k)_{k\geq1}$ is strictly increasing and bounded. The solution to (125) is then constructed by

$$(z(t),T(t)) := (z_k(t),T_k(t)), \quad t\in[0,t_k), \tag{131}$$

$$t_\chi := \sup_{k\geq 1} t_k = \lim_{k\to\infty} t_k < \infty. \tag{132}$$

Note that (131) is indeed well-defined by uniqueness of solutions to (127), and that the identities (124)–(125) hold true by construction. Hence, it remains to verify the existence of solutions to (127) with the asserted properties.

Fix an integer $k\geq 1$. In order to apply the Picard–Lindelöf argument, we have to show that for given $t\in(0,\infty)$, the function $(z,T)\to F_k(z,T,t)$ is globally Lipschitz with Lipschitz constant independent of t. For notational convenience, we abbreviate the truncation by $\widehat{T} := \{T, \frac{1}{2}r_0^2(1-\frac{1}{k})\}$. First, we compute

$$\frac{1}{r_{\widehat{T}}^2(t)}\frac{1}{\mathcal{H}^1(\bar{I}^{z,\widehat{T}}(t))} = \frac{1}{2\pi}\frac{1}{r_{\widehat{T}}^3(t)}\frac{2\pi r_{\widehat{T}}(t)}{\mathcal{H}^1(\bar{I}^{0,\widehat{T}}(t))},$$

so that the normalization factor has the required Lipschitz regularity due to the action of the truncation and the smoothness of the evolution of $\bar{\chi}$. Second, since the Jacobian of the tubular neighborhood diffeomorphism

$$x\mapsto (P_{\bar{I}^{z,\widehat{T}}}(x,t), \mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(x,t))$$

is given by $x\mapsto 1/(1-(H_{\bar{I}^{z,\widehat{T}}}\circ P_{\bar{I}^{z,\widehat{T}}})(x,t)\,\mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(x,t))$, plugging in the definition (67) together with a change of variables yields

$$\begin{aligned}
&\int_{\bar{I}^{z,\widehat{T}}(t)} \rho_+(\cdot,t;z,\widehat{T})\mathrm{n}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\,d\mathcal{H}^1\\
&= \int_{\{0\leq \mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\leq r_{\widehat{T}}(t)/8\}} \big(1-(H_{\bar{I}^{z,\widehat{T}}}\circ P_{\bar{I}^{z,\widehat{T}}})(\cdot,t)\,\mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\big)\\
&\qquad\qquad \big(\bar{\chi}_1^{z,\widehat{T}}-\chi_1\big)(\cdot,t)\zeta\Big(\frac{\mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)}{r_{\widehat{T}}(t)}\Big)\nabla\,\mathrm{sdist}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\,dx.
\end{aligned}$$

Shifting variables in space, we obtain from the relations (21)–(23)

$$\begin{aligned}
&\int_{\bar{I}^{z,\widehat{T}}(t)} \rho_+(\cdot,t;z,\widehat{T})\mathrm{n}_{\bar{I}^{z,\widehat{T}}}(\cdot,t)\,d\mathcal{H}^1\\
&= \int_{\{0\leq \mathrm{sdist}_{\bar{I}^{0,\widehat{T}}}(\cdot,t)\leq r_{\widehat{T}}(t)/8\}} \big(1-(H_{\bar{I}^{0,\widehat{T}}}\circ P_{\bar{I}^{0,\widehat{T}}})(\cdot,t)s_{\bar{I}^{0,\widehat{T}}}(\cdot,t)\big)\\
&\qquad\qquad \big(\bar{\chi}_1^{0,\widehat{T}}-\chi_1^{-z,\mathrm{id}}\big)(\cdot,t)\zeta\Big(\frac{\mathrm{sdist}_{\bar{I}^{0,\widehat{T}}}(\cdot,t)}{r_{\widehat{T}}(t)}\Big)\nabla\,\mathrm{sdist}_{\bar{I}^{0,\widehat{T}}}(\cdot,t)\,dx,
\end{aligned}$$

and the required estimate follows from this representation by smoothness of the evolution of $\bar{\chi}$, the action of the truncation, and Lipschitz continuity of translations of volumes. Since an analogous formula also holds for ρ_+ replaced by ρ_-, this concludes the proof (by resorting to Banach's fixed point theorem exactly as in the proof of the Picard-Lindelöf theorem). □

6.2. **Proof of Lemma 12: Bounds for space-time shifts.** Our goal is to prove (92)–(93). Fix $t\in(0,t_\chi)$. Note that from (61), Construction 2, a change to tubular neighborhood coordinates, $|H_{\bar{I}^{z,T}}(\cdot,t)| \leq 2/r_T(t)$, and (69) it follows that

$$\begin{aligned}
&E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}](t) \\
&\geq \int_{\{\mathrm{dist}(\cdot,\bar{I}^{z,T}(t))<r_T(t)/8\}} \big|\chi_1(\cdot,t) - \bar{\chi}_1^{z,T}(\cdot,t)\big|\big|\vartheta_1(\cdot,t)\big|\,dx \\
&= \int_{\bar{I}^{z,T}(t)} \int_{-\frac{r_T(t)}{8}}^{\frac{r_T(t)}{8}} \frac{(\chi_1 - \bar{\chi}_1^{z,T})\big(\cdot + s\mathrm{n}_{\bar{I}^{z,T}}(\cdot,t),t\big)}{1 - H_{\bar{I}^{z,T}}(\cdot,t)s} \frac{-s}{r_T^2(t)}\,ds d\mathcal{H}^1 \qquad (133) \\
&\geq \frac{4}{5} \int_{\bar{I}^{z,T}(t)} \frac{1}{2}\frac{1}{r_T^2(t)} \Big(\rho_+^2(\cdot,t;z,T) + \rho_-^2(\cdot,t;z,T)\Big)\,d\mathcal{H}^1 \\
&\geq \frac{1}{10} \int_{\bar{I}^{z,T}(t)} \Big(\frac{\rho(\cdot,t;z,T)}{r_T(t)}\Big)^2\,d\mathcal{H}^1.
\end{aligned}$$

Hence, plugging in (70), recalling (16), and using Jensen's inequality

$$\begin{aligned}
\frac{1}{r_0}|z(t)| &\leq \int_0^t \frac{1}{r_0}|\dot{z}(s)|\,ds \\
&\lesssim (1+\delta_{\mathrm{asymp}})\frac{1}{r_0}\int_0^t \frac{1}{r_T^2(s)} \Big(\fint_{\bar{I}^{z,T}(s)} \big|\rho(\cdot,s;z,T)\big|^2\,d\mathcal{H}^1\Big)^{\frac{1}{2}}\,ds \qquad (134) \\
&\lesssim (1+\delta_{\mathrm{asymp}})\frac{1}{r_0}\int_0^t \frac{1}{r_T^{3/2}(s)} \Big(\int_{\bar{I}^{z,T}(s)} \Big(\frac{\rho(\cdot,s;z,T)}{r_T(s)}\Big)^2\,d\mathcal{H}^1\Big)^{\frac{1}{2}}\,ds.
\end{aligned}$$

Inserting the estimate (134) into (133) and afterwards exploiting the assumption (90) together with (91) and (71) further entail

$$\frac{1}{r_0}|z(t)| \lesssim \sqrt{\frac{1}{r_0}E[\mathcal{V}_0,\chi_0|\bar{\chi}_0]} \int_0^t \frac{1}{r_T^2(s)}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}\,ds,$$

which in turn by (71) upgrades to

$$\begin{aligned}
\frac{1}{r_0}|z(t)| &\lesssim \sqrt{\frac{1}{r_0}E[\mathcal{V}_0,\chi_0|\bar{\chi}_0]} \int_0^t \frac{1}{r_T^2(s)}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}(1+\dot{\mathfrak{T}}(s))\,ds \\
&= -\sqrt{\frac{1}{r_0}E[\mathcal{V}_0,\chi_0|\bar{\chi}_0]} \int_0^t \frac{d}{ds}\Big(\frac{r_T(s)}{r_0}\Big)^{\frac{1}{2}}\,ds \\
&\lesssim \sqrt{\frac{1}{r_0}E[\mathcal{V}_0,\chi_0|\bar{\chi}_0]}\Big(1 - \Big(\frac{r_T(t)}{r_0}\Big)^{\frac{1}{2}}\Big).
\end{aligned}$$

Now, choosing $\delta_{\mathrm{err}} \ll 1$ such that the implicit constant in the last estimate gets canceled, we obtain the claim for the path of translations z. Analogously, one derives a bound of same type for $\frac{1}{T_{ext}}|\mathfrak{T}(t)|$. □

7. Reduction to perturbative graph setting

7.1. **Proof of Proposition 4: Strategy and intermediate results.** We fix $\Lambda > 0$ and let $t \in \mathcal{T}_{\mathrm{reg}}(\Lambda)$, namely $t \in (0,t_\chi)$ such that (76), i.e.,

$$\int_{\mathbb{R}^d} |H_\mu(\cdot,t)|^2\,d\mu_t < \Lambda\frac{2\pi}{r_T(t)},$$

holds. Given $C_\zeta \geq 1$ from Construction 1 and given any $C, C' \geq 1$ (representing the constants from (80)–(81)), we aim to find a constant $\delta \ll_{\Lambda,C,C',C_\zeta} \frac{1}{2}$ such that the assumption (77), i.e.,

$$E[\mathcal{V},\chi|\bar\chi^{z,T}](t)\le \delta r_T(t)$$

implies the conclusions of Proposition 4. From now on, let us suppress the dependence of all quantities on t.

The proof of Proposition 4 leverages on two results from the literature (for the precise statements, see Theorem 23 and Theorem 24 in Appendix A):

- Allard's regularity theory [22, Chapter 5, Theorem 23.1 and Remark 23.2(a)]
- The decomposition of the reduced boundary of a set of finite perimeter in $\mathbb{R}^2$ into a countable family of rectifiable Jordan-Lipschitz curves [2, Section 6, Theorem 4].

The idea for the proof is then roughly speaking the following. Our assumptions (76)–(77) and Allard's regularity theory first provide us with a scale $\varrho \ll r_T$ (uniform in $x_0\in\operatorname{supp}\mu$) such that $\operatorname{supp}\mu$ admits a local graph representation on that scale at each $x_0\in\operatorname{supp}\mu$. In addition, the assumption (77) together with the coercivity properties of the error functional $E[\mathcal{V},\chi|\bar\chi^{z,T}]$ allow to show that any part of the set $\operatorname{supp}\mu$ not being in accordance with the asserted graph representation of Proposition 4 has to be of sufficiently small mass. In fact, by a suitable choice of the constant δ from assumption (77) and exploiting the decomposition result from [2] (i.e., the Lipschitz parametrization of the curves), one may trap any undesired behavior of $\operatorname{supp}\mu$ within balls of radius $\varrho/2$. This, however, contradicts the local graph property of $\operatorname{supp}\mu$ within balls of radius ϱ.

Keeping this heuristic in mind, we proceed by stating several intermediate results which combined entail a proof of Proposition 4. As a first step, we ensure that our assumptions (76)–(77) imply the applicability of Allard's regularity theory.

Lemma 15 (Applicability of Allard's regularity theory)**.** *Let $(\varepsilon,\gamma,C_{Allard})$ be the constants from Theorem 23. There exist $\tilde\varepsilon\in(0,\varepsilon)$, $\delta_{\mathrm{asymp}}\ll 1$, $\delta\ll_{\tilde\varepsilon,\Lambda} 1$ and $\tilde C\gg 1$ such that, for all $x_0\in\operatorname{supp}\mu$ and*

$$\begin{cases} G=\operatorname{Tan}_{P_{\bar I^{z,T}}(x_0)}\bar I^{z,T} & \text{if } x_0\in\{|\xi^{z,T}|\le 1/2\},\\ G \text{ arbitrary} & \text{if } x_0\in\{|\xi^{z,T}|> 1/2\},\end{cases}$$

the assumptions of Allard's regularity theorem (see Theorem 23 for $p=2$) are fulfilled at scale $\varrho:=\frac{1}{\tilde C}\frac{\tilde\varepsilon^2}{2\pi(\Lambda+1)}r_T$ in the stronger form of

$$\frac{\mu(B_\varrho(x_0))}{\mathrm{Vol}_1\varrho}\le 1+\tilde\varepsilon,\quad \Big(\int_{B_\varrho(x_0)}|H_\mu|^2\,dx\Big)^{\frac12}\varrho^{\frac12}\le\tilde\varepsilon,\quad E_{\mathrm{tilt}}[x_0,\varrho,G]\le\tilde\varepsilon^2.$$

Furthermore, one may choose $\tilde\varepsilon\in(0,\varepsilon)$, $\delta_{\mathrm{asymp}}\ll 1$ such that the following properties are satisfied:

- *$\tilde\varepsilon\in(0,\varepsilon)$ is such that*

$$\tilde\varepsilon\le\frac23\frac{1}{2C_{Allard}C'},\quad \tilde\varepsilon\le\frac14\frac{1}{2C_{Allard}}\frac{1}{16\max\{C,C_\zeta\}},\tag{135}$$

- *for all $x,\tilde x_0\in\bar I^{z,T}$ such that $|(x-\tilde x_0)\cdot\tau_{\bar I^{z,T}}(\tilde x_0)|\le\gamma\varrho$, it holds*

$$\frac{|(x-\tilde x_0)\cdot \mathrm{n}_{\bar I^{z,T}}(\tilde x_0)|}{r_T}\le\frac14\frac{1}{16\max\{C,C_\zeta\}},\tag{136}$$

- *defining* $\tilde\alpha$ *by* $2C_{Allard}\tilde\varepsilon = \tan\tilde\alpha$, *for all* $x_0 \in \{|\xi^{z,T}| \leq 1/2\}$ *and all* $x \in \partial B_{\gamma\varrho/2}(x_0)$ *satisfying* $|(x-x_0)\cdot\tau_{\bar I^{z,T}}(P_{\bar I^{z,T}}(x_0))| \geq \frac{\gamma\varrho}{2}\cos\tilde\alpha$, *it holds*

$$|P_{\bar I^{z,T}}(x) - P_{\bar I^{z,T}}(x_0)| \geq \frac{1}{2}\frac{\gamma\varrho}{2}\cos\tilde\alpha, \tag{137}$$

where $C, C' > 0$ *are from Proposition 4, and* $C_\zeta > 0$ *is from Construction 1.*

We remark that the first bound in (135) will be needed to prove (81), the second bound in (135) together with (136) will be needed to prove (80), whereas (137) will be needed to prove Lemma 18 below.

As a second step, we show that the geometry of the varifold-BV solution $(\mathcal{V}, \chi)$ reduces to the geometry of a two-phase BV solution.

Lemma 16 (No other phases, hidden boundaries, and higher-multiplicity interfaces)**.** *It holds*

$$\chi_2 = ... = \chi_{P-1} = 0, \tag{138}$$

and

$$\mathcal{V} = (\mathcal{H}^1 \llcorner \operatorname{supp}|\nabla\chi_1|) \otimes (\delta_{\operatorname{Tan}_x \operatorname{supp}|\nabla\chi_1|})_{x\in\operatorname{supp}|\nabla\chi_1|}. \tag{139}$$

Next, we guarantee that the (remaining) interface of the weak solution is not located too far away from where we expect it to be (i.e., close to the interface of the strong solution).

Lemma 17 (No interface far away from $\bar I^{z,T}$)**.** *It holds*

$$\operatorname{supp}|\nabla\chi_1| \subseteq \{|\xi^{z,T}| > 1/2\} \subseteq \{\operatorname{dist}(\cdot, \bar I^{z,T}) \leq r_T/4\}. \tag{140}$$

So far, we only argued that certain features of the weak solution (contradicting the conclusions of Proposition 4) are not present. We now turn to the part of the argument guaranteeing in turn the existence of a subset of the interface satisfying the required graph representation.

Lemma 18 (Construction of a graph candidate)**.** *There exists a Jordan-Lipschitz curve* $J \subseteq \operatorname{supp}|\nabla\chi_1| \subseteq \{|\xi^{z,T}| > 1/2\}$ *such that* J *can be considered as a graph over* $\bar I^{z,T}$. *In particular, there exists a height function* $h\colon \bar I^{z,T} \to [-r_T/4, r_T/4]$ *such that*

$$J = \{x \in \bar I^{z,T} : x + h(x)\mathrm{n}_{\bar I^{z,T}}(x)\}. \tag{141}$$

We then show that the previously found candidate for the graph representation in fact saturates the whole interface of the weak solution.

Lemma 19 (Interface is a graph over $\bar I^{z,T}$)**.** *It holds*

$$\operatorname{supp}|\nabla\chi_1| = J, \tag{142}$$

where J *is the Jordan-Lipschitz curve from Lemma 18.*

Finally, we show that the associated height funcion h over $\bar I^{z,T}$ satisfies the bounds (80)–(81) from the conclusions of Proposition 4.

Lemma 20 (Height function estimates)**.** *The height function* $h : \bar I^{z,T} \mapsto [-r_T/4, r_T/4]$ *satisfies the regularity* (78), *i.e.,*

$$h \in H^2(\bar I^{z,T})$$

and the bounds (80)–(81)*, namely*

$$\|h\|_{L^\infty(\bar{I}^{z,T})} \leq \frac{r_T}{16\max\{C, C_\zeta\}}, \tag{143}$$

$$\|h'\|_{L^\infty(\bar{I}^{z,T}(t))} \leq \frac{1}{C'}. \tag{144}$$

Recalling the claims of Proposition 4, it is immediate that its proof simply follows now from a combination of the previous lemmas, so that at this stage it remains to provide a proof for all the intermediate results of this subsection (which is done in the next subsection).

As a technical ingredient for some of the previous auxiliary results, a specific coercivity property of the error functional is exploited. More precisely, we will use the fact that our interface error E_{int} (cf. (65)) controls the folding of the interface in the following sense.

Lemma 21 (Error control)**.** *Let $\Omega_1 \subseteq \mathbb{R}^2$ be a set of finite perimeter such that $\partial^*\Omega_1 = \operatorname{supp}|\nabla\chi_1|$. Fix $x_0 \in \{\operatorname{dist}(\cdot, \bar{I}^{z,T}) \leq r_T/4\}$ and consider $\varrho \ll r_T/4$ such that, for all $x \in B_\varrho(x_0)$, it holds $|\xi^{z,T}(x) - \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \leq 1/4$. Define $G_{x_0} := x_0 + \operatorname{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}$ and denote by $P_{G_{x_0}}$ the nearest point projection onto G_{x_0}. For $x \in \mathbb{R}^2$, we denote by $(\Omega_1)_{P_{G_{x_0}}(x)}$ the one-dimensional slice*

$$\Omega_1 \cap \{P_{G_{x_0}}(x) + y\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) : |y| \leq r_T/2\}.$$

Then, there exists a constant $C_{\mathrm{err}} > 0$ such that

$$\mathcal{H}^1\Big(B_\varrho(x_0) \cap \partial^*\Omega_1 \cap \{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) > 1\}\Big) \leq C_{\mathrm{err}} E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}]. \tag{145}$$

7.2. **Proofs of the intermediate Lemmas.** We provide the proofs of the several intermediate results from the previous subsection.

Proof of Lemma 15 (Applicability of Allard's regularity theory). Let $(\varepsilon, \gamma, C_{Allard})$ be the constants from Theorem 23. One may choose $\tilde{\varepsilon} \in (0, \varepsilon)$ so that the bounds in (135) are satisfied. In addition, the bounds (136)–(137) can be guaranteed by choosing $\tilde{\varepsilon} \ll \varepsilon$ and $\delta_{\mathrm{asymp}} \ll 1$ due to the uniform smoothness of the ball.

For the proof of the remaining assertions we distnguish between two cases:

(a) $x_0 \in \operatorname{supp}\mu \cap \{|\xi^{z,T}| \leq 1/2\}$, i.e., points located sufficiently far away from the interface $\bar{I}^{z,T}$,
(b) $x_0 \in \operatorname{supp}\mu \cap \{|\xi^{z,T}| > 1/2\}$, i.e., points located sufficiently close to $\bar{I}^{z,T}$.

Case (a): Let $\tilde{C} > 1$ and let $\varrho = \frac{1}{\tilde{C}} \frac{\tilde{\varepsilon}^2}{2\pi(\Lambda+1)} r_T$. From the definition (76) it follows that

$$\Big(\int_{B_\varrho(x_0)} |H_\mu|^2 \, d\mu\Big)\varrho < \Lambda\frac{2\pi}{r_T}\varrho < \tilde{\varepsilon}^2.$$

We may further assume that $\tilde{\varepsilon} \in (0, \varepsilon)$ is small enough such that for $x_0 \in \{|\xi^{z,T}| \leq 1/2\}$ it holds $B_\varrho(x_0) \subseteq \{|\xi^{z,T}| \leq 3/4\}$. Hence, recalling (59), we have

$$\begin{aligned}
\mu(B_\varrho(x_0)) &\leq \mu(B_\varrho(x_0) \cap \{\omega \leq 1/2\}) + \mu(B_\varrho(x_0) \cap \{\omega = 1\}) \\
&\leq 2\int_{\mathbb{R}^2 \cap \{\omega \leq 1/2\}} 1 - \omega \, d\mu + 4\int_{I_{1,P}} 1 - \mathrm{n}_{P,1} \cdot \xi^{z,T} \, d\mathcal{H}^1 \\
&\quad + 2\sum_{i,j \notin \{1,P\}} \frac{1}{2}\int_{I_{i,j}} 1 - \mathrm{n}_{i,j} \cdot \xi^{z,T}_{i,j} \, d\mathcal{H}^1
\end{aligned}$$

$$\leq 4E_{\mathrm{int}}[\mathcal{V},\chi|\chi^{z,T}],$$

where we used the fact that $|\xi^{z,T}_{i,j}| \leq 1/2$ for $\{i,j\} \notin \{1,P\}$. It follows that

$$\frac{\mu(B_\varrho(x_0))}{\mathrm{Vol}_1\varrho} \leq \frac{8\pi(\Lambda+1)\tilde{C}}{\mathrm{Vol}_1\tilde{\varepsilon}^2 r_T} E_{\mathrm{int}}[\mathcal{V},\chi|\chi^{z,T}]$$

and that there exists $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$ such that the assumption (77) implies

$$\frac{\mu(B_\varrho(x_0))}{\mathrm{Vol}_1\varrho} \leq 1+\tilde{\varepsilon}.$$

Similarly, one can prove that there exists $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$ such that the assumption (77) implies

$$E_{\mathrm{tilt}}[x_0,\varrho,\mathbb{R}\times\{0\}] \leq C_{\mathrm{tilt}}\varrho^{-1}\mu(B_\varrho(x_0)) \leq \tilde{\varepsilon}^2$$

for some constant $C_{\mathrm{tilt}} > 0$.

Case (b): The estimate for the curvature term works as in case (a) as the argument does not rely on the assumption on $x_0 \in \operatorname{supp}\mu$.

By uniform smoothness of the ball, one may choose $\tilde{\varepsilon} \ll \varepsilon$, $\delta_{\mathrm{asymp}} \ll 1$, and $\tilde{C} \gg 1$ such that for all $x_0 \in \{|\xi^{z,T}| > 1/2\}$ and for all $x \in B_\varrho(x_0)$ it holds

$$|\xi^{z,T}(x) - \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \leq \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8}. \tag{146}$$

We have that

$$\mu(B_\varrho(x_0)) \leq 2E_{\mathrm{int}}[\mathcal{V},\chi|\chi^{z,T}] + \int_{I_{1,P}\cap B_\varrho(x_0)} 1\,d\mathcal{H}^1,$$

where the second term has to be estimated. Hence, we further decompose

$$\begin{aligned} B_\varrho(x_0)\cap I_{1,P} = &\Big(B_\varrho(x_0)\cap\Big\{x\in I_{1,P} : \mathrm{n}_{P,1}(x)\cdot\xi^{z,T}(x) \geq \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8}\Big\}\Big)\\ &\cup\Big(B_\varrho(x_0)\cap\Big\{x\in I_{1,P} : \mathrm{n}_{P,1}(x)\cdot\xi^{z,T}(x) < \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8}\Big\}\Big)\\ &\subseteq M^{(1)}_{x_0}\cup M^{(2)}_{x_0}, \end{aligned}$$

where

$$\begin{aligned} M^{(1)}_{x_0} &:= B_\varrho(x_0)\cap\Big\{x\in\operatorname{supp}|\nabla\chi_1| : \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\xi^{z,T}(x) \geq \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8}\Big\},\\ M^{(2)}_{x_0} &:= B_\varrho(x_0)\cap\Big\{x\in I_{1,P} : \mathrm{n}_{P,1}(x)\cdot\xi^{z,T}(x) < \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8}\Big\}. \end{aligned}$$

Using Lemma 21 and the notation there adopted, we estimate

$$\begin{aligned} \mathcal{H}^1(M^{(1)}_{x_0}) &= \mathcal{H}^1(M^{(1)}_{x_0}\cap\{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) > 1\})\\ &\quad + \mathcal{H}^1(M^{(1)}_{x_0}\cap\{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) = 1\})\\ &\leq C_{\mathrm{err}}E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}] + \mathcal{H}^1(M^{(1)}_{x_0}\cap\{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) = 1\}). \end{aligned}$$

Note that by (146) and by the definition of $M^{(1)}_{x_0}$ it holds $\mathrm{n}_{\partial\Omega_1}(x)\cdot\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) \geq \frac{1}{1+\tilde{\varepsilon}/8}$ for all $x\in M^{(1)}_{x_0}$. As a consequence, the coarea formula gives

$$\mathcal{H}^1(M^{(1)}_{x_0}\cap\{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) = 1\}) \leq (1+\tfrac{\tilde{\varepsilon}}{8})\mathcal{H}^1(B_\varrho(P_{\bar{I}^{z,T}}(x_0))) \leq (1+\tfrac{\tilde{\varepsilon}}{8})\mathrm{Vol}_1\varrho.$$

Therefore, we obtain

$$\mathcal{H}^1(M_{x_0}^{(1)}) \leq C_{\mathrm{err}} E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}] + (1+\tfrac{\tilde{\varepsilon}}{8})\mathrm{Vol}_1\varrho.$$

Furthermore, one may estimate

$$\mathcal{H}^1(M_{x_0}^{(2)}) \leq \frac{16}{\tilde{\varepsilon}} \int_{I_{1,P}} 1 - \mathrm{n}_{P,1}\cdot\xi^{z,T}\,d\mathcal{H}^1 \leq \frac{16}{\tilde{\varepsilon}} E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}].$$

Collecting the estimates above, we have

$$\mu(B_\varrho(x_0)) \leq \Big(2 + C_{\mathrm{err}} + \frac{16}{\tilde{\varepsilon}}\Big) E_{\mathrm{int}}[\mathcal{V},\chi|\bar{\chi}^{z,T}] + \Big(1+\frac{\tilde{\varepsilon}}{8}\Big)\mathrm{Vol}_1\varrho,$$

whence we can conclude that there esists $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$ such that the assumption (77) implies

$$\frac{\mu(B_\varrho(x_0))}{\mathrm{Vol}_1\varrho} \leq 1+\tilde{\varepsilon}.$$

It remains to prove the estimate for the tilt excess $E_{\mathrm{tilt}}[x_0,\varrho,\mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}]$. First, recalling (59), we notice that

$$\begin{aligned}
&E_{\mathrm{tilt}}[x_0,\varrho,\mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}] \\
&\lesssim \varrho^{-1}\mu(B_\varrho(x_0)\cap\{\omega\leq 1/2\}) + \varrho^{-1}\sum_{\{i,j\}\neq\{1,P\}}\mathcal{H}^1(I_{i,j}) \\
&\quad + \varrho^{-1}\int_{B_\varrho(x_0)\cap I_{1,P}} |\mathrm{n}_{P,1}(x) - \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))|^2\,d\mathcal{H}^1 \\
&\lesssim \varrho^{-1}E_{\mathrm{int}}[\mathcal{V},\chi|\bar{\chi}^{z,T}] + \varrho^{-1}R_{tilt},
\end{aligned}$$

where

$$R_{tilt} := \int_{B_\varrho(x_0)\cap I_{1,P}} 1 - \mathrm{n}_{P,1}(x)\cdot\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))\,d\mathcal{H}^1.$$

In order to estimate R_{tilt}, we decompose

$$B_\varrho(x_0)\cap I_{1,P} \subseteq N_{x_0}^{(1)}\cup N_{x_0}^{(2)}\cup N_{x_0}^{(3)},$$

where

$$\begin{aligned}
N_{x_0}^{(1)} &:= B_\varrho(x_0)\cap\Big\{x\in\partial^*\Omega_1 : \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\xi^{z,T}(x)\geq\frac12,\ \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)})=1\Big\},\\
N_{x_0}^{(2)} &:= B_\varrho(x_0)\cap\Big\{x\in\partial^*\Omega_1 : \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\xi^{z,T}(x)\geq\frac12,\ \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)})>1\Big\},\\
N_{x_0}^{(3)} &:= B_\varrho(x_0)\cap\{x\in I_{1,P} : \mathrm{n}_{P,1}(x)\cdot\xi^{z,T}(x)<1/2\}.
\end{aligned}$$

By previous arguments (in particular the one using Lemma 21), one may infer

$$\begin{aligned}
R_{tilt} &\leq \int_{N_{x_0}^{(1)}} 1 - \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))\,d\mathcal{H}^1 + 2\mathcal{H}^1(N_{x_0}^{(2)}\cup N_{x_0}^{(3)})\\
&\leq R'_{tilt} + C''E_{\mathrm{int}}[\mathcal{V},\chi|\bar{\chi}^{z,T}],
\end{aligned}$$

for some constant $C''>0$, where

$$\begin{aligned}
R'_{tilt} &:= \int_{N_{x_0}^{(1)}} 1 - \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))\,d\mathcal{H}^1\\
&\leq \int_{N_{x_0}^{(1)}} 1 - \frac{\nabla\chi_1}{|\nabla\chi_1|}(x)\cdot\xi^{z,T}(x)\,d\mathcal{H}^1 + \frac{\tilde{\varepsilon}}{16}\mathcal{H}^1(N_{x_0}^{(1)})
\end{aligned}$$

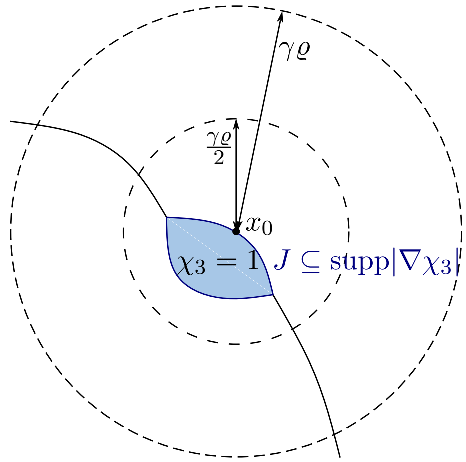

FIGURE 4. If $\mathcal{H}^1(\operatorname{supp}|\nabla\chi_3|) > 0$, then $\operatorname{int}(J) \subseteq B_{\gamma\varrho/2}(x_0)$ for the Jordan-Lipschitz curve $J \subseteq \operatorname{supp}|\nabla\chi_3|$, contradicting the graph property of $\operatorname{supp}\mu$.

$$\leq C''' E_{\mathrm{int}}[\chi|\chi^{z,T}] + \frac{\tilde{\varepsilon}}{16}\mathcal{H}^1(N_{x_0}^{(1)})$$

for some constant $C''' > 0$, where the last inequality can be justified arguing as above. Furthermore, note that for any $x \in N_{x_0}^{(1)}$ we have

$$\frac{\nabla\chi_1}{|\nabla\chi_1|}(x) \cdot \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) \geq \frac{\nabla\chi_1}{|\nabla\chi_1|}(x) \cdot \xi^{z,T}(x) - |\xi^{z,T}(x) - \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))|$$
$$\geq \frac{1}{2} - \frac{\tilde{\varepsilon}/16}{1+\tilde{\varepsilon}/8} \geq \frac{1}{4}$$

whence we can deduce $\mathcal{H}^1(N_{x_0}^{(1)}) \leq 8\pi\varrho$, due to the coarea formula. Collecting the estimates above, we can conclude that there exists $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$ such that the assumption (77) implies

$$E_{\mathrm{tilt}}[x_0, \varrho, \operatorname{Tan}_{P_{\bar{I}^{z,T}}x_0}\bar{I}^{z,T}] \leq \tilde{\varepsilon}.$$

This shows Lemma 15. □

Proof of Lemma 16 (No other phases, hidden boundaries, and higher-multiplicity interfaces). In order to prove (138), we argue by contradiction. Assume there exists $i \in \{2, ..., P-1\}$ such that $\mathcal{H}^1(\operatorname{supp}|\nabla\chi_i|) > 0$. Fix $x_0 \in \operatorname{supp}|\nabla\chi_i|$. By Lemma 15 and Theorem 23, $\operatorname{supp}\mu$ is a graph within $B_{\gamma\varrho}(x_0)$ for $\varrho = \frac{1}{\tilde{C}}\frac{\tilde{\varepsilon}^2}{2\pi(\Lambda+1)}r_T$. In addition, by Theorem 24, there exists a Jordan-Lipschitz curve $J \subseteq \operatorname{supp}|\nabla\chi_i|$ such that $x_0 \in J$. By choosing $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$, one may ensure that $\operatorname{int}(J) \subseteq B_{\gamma\varrho/2}(x_0)$ (see Figure 4 for an example): indeed, we recall that

$$\mathcal{H}^1(\operatorname{supp}|\nabla\chi_i|) = \sum_{j\neq i}\int_{I_{i,j}} 1\,d\mathcal{H}^1 \leq 2\sum_{j\neq i}\int_{I_{i,j}} 1 - \mathrm{n}_{i,j}\cdot\xi_{i,j}^{z,T}\,d\mathcal{H}^1 \leq 2E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}],$$

where we used that $|\xi_{i,j}^{z,T}| \leq 1/2$ for $\{i,j\} \neq \{1,P\}$. However, $\operatorname{int}(J) \subseteq B_{\gamma\varrho/2}(x_0)$ is a contradiction to the graph property of $\operatorname{supp}\mu$ within $B_{\gamma\varrho}(x_0)$.

It remains to prove (139). By Lemma 15 and [22, Remark 23.2(2)], every $x_0 \in \operatorname{supp}\mu$ is a point of unit density. Hence, $x_0 \in \frac{1}{2}\sum_{i=1}^{P} \operatorname{supp}|\nabla\chi_i| = \operatorname{supp}|\nabla\chi_1|$ due to (138). It follows that $\mu = \mathcal{H}^1 \llcorner \operatorname{supp}|\nabla\chi_1|$, whence we deduce (139) by the rectifiability of $\mathcal{V}$. ☐

Proof of Lemma 17 (No interface far away from $\bar{I}^{z,T}$). We assume by contradiction that there exists $x_0 \in \operatorname{supp}|\nabla\chi_1|$ such that $x_0 \in \{|\xi^{z,T}| \leq 1/2\}$. From Theorem 23, Lemma 15, and Lemma 16, it follows that $\operatorname{supp}\mu = \operatorname{supp}|\nabla\chi_1|$ is locally a graph around x_0 on a scale $\gamma\varrho$ over, say, $G_{x_0} = x_0 + (\mathbb{R}\times\{0\})$, where $\varrho = \frac{1}{C}\frac{\tilde{\varepsilon}^2}{2\pi(\Lambda+1)} r_T$. On the other hand, by Theorem 24, there exists a Jordan-Lipschitz curve $J \subseteq \operatorname{supp}|\nabla\chi_1|$ such that $x_0 \in J$. Without loss of generality, we may assume that $\tilde{\varepsilon} \ll 1$ such that $x_0 \in \{|\xi^{z,T}| < 1/2\}$ implies $B_{\gamma\varrho}(x_0) \subseteq \{|\xi^{z,T}| < 3/4\}$. In particular, we have

$$\mathcal{H}^1(J \cap B_{\gamma\varrho}(x_0)) \leq 4\int_{J\cap B_{\gamma\varrho}(x_0)} 1 - \frac{\nabla\chi_1}{|\nabla\chi_1|}\cdot\xi^{z,T}\,d\mathcal{H}^1 \lesssim E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}] \lesssim \delta r_T.$$

As a consequence, for suitably small $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$, it follows from the continuity of J that $\operatorname{int}(J) \subseteq B_{\gamma\tilde{\rho}/2}(x_0)$, which is a contradiction to the graph property of $\operatorname{supp}|\nabla\chi_1|$ within $B_{\gamma\varrho}(x_0)$. The last inclusion in (140) follows from Construction 2. ☐

Proof of Lemma 18 (Construction of a graph candidate). We proceed in two steps.

Step 1: We claim that there exist $\delta_{\mathrm{asymp}} \ll 1$ and $\delta \ll 1$ such that

$$\mathcal{H}^1(\operatorname{supp}|\nabla\chi_1|) \geq \tfrac{1}{2}\mathcal{H}^1(\bar{I}^{z,T}). \tag{147}$$

In order to prove (147), we argue by contradiction, namely we assume that

$$\mathcal{H}^1(\operatorname{supp}|\nabla\chi_1|) < \tfrac{1}{2}\mathcal{H}^1(\bar{I}^{z,T}).$$

By the isoperimetric inequality, we have

$$\int_{\mathbb{R}^2}\chi_1\,dx \leq \tfrac{1}{4\pi}\Big(\tfrac{1}{2}\mathcal{H}^1(\bar{I}^{z,T})\Big)^2 \leq \big(\tfrac{9}{16}\big)^2\pi r_T^2,$$

where we used the fact that $\mathcal{H}^1(\bar{I}^{z,T}) \leq \frac{9}{8}2\pi r_T$ for a suitably small $\delta_{\mathrm{asymp}} \ll 1$. Furthermore, for a suitably small $\delta_{\mathrm{asymp}} \ll 1$, we have

$$\int_{\mathbb{R}^2}\bar{\chi}^{z,T}\,dx \geq \big(\tfrac{3}{4}\big)^2\pi r_T^2.$$

By the triangle inequality, it follows that

$$\|\chi_1 - \bar{\chi}^{z,T}\|_{L^1} \geq \int_{\mathbb{R}^2}\bar{\chi}^{z,T}\,dx - \int_{\mathbb{R}^2}\chi_1\,dx \geq \Big(\big(\tfrac{3}{4}\big)^2 - \big(\tfrac{9}{16}\big)^2\Big)\pi r_T^2.$$

On the other side, we claim that

$$\frac{1}{r_T^3}\|\chi_1 - \bar{\chi}^{z,T}\|_{L^1}^2 \lesssim E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}]. \tag{148}$$

Indeed, by change of variables and Lemma 17, we have

$$\begin{aligned}&\frac{1}{r_T^3}\|\chi_1 - \bar{\chi}^{z,T}\|_{L^1}^2\\ &= \frac{1}{r_T^3}\Big(\int_{\bar{I}^{z,T}}\int_{-\frac{r_T}{4}}^{\frac{r_T}{4}} \frac{1}{1 - H_{\bar{I}^{z,T}}(x)s}|\chi_1 - \bar{\chi}^{z,T}|(x + s\mathrm{n}_{\bar{I}^{z,T}}(x))\,ds dx\Big)^2\end{aligned}$$

$$\lesssim \frac{1}{r_T} \fint_{\bar{I}^{z,T}} \Big(\int_{-\frac{r_T}{4}}^{\frac{r_T}{4}} \frac{1}{1 - H_{\bar{I}^{z,T}}(x)s} |\chi_1 - \bar{\chi}^{z,T}|(x + s\mathrm{n}_{\bar{I}^{z,T}}(x)) \, ds \Big)^2 dx$$

$$\lesssim \int_{\bar{I}^{z,T}} \int_{-\frac{r_T}{4}}^{\frac{r_T}{4}} \frac{1}{1 - H_{\bar{I}^{z,T}}(x)s} (|\chi_1 - \bar{\chi}^{z,T}||\vartheta|)(x + s\mathrm{n}_{\bar{I}^{z,T}}(x)) \, ds dx \lesssim E_{\mathrm{bulk}}[\chi|\bar{\chi}^{z,T}].$$

where in the last line we used Fubini's theorem by bisecting $[-\frac{r_T}{4}, \frac{r_T}{4}]^2$ into two triangles (cf. the argument in [10, Proof of Theorem 1]). Hence, by choosing $\delta \ll 1$ in assumption (77), we obtain

$$\|\chi_1 - \bar{\chi}^{z,T}\|_{L^1} < \Big(\big(\tfrac{3}{4}\big)^2 - \big(\tfrac{9}{16}\big)^2 \Big) \pi r_T^2,$$

whence the contradiction follows.

Step 2: From the previous step, there exists $x_0 \in \operatorname{supp} |\nabla \chi_1|$ and, by Theorem 24, there exists a Jordan-Lipschitz curve $J \subseteq \operatorname{supp} |\nabla \chi_1|$ such that $x_0 \in J$. By Theorem 23, Lemma 15, and Lemma 16, we know that $\operatorname{supp} \mu = \operatorname{supp} |\nabla \chi_1|$ can be represented within $B_{\gamma\varrho}(x_0)$ as a graph over $x_0 + \mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)} \bar{I}^{z,T}$ with height function u such that $\operatorname{supp} |\nabla u| \leq 2\tilde{C}\tilde{\varepsilon} =: \tan \tilde{\alpha}$, $\tilde{\alpha} \in (0, \pi/2)$. Hence, for $x_1^{\pm} \in x_0 + \mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)} \bar{I}^{z,T}$ such that $\operatorname{sign}((x_1^{\pm} - x_0) \cdot \tau_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) = \pm 1$ and $x_1^{\pm} + u(x_1^{\pm}) \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) \in \partial B_{\gamma\varrho/2}(x_0)$, it follows $|(x_1^{\pm} - x_0) \cdot \tau_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \geq \frac{\gamma\tilde{\varrho}}{2} \cos \tilde{\alpha}$. Consequently, from Lemma 15 we deduce

$$|P_{\bar{I}^{z,T}}(x_1^{\pm}) - P_{\bar{I}^{z,T}}(x_0)| \geq \frac{1}{2} \frac{\gamma\varrho}{2} \cos \tilde{\alpha}. \tag{149}$$

Since $\operatorname{supp} |\nabla \chi_1| \cap B_{\gamma\varrho}(x_0) = J \cap B_{\gamma\varrho}(x_0)$, one may continue with $x_1 = x_1^+$ and iterate the above reasoning to conclude that J wriggles around $\bar{I}^{z,T}$ in the sense of

$$\big(\{\bar{\chi}^{z,T} = 1\} \setminus \{\mathrm{dist}(\cdot, \bar{I}^{z,T}) \leq \delta r_T/4\}\big) \subseteq \mathrm{int}(J),$$
$$\big(\{\bar{\chi}^{z,T} = 0\} \setminus \{\mathrm{dist}(\cdot, \bar{I}^{z,T}) \leq \delta r_T/4\}\big) \subseteq \mathrm{ext}(J).$$

We notice that the iteration stops after finitely many steps due to (149). At last, we argue that there exists a height function $h : \bar{I}^{z,T} \to [-r_T/4, r_T/4]$ such that (141) holds. This directly follows from the compactness of $\bar{I}^{z,T}$ together with the fact that, for any $x_0 \in J$, there exists an open neighborhood $\mathcal{W}_{x_0} \ni x_0$ such that

$$\begin{aligned} J \cap \mathcal{W}_{x_0} &\to \bar{I}^{z,T} \\ x &\mapsto P_{\bar{I}^{z,T}}(x) = x - s(x) \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x)) \end{aligned} \tag{150}$$

is a manifold diffeomorphism onto its image, due to an application of the inverse function theorem. □

Proof of Lemma 19 (Interface is a graph over $\bar{I}^{z,T}$). Recall from Lemma 17 that $\operatorname{supp} |\nabla \chi_1| \subseteq \{|\xi^{z,T}| > 1/2\} \subseteq \{\mathrm{dist}(\cdot, \bar{I}^{z,T}) \leq r_T/4\}$. Assume by contradiction that there exists a nontrivial Jordan-Lipschitz curve J' from the decomposition of Theorem 24 such that $J' \subseteq \operatorname{supp} |\nabla \chi_1|$ and $J' \neq J$, i.e., either $\mathrm{int}(J') \cap \mathrm{int}(J) = \emptyset$, or $\mathrm{int}(J') \subset \mathrm{int}(J)$, or $\mathrm{int}(J) \subset \mathrm{int}(J')$. Fix $x_0 \in J'$. Since J is a graph over $\bar{I}^{z,T}$ by Lemma 18, using the notation from Lemma 21, it follows that

$$B_\varrho(x_0) \cap J' \subseteq \{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) > 1\}.$$

Recall from Lemma 15 that ϱ is chosen such that the hypothesis of Lemma 21 applies, hence $\mathcal{H}^1(B_\varrho(x_0) \cap J') \lesssim E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}]$. By the continuity of J', for suitably small $\delta \ll_{\tilde{\varepsilon},\Lambda} 1$, one may infer that $\mathrm{int}(J') \subseteq B_{\gamma\varrho/2}(x_0)$ (see Figure 5). On the other

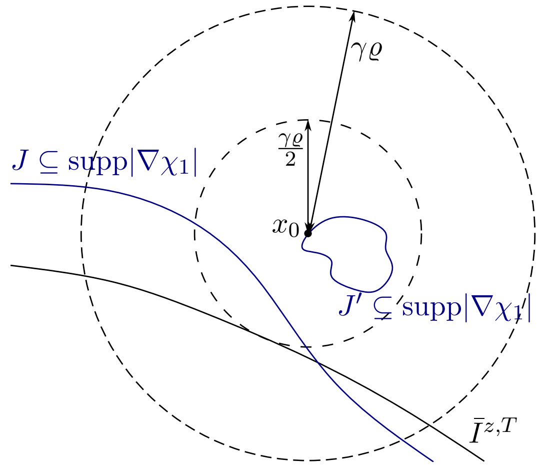

FIGURE 5. Interface is a graph over $\bar{I}^{z,T}$

side, by Theorem 23, Lemma 15, and Lemma 17, we have that $\operatorname{supp}\mu = \operatorname{supp}|\nabla\chi_1|$ is a graph within $B_{\gamma\varrho}(x_0)$, which provides the contradiction. □

Proof of Lemma 20 (Height function estimates). Step 1: Regularity (78). Fix $x_0 \in \operatorname{supp}|\nabla\chi_1|$. By Theorem 23, Lemma 15, Lemma 16, and Lemma 19 we know that $\operatorname{supp}\mu = \operatorname{supp}|\nabla\chi_1| = J$ can be represented within $B_{\gamma\varrho}(x_0)$ as a graph over $G_{x_0} := x_0 + \operatorname{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}$ with height function u. Furthermore, we know by Lemma 18 that there exists a height function $h\colon \bar{I}^{z,T} \to [-r_T/4, r_T/4]$ such that (141) holds.

Define $\mathcal{U}_{x_0} := P_{x_0}(G_{x_0} \cap B_{\gamma\varrho}(x_0))$, and we first claim that

$$u \in H^2(\mathcal{U}_{x_0}). \tag{151}$$

In order to prove (151), we fix $g \in C^\infty_{cpt}(\mathcal{U}_{x_0})$ and then test (12) with $B(x) = g(P_{\bar{I}^{z,T}}(x))\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))$. Using $H_\mu = (H_\mu \cdot \mathrm{n}_{\operatorname{supp}\mu})\mathrm{n}_{\operatorname{supp}\mu}$, the coarea formula, and the coordinates induced by the height function u, we obtain

$$\begin{aligned} &-\int_{\mathcal{U}_{x_0}} (H_\mu \cdot \mathrm{n}_{\operatorname{supp}\mu})(x + u(x)\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)))g(x)\,d\mathcal{H}^1 \\ &= \int_{\mathcal{U}_{x_0}} \frac{u'}{\sqrt{1+(u')^2}} g'\,d\mathcal{H}^1, \end{aligned}$$

whence we deduce $\frac{u'}{\sqrt{1+(u')^2}} \in H^1(\mathcal{U}_{x_0})$ due to the assumption (76) of controlled dissipation. The regularity (151) then follows from Lemma 15 and Theorem 23, in particular from the estimate (203) (recall in this context also the regularity $u \in C^{1,\frac{1}{2}}$).

In a second step, we argue that one may capitalize on (151) to show that there is an open neighborhood $\bar{\mathcal{U}}_{x_0}$ of $P_{\bar{I}^{z,T}}(x_0)$ in $\bar{I}^{z,T}$ such that

$$h \in H^2(\bar{\mathcal{U}}_{x_0}), \tag{152}$$

whence one may deduce (78) by compactness of $\bar{I}^{z,T}$. Indeed, define $\bar{\mathcal{U}}_{x_0} := \iota(\mathcal{U}_{x_0})$, where $\iota : \mathcal{U}_{x_0} \to \bar{I}^{z,T}$ is given by $x \mapsto P_{\bar{I}^{z,T}}(x + u(x)\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)))$. Since the map ι is a chart for $\bar{I}^{z,T}$, we conclude (152) from the formula $h(\iota(x)) = s(x + u(x)\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x)))$ for any $x \in \mathcal{U}_{x_0}$, the regularity of the signed distance function, and (151).

Step 2: Estimate (143) *for* $\sup|h|$. The idea here is to exploit that if (143) would not be satisfied, then one accumulates too much L^1-error between the two phases χ_1 and $\bar{\chi}_1$ in contradiction with the smallness of the overall error (77).

Hence, we assume by contradiction that there exists $x_0 \in \operatorname{supp}|\nabla\chi_1|$ such that

$$\|h\|_{L^\infty(\bar{I}^{z,T})} > \frac{r_T}{16\max\{C, C_\zeta\}}.$$

Recall that by Theorem 23, Lemma 15, and Lemma 19, we know that $\operatorname{supp}\mu \cap B_{\gamma\varrho}(x_0)$ can be represented as a graph over $(x_0 + \mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}) \cap B_{\gamma\varrho}(x_0)$ with height function u, where $\varrho = \frac{1}{\tilde{C}}\frac{\tilde{\varepsilon}^2}{2\pi(\Lambda+1)}r_T$. From Theorem 23 and Lemma 15, more precisely from (202), (203) and (135), it follows that

(153)
$$\sup|u| \leq 2\varrho\tilde{C}\tilde{\varepsilon} \leq \frac{1}{4}\frac{r_T}{16\max\{C, C_\zeta\}}, \quad \sup|u'| \leq 2\tilde{C}\tilde{\varepsilon} =: \tan\tilde{\alpha},\ \tilde{\alpha} \in (0, \pi/2).$$

In particular, we have (see Figure 6)

$$\begin{aligned}\partial B_{\gamma\varrho}(x_0) \cap \{y + u(y)\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0)) : y \in (x_0 + \mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}) \cap B_{\gamma\varrho}(x_0)\} \\ \subseteq \Big\{x \in \mathbb{R}^2 : |(x - x_0)\cdot \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \leq \frac{1}{4}\frac{r_T}{16\max\{C, C_\zeta\}}, \\ |(x - x_0)\cdot \tau_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \geq \gamma\varrho\cos\tilde{\alpha}\Big\}.\end{aligned}$$

Furthermore, by (136) from Lemma 15, we deduce that for all $x \in \bar{I}^{z,T}$ such that $|(x - P_{\bar{I}^{z,T}}(x_0))\cdot\tau_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \leq \gamma\varrho$, it holds

$$|(x - P_{\bar{I}^{z,T}}(x_0))\cdot \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))| \leq \frac{1}{4}\frac{r_T}{16\max\{C, C_\zeta\}}.$$

Hence, we have

$$\|\chi_1 - \bar{\chi}^{z,T}\|_{L^1} \geq \gamma\varrho\frac{1}{2}\cos\tilde{\alpha}\frac{r_T}{16\max\{C, C_\zeta\}}.$$

The contradiction follows from (148) and by choosing $\delta \ll_{\tilde{\varepsilon},\Lambda,C,C_\zeta} 1$ suitably small in assumption (77).

Step 3: Estimate (144) *for* $\sup|h'|$. Let $\bar{\gamma}$ be an arc-lenght parametrization of $\bar{I}^{z,T}$ so that $\operatorname{supp}|\nabla\chi_1|$ admits the parametrization $\gamma_h := (\mathrm{Id} + h\mathrm{n}_{\bar{I}^{z,T}}) \circ \bar{\gamma}$. Fix $x_0 \in \operatorname{supp}|\nabla\chi_1|$ and denote by u the associated height function given by Theorem 23 and Lemma 15 on scale $\gamma\varrho$. In other words, locally around x_0, we have a second parametrization $\gamma_u := (\mathrm{Id} + u\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))) \circ \bar{\gamma}_{x_0}$, where $\bar{\gamma}_{x_0}$ is an arc-length parametrization of $(x_0 + \mathrm{Tan}_{P_{\bar{I}^{z,T}}(x_0)}\bar{I}^{z,T}) \cap B_{\gamma\varrho}(x_0)$. One may compute

$$\frac{1}{\sqrt{1 + \Big(\frac{h'(P_{\bar{I}^{z,T}}(x_0))}{1 - (H_{\bar{I}^{z,T}}h)(P_{\bar{I}^{z,T}}(x_0))}\Big)^2}} = \frac{1}{\sqrt{1 + (u'(x_0))^2}}$$

as both terms equals $\frac{\nabla\chi_1}{|\nabla\chi_1|}(x_0)\cdot\mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))$. Note that this gives us a relation expressing $|h'(P_{\bar{I}^{z,T}}(x_0))|$ in terms of $|u'(x_0)|$. In particular, by choosing $\delta_{\mathrm{asymp}} \ll 1$ suitably small and using $|h(P_{\bar{I}^{z,T}}(x_0))| \leq \frac{1}{4}r_T$, we obtain $|h'(P_{\bar{I}^{z,T}}(x_0))| \leq \frac{3}{2}|u'(x_0)|$

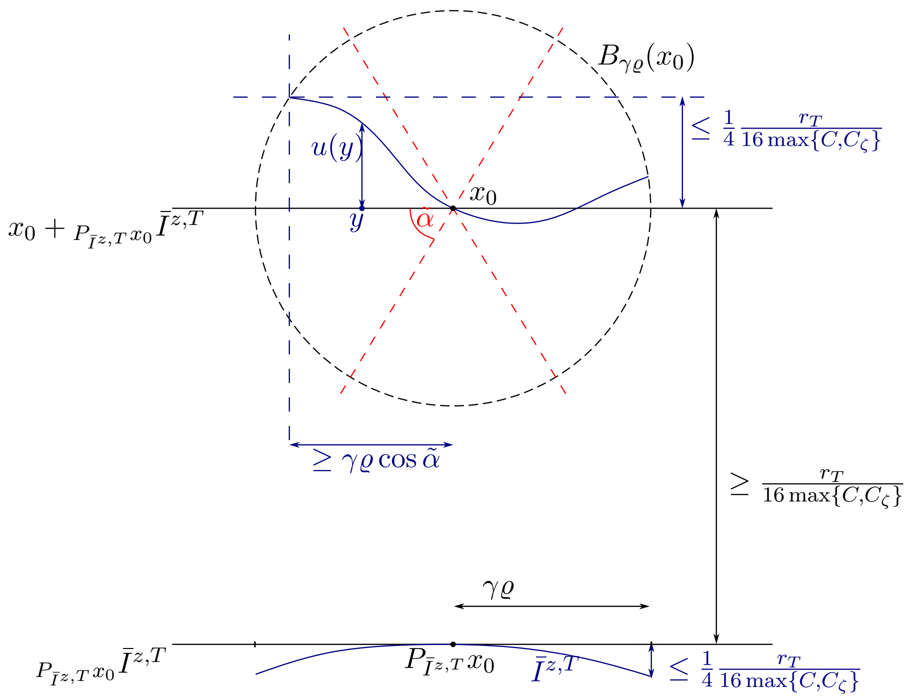

FIGURE 6. Height function estimates

for all $x_0 \in \operatorname{supp}|\nabla\chi_1|$. From (153) by means of (135) one may infer that $\sup|h'| \leq \frac{3}{2}\sup|u'| \leq 1/C'$. □

Proof of Lemma 21 (Error control). Define $\mathrm{n}_{x_0} := \mathrm{n}_{\bar{I}^{z,T}}(P_{\bar{I}^{z,T}}(x_0))$ and the set

$$A_{x_0} := B_\varrho(x_0) \cap \partial^*\Omega_1 \cap \{x : \mathcal{H}^0(\partial^*(\Omega_1)_{P_{G_{x_0}}(x)}) > 1\},$$

which we then decompose as

$$A_{x_0} = (A_{x_0} \cap \{\mathrm{n}_{\partial^*\Omega_1} \cdot \mathrm{n}_{x_0} \geq 1/2\}) \cup (A_{x_0} \cap \{\mathrm{n}_{\partial^*\Omega_1} \cdot \mathrm{n}_{x_0} < 1/2\}) =: A^{(1)}_{x_0} \cup A^{(2)}_{x_0}.$$

Since $\mathrm{n}_{\partial^*\Omega_1}(x)\cdot\xi^{z,T}(x) \leq |\xi^{z,T}(x) - \mathrm{n}_{x_0}(x)| + \mathrm{n}_{\partial^*\Omega_1}(x)\cdot\mathrm{n}_{x_0} \leq 3/4$ for any $x \in A^{(2)}_{x_0}$, then we obtain $1 - \mathrm{n}_{\partial^*\Omega_1}(x)\cdot\xi^{z,T}(x) \geq 1/4$ for any $x \in A^{(2)}_{x_0}$. Hence, we have

$$\begin{aligned}
\mathcal{H}^1(A^{(2)}_{x_0}) &\leq 4\int_{\partial^*\Omega_1} 1 - \mathrm{n}_{\partial^*\Omega_1}\cdot\xi^{z,T}\,d\mathcal{H}^1 \\
&\leq 4\int_{I_{1,P}} 1 - \mathrm{n}_{P,1}\cdot\xi^{z,T}\,d\mathcal{H}^1 + 8\sum_{j\notin\{1,P\}}\int_{I_{1,j}} 1\,d\mathcal{H}^1 \\
&\leq 4\int_{I_{1,P}} 1 - \mathrm{n}_{P,1}\cdot\xi^{z,T}\,d\mathcal{H}^1 + 16\sum_{i,j\notin\{1,P\}}\int_{I_{i,j}} 1 - \mathrm{n}_{i,j}\cdot\xi^{z,T}_{i,j}\,d\mathcal{H}^1 \\
&\leq 16E_{\mathrm{int}}[\chi|\bar{\chi}^{z,T}],
\end{aligned}$$

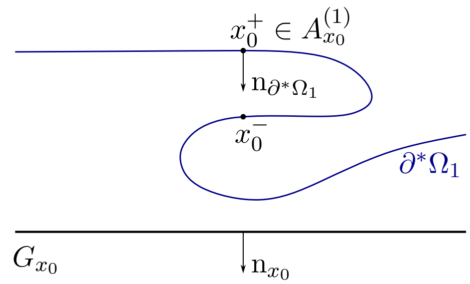

FIGURE 7. Error control by $E_{\mathrm{int}}[\chi|\bar\chi^{z,T}]$

where we use the fact that $|\xi^{z,T}_{i,j}| \leq 1/2$ for $\{i,j\} \neq \{1,P\}$. Moreover, using the coarea formula, we obtain

$$\begin{aligned}\mathcal{H}^1(A^{(1)}_{x_0}) &\leq 2\int_{B_\varrho(P_{\bar I^{z,T}}x_0)\cap G_{x_0}} \Big(\sum_{y\colon x+y\mathrm{n}_{x_0}\in A^{(1)}_{x_0}} 1\Big)\,d\mathcal{H}^1\\ &\leq 4\int_{B_\varrho(P_{\bar I^{z,T}}x_0)\cap G_{x_0}} \Big(\sum_{y\colon x+y\mathrm{n}_{x_0}\in A^{(2)}_{x_0}} 1\Big)\,d\mathcal{H}^1\\ &\leq 4\mathcal{H}^1(A^{(2)}_{x_0}),\end{aligned}$$

where the second inequality follows from the fact that, for any $x_0^+ \in A^{(1)}_{x_0}$, one may associate x_0^- such that $\mathrm{n}_{\partial^*\Omega_1}(x_0^-)\cdot \mathrm{n}_{x_0} \leq 0$ (see Figure 7). The claim follows by combining the two estimates above. □

7.3. **Proof of Lemma 5: Error functionals in perturbative regime.** The estimates (83) follow directly from the change of variables (184) to tubular neighborhood coordinates, the definition (61) of E_{bulk}, the identity (170) for the weight ϑ in the vicinity of the interface as well as the height function estimate (80), whereas the estimates (84) are immediate consequences of the coarea formula (183), the definition (60) of E_{int}, the representation (168) for the vector field ξ near the interface, the linearization (178) for the normal of the weak solution as well as the height function estimates (80)–(81). □

8. AUXILIARY COMPUTATIONS IN PERTURBATIVE REGIME

Let (ξ,ϑ,B) be the maps from Construction 2, and let (t_χ,z,T) be the space-time shifts from Lemma 3. Fix $t\in(0,t_\chi)$ and assume the existence of an height function $h(\cdot,t)$ satisfying the properties as in the conclusions of Proposition 4. For ease of notation, we will drop in the following any dependence on the time t. Furthermore, we will abbreviate in the tubular neighborhood $\{\mathrm{dist}(\cdot,\bar I^{z,T}) < r_T/2\}$

$$\begin{aligned} s_{\bar I^{z,T}} &:= \mathrm{sdist}_{\bar I^{z,T}},\\ \bar{\mathrm{n}}_{\bar I^{z,T}} &:= \mathrm{n}_{\bar I^{z,T}}\circ P_{\bar I^{z,T}}, \quad \bar\tau_{\bar I^{z,T}} := \tau_{\bar I^{z,T}}\circ P_{\bar I^{z,T}},\\ \bar H_{\bar I^{z,T}} &:= H_{\bar I^{z,T}}\circ P_{\bar I^{z,T}}.\end{aligned}$$

Finally, define $I := I_{1,P}$, which is by assumption subject to the graph representation (78)–(81), and denote $V := V_1$, $\mathrm{n} := \mathrm{n}_{P,1}$ as well as, by slight abuse of notation, $\chi := \chi_1$ and $\bar\chi := \bar\chi_1$.

Lemma 22. *For given $\delta_{\mathrm{err}} \in (0,1)$, one may choose the constants $C, C' \gg_{\delta_{\mathrm{err}}} 1$ from (80)–(81) such that the individual terms of the stability estimates (63) and (64) are estimated as follows:*

$$\begin{aligned} &-\int_I \big(\partial_t \xi^{z,T} + (B^{z,T}\cdot\nabla)\xi^{z,T} + (\nabla B^{z,T})^\mathsf{T}\xi^{z,T}\big)\cdot(\mathrm{n}-\xi^{z,T})\,d\mathcal{H}^1 \\ &\qquad\leq -\int_{\bar I^{z,T}} H^2_{\bar I^{z,T}} h(\mathrm{n}_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1 \\ &\qquad\quad -\int_{\bar I^{z,T}} H'_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\,d\mathcal{H}^1 \\ &\qquad\quad + \int_{\bar I^{z,T}} \delta_{\mathrm{err}}\Big(\frac{1}{r_T}\big|(\tau_{\bar I^{z,T}}\cdot\dot z)h'\big| + \big|H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\big|\Big)\,d\mathcal{H}^1, \end{aligned} \tag{154}$$

$$-\int_I \big(\partial_t \xi^{z,T} + (B^{z,T}\cdot\nabla)\xi^{z,T}\big)\cdot\xi^{z,T}\,d\mathcal{H}^1 = 0, \tag{155}$$

$$\begin{aligned} &\int_I \frac{1}{2}\big|\nabla\cdot\xi^{z,T} + B^{z,T}\cdot\xi^{z,T}\big|^2\,d\mathcal{H}^1 \\ &\qquad\leq \int_{\bar I^{z,T}} \frac{1}{2}H^4_{\bar I^{z,T}}h^2 + \delta_{\mathrm{err}}\frac{1}{r_T^4}h^2\,d\mathcal{H}^1, \end{aligned} \tag{156}$$

$$-\int_I \frac{1}{2}\big|B^{z,T}\cdot\xi^{z,T}\big|\big(1-|\xi^{z,T}|^2\big)\,d\mathcal{H}^1 = 0, \tag{157}$$

$$\begin{aligned} &-\int_I (1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot\xi^{z,T}(B^{z,T}\cdot\xi^{z,T})\,d\mathcal{H}^1 \\ &\qquad\leq \int_{\bar I^{z,T}} \frac{1}{2}H^2_{\bar I^{z,T}}(h')^2 + \delta_{\mathrm{err}}\frac{1}{r_T^2}(h')^2\,d\mathcal{H}^1, \end{aligned} \tag{158}$$

$$\int_I \big((\mathrm{Id}-\xi^{z,T}\otimes\xi^{z,T})B^{z,T}\big)\cdot(V+\nabla\cdot\xi^{z,T})\mathrm{n}\,d\mathcal{H}^1 = 0, \tag{159}$$

$$\begin{aligned} &\int_I (1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot B^{z,T}\,d\mathcal{H}^1 \\ &\qquad\leq -\int_{\bar I^{z,T}} \frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1 + \int_{\bar I^{z,T}} \delta_{\mathrm{err}}\frac{1}{r_T^2}(h')^2\,d\mathcal{H}^1, \end{aligned} \tag{160}$$

$$\begin{aligned} &-\int_I (\mathrm{n}-\xi^{z,T})\otimes(\mathrm{n}-\xi^{z,T}) : \nabla B^{z,T}\,d\mathcal{H}^1 \\ &\qquad\leq \int_{\bar I^{z,T}} H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1 + \int_{\bar I^{z,T}} \delta_{\mathrm{err}}\Big(\frac{1}{r_T^2} + \big|H'_{\bar I^{z,T}}\big|\Big)(h')^2\,d\mathcal{H}^1, \end{aligned} \tag{161}$$

$$\begin{aligned} &-\int_I \frac{1}{2}\big|V+\nabla\cdot\xi^{z,T}\big|^2\,d\mathcal{H}^1 \\ &\qquad\leq -\int_{\bar I^{z,T}} \frac{1}{2}(h'')^2\,d\mathcal{H}^1 \\ &\qquad\quad + \int_{\bar I^{z,T}} \delta_{\mathrm{err}}\Big((h'')^2 + \frac{1}{r_T^2}(h')^2 + \Big((H'_{\bar I^{z,T}})^2 + \frac{1}{r_T^4}\Big)h^2\Big)\,d\mathcal{H}^1 \end{aligned} \tag{162}$$

$$\begin{aligned} &-\int_I \frac{1}{2}\big|V\mathrm{n}-(B^{z,T}\cdot\xi^{z,T})\xi^{z,T}\big|^2\,d\mathcal{H}^1 \\ &\qquad\leq -\int_{\bar I^{z,T}} \frac{1}{2}\Big((h'')^2 + H^4_{\bar I^{z,T}}h^2 - H^2_{\bar I^{z,T}}(h')^2\Big)\,d\mathcal{H}^1 \end{aligned} \tag{163}$$

$$+\int_{\bar{I}^{z,T}} 2H_{\bar{I}^{z,T}}H'_{\bar{I}^{z,T}}hh'\,d\mathcal{H}^1$$
$$+\int_{\bar{I}^{z,T}} \delta_{\mathrm{err}}\Big((h'')^2+\frac{1}{r_T^2}(h')^2+\Big((H'_{\bar{I}^{z,T}})^2+\frac{1}{r_T^4}\Big)h^2\Big)\,d\mathcal{H}^1,$$

$$\int_I \vartheta^{z,T}(B^{z,T}\cdot\xi^{z,T}-V)\,d\mathcal{H}^1$$
$$\leq \int_{\bar{I}^{z,T}}\frac{H^2_{\bar{I}^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1-\int_{\bar{I}^{z,T}}\frac{1}{r_T^2}(h')^2\,d\mathcal{H}^1 \tag{164}$$
$$+\int_{\bar{I}^{z,T}} \delta_{\mathrm{err}}\Big((h'')^2+\frac{1}{r_T^2}(h')^2+\Big((H'_{\bar{I}^{z,T}})^2+\frac{1}{r_T^4}\Big)h^2\Big)\,d\mathcal{H}^1,$$

$$\int_I \vartheta^{z,T}B^{z,T}\cdot(\mathrm{n}-\xi^{z,T})\,d\mathcal{H}^1$$
$$\leq \int_{\bar{I}^{z,T}}\delta_{\mathrm{err}}\Big(\frac{1}{r_T^4}h^2+\frac{1}{r_T^2}(h')^2\Big)\,d\mathcal{H}^1, \tag{165}$$

$$\int_{\mathbb{R}^2}(\chi-\bar{\chi}^{z,T})\vartheta^{z,T}\nabla\cdot B^{z,T}\,dx$$
$$\leq -\int_{\bar{I}^{z,T}}\frac{1}{2}\frac{H^2_{\bar{I}^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1+\int_{\bar{I}^{z,T}}\delta_{\mathrm{err}}\frac{1}{r_T^4}h^2\,d\mathcal{H}^1, \tag{166}$$

$$\int_{\mathbb{R}^2}(\chi-\bar{\chi}^{z,T})\big(\partial_t\vartheta^{z,T}+(B^{z,T}\cdot\nabla)\vartheta^{z,T}\big)\,dx$$
$$\leq \int_{\bar{I}^{z,T}}\frac{1}{r_T^4}h^2\,d\mathcal{H}^1 \tag{167}$$
$$-\int_{\bar{I}^{z,T}}\frac{1}{r_T^2}H_{\bar{I}^{z,T}}h\dot{\mathfrak{T}}\,d\mathcal{H}^1-\int_{\bar{I}^{z,T}}\frac{1}{r_T^2}h(\mathrm{n}_{\bar{I}^{z,T}}\cdot\dot{z})\,d\mathcal{H}^1$$
$$+\int_{\bar{I}^{z,T}}\delta_{\mathrm{err}}\Big(\frac{1}{r_T^3}\big|\dot{\mathfrak{T}}h\big|+\frac{1}{r_T^2}\big|(\mathrm{n}_{\bar{I}^{z,T}}\cdot\dot{z})h\big|+\frac{1}{r_T^4}h^2\Big)\,d\mathcal{H}^1.$$

Proof. We proceed in several steps.

Step 1: Properties of gradient flow calibration. Thanks to the height function estimate (80), the definitions (72)–(74) of the gradient flow calibration, and the identities (21)–(23) for the shifted geometry, it holds on $I\subset\{\mathrm{dist}(\cdot,\bar{I}^{z,T})<r_T/8\}$

$$\xi^{z,T}=\bar{\mathrm{n}}_{\bar{I}^{z,T}}=\nabla s_{\bar{I}^{z,T}}, \tag{168}$$
$$B^{z,T}=\bar{H}_{\bar{I}^{z,T}}\bar{\mathrm{n}}_{\bar{I}^{z,T}}, \tag{169}$$
$$\vartheta^{z,T}=-\frac{s_{\bar{I}^{z,T}}}{r_T^2}. \tag{170}$$

In particular, because of

$$\nabla P_{\bar{I}^{z,T}}=\mathrm{Id}-\bar{\mathrm{n}}_{\bar{I}^{z,T}}\otimes\bar{\mathrm{n}}_{\bar{I}^{z,T}}-s_{\bar{I}^{z,T}}\nabla\bar{\mathrm{n}}_{\bar{I}^{z,T}},$$

we obtain by direct computation throughout $\{\mathrm{dist}(\cdot,\bar{I}^{z,T})<r_T/8\}$

$$\nabla\xi^{z,T}=-\frac{\bar{H}_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}\bar{\tau}_{\bar{I}^{z,T}}\otimes\bar{\tau}_{\bar{I}^{z,T}}, \tag{171}$$
$$\nabla\cdot\xi^{z,T}=-\frac{\bar{H}_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}, \tag{172}$$

$$\nabla B^{z,T} = -\frac{\bar{H}^2_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}\bar{\tau}_{\bar{I}^{z,T}}\otimes\bar{\tau}_{\bar{I}^{z,T}} + \frac{\bar{H}'_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}\bar{\mathrm{n}}_{\bar{I}^{z,T}}\otimes\bar{\tau}_{\bar{I}^{z,T}}, \tag{173}$$

$$\nabla\cdot B^{z,T} = -\frac{\bar{H}^2_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}, \tag{174}$$

$$\nabla\vartheta^{z,T} = -\frac{1}{r_T^2}\bar{\mathrm{n}}_{\bar{I}^{z,T}}. \tag{175}$$

Note that these computations are justified thanks to $|1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}| \geq 1/2$ being valid throughout $\{\mathrm{dist}(\cdot,\bar{I}^{z,T}) < r_T/8\}$, which in turn follows from $\bar{H}_{\bar{I}^{z,T}} \leq 2/r_T$ since, by assumption, $r_T/2$ is an admissible tubular neighborhood width for $\bar{I}^{z,T}$ (cf. Definition 4). Within $\{\mathrm{dist}(\cdot,\bar{I}^{z,T}) < r_T/8\}$, we also record the following simplifications of (104) and (106):

$$\partial_t\xi^{z,T} = -\big(1+\dot{\mathfrak{T}}\big)\frac{H'_{\bar{I}^{z,T}}\circ P_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}\bar{\tau}_{\bar{I}^{z,T}} + \big(\bar{\tau}_{\bar{I}^{z,T}}\cdot\dot{z}\big)\frac{\bar{H}_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}}}\bar{\tau}_{\bar{I}^{z,T}}, \tag{176}$$

$$\partial_t\vartheta^{z,T} = -\frac{1+\dot{\mathfrak{T}}}{r_T^2}\Big(2\frac{s_{\bar{I}^{z,T}}}{r_T^2} - \bar{H}_{\bar{I}^{z,T}}\Big) + \frac{\bar{\mathrm{n}}_{\bar{I}^{z,T}}\cdot\dot{z}}{r_T^2}. \tag{177}$$

Step 2: Identities for geometric quantities of perturbed interface. First, we define $(\mathfrak{h},\mathfrak{h}',\mathfrak{h}'') := (h,h',h'')\circ P_{\bar{I}^{z,T}}$. Then, denoting by $o(1)$ any quantity $f(\bar{H}_{\bar{I}^{z,T}}\mathfrak{h},\mathfrak{h}')$ such that $f\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $f(x_1,x_2)\to 0$ whenever $|(x_1,x_2)|\to 0$, we claim that along I

$$\mathrm{n} = \bigg(1+\Big(-\frac{1}{2}+o(1)\Big)(\mathfrak{h}')^2\bigg)\bar{\mathrm{n}}_{\bar{I}^{z,T}} - \big(1+o(1)\big)\mathfrak{h}'\bar{\tau}_{\bar{I}^{z,T}}, \tag{178}$$

$$V = \frac{\bar{H}_{\bar{I}^{z,T}}}{1-\bar{H}_{\bar{I}^{z,T}}\mathfrak{h}} + \mathfrak{h}'' + o(1)\mathfrak{h}'' + o(1)\mathfrak{h}'\bar{H}_{\bar{I}^{z,T}} + o(1)\mathfrak{h}\bar{H}'_{\bar{I}^{z,T}}. \tag{179}$$

For a proof of (178)–(179), it is computationally convenient to represent the interface I as the image of the curve $\gamma_h := (\mathrm{id}+h\bar{\mathrm{n}}_{\bar{I}^{z,T}})\circ\bar{\gamma}^{z,T}$, where $\bar{\gamma}$ is an arc-length parametrization of $\bar{I}(T^{-1}(\cdot))$ such that $\bar{\tau}_{\bar{I}^{z,T}}\circ\bar{\gamma}^{z,T} = (\bar{\gamma}^{z,T})'$. Then

$$\gamma_h' = \bigg(\big(1-\bar{H}_{\bar{I}^{z,T}}h\big)\bar{\tau}_{\bar{I}^{z,T}} + h'\bar{\mathrm{n}}_{\bar{I}^{z,T}}\bigg)\circ\bar{\gamma}^{z,T}, \tag{180}$$

hence (recall that $J\in\mathbb{R}^{2\times 2}$ denotes counter-clockwise rotation by 90°)

$$\mathrm{n}\circ\gamma_h = J\frac{\gamma_h'}{|\gamma_h'|} = \Bigg(\frac{\big(1-\bar{H}_{\bar{I}^{z,T}}h\big)\bar{\mathrm{n}}_{\bar{I}^{z,T}} - h'\bar{\tau}_{\bar{I}^{z,T}}}{\sqrt{\big(1-\bar{H}_{\bar{I}^{z,T}}h\big)^2+(h')^2}}\Bigg)\circ\bar{\gamma}^{z,T}, \tag{181}$$

so that (178) follows from Taylor expansion with respect to the variables $\bar{H}_{\bar{I}^{z,T}}h$ and h'. By virtue of H^2 regularity of the height function h and V being the distributional curvature of I due to [9, Definition 13, item iii)] and $\chi_i\equiv 0$ for all $i\notin\{1,P\}$, we deduce $V = H_{\gamma_h}$. In other words,

$$\begin{aligned} V\circ\gamma_h &= \frac{\gamma_h''\cdot J\gamma_h'}{|\gamma_h'|^3} \\ &= \Bigg(\frac{h'\big(\bar{H}'_{\bar{I}^{z,T}}h + 2\bar{H}_{\bar{I}^{z,T}}h'\big) + \big(1-\bar{H}_{\bar{I}^{z,T}}h\big)\big(h''+\bar{H}_{\bar{I}^{z,T}}(1-\bar{H}_{\bar{I}^{z,T}}h)\big)}{\sqrt{\big(1-\bar{H}_{\bar{I}^{z,T}}h\big)^2+(h')^2}^{\,3}}\Bigg)\circ\bar{\gamma}^{z,T}, \end{aligned} \tag{182}$$

so that (179) again follows from Taylor expansion with respect to the variables $\bar{H}_{\bar{I}^{z,T}}h$, h' and h''.

Step 3: Change of variables formula. Let $g\colon I \to \mathbb{R}$ be integrable. Then, by the coarea formula

$$\int_I g\,d\mathcal{H}^1 = \int_{\bar{I}^{z,T}} g\circ(\mathrm{id}{+}h\bar{\mathrm{n}}_{\bar{I}^{z,T}})\sqrt{\big(1{-}H_{\bar{I}^{z,T}}h\big)^2+(h')^2}\,d\mathcal{H}^1. \tag{183}$$

Furthermore, since the Jacobian of the tubular neighborhood diffeomorphism $x\mapsto(P_{\bar{I}^{z,T}}(x), s_{\bar{I}^{z,T}})$ is given by $1/(1{-}\bar{H}_{\bar{I}^{z,T}}s_{\bar{I}^{z,T}})$, we also obtain for any integrable $G\colon\mathbb{R}^2\to\mathbb{R}$ with $\operatorname{supp}G\subset\{\mathrm{dist}(\cdot,\bar{I}^{z,T})<r_T/4\}$ by the area formula and the assumed conclusions of Proposition 4

$$\int_{\mathbb{R}^2}(\chi-\bar{\chi}^{z,T})G\,dx = -\int_{\bar{I}^{z,T}}\int_0^h \frac{G\big(x{+}s\bar{\mathrm{n}}_{\bar{I}^{z,T}}(x)\big)}{1{-}\bar{H}_{\bar{I}^{z,T}}(x)s}\,ds d\mathcal{H}^1(x). \tag{184}$$

Step 4: Collecting further auxiliary identities. We obtain from (178) and (168) that, along I,

$$\mathrm{n}-\xi^{z,T} = o(1)\mathfrak{h}'\bar{\mathrm{n}}_{\bar{I}^{z,T}} - \big(1{+}o(1)\big)\mathfrak{h}'\bar{\tau}_{\bar{I}^{z,T}}, \tag{185}$$

$$1-\mathrm{n}\cdot\xi^{z,T} = \Big(\frac{1}{2}+o(1)\Big)(\mathfrak{h}')^2. \tag{186}$$

In addition, we infer from (179) and (172) that, along I,

$$V+\nabla\cdot\xi^{z,T} = \mathfrak{h}''+o(1)\mathfrak{h}''+o(1)\mathfrak{h}'\bar{H}_{\bar{I}^{z,T}}+o(1)\mathfrak{h}\bar{H}'_{\bar{I}^{z,T}}, \tag{187}$$

as well as from (178), (179) and (168)–(169)

$$\begin{aligned}
&V\mathrm{n}-(B^{z,T}\cdot\xi^{z,T})\xi^{z,T}\\
&\quad= V\mathrm{n}-B^{z,T}\\
&\quad= \big(\bar{H}^2_{\bar{I}^{z,T}}\mathfrak{h}{+}\mathfrak{h}''\big)\bar{\mathrm{n}}_{\bar{I}^{z,T}} - \bar{H}_{\bar{I}^{z,T}}\mathfrak{h}'\bar{\tau}_{\bar{I}^{z,T}}\\
&\qquad+\big(o(1)\mathfrak{h}''+o(1)\bar{H}_{\bar{I}^{z,T}}\mathfrak{h}'+o(1)(\bar{H}'_{\bar{I}^{z,T}}\circ P_{\bar{I}^{z,T}})\mathfrak{h}+o(1)\bar{H}^2_{\bar{I}^{z,T}}\mathfrak{h}\big)\bar{\mathrm{n}}_{\bar{I}^{z,T}}\\
&\qquad+\big(o(1)\mathfrak{h}''+o(1)\bar{H}_{\bar{I}^{z,T}}\mathfrak{h}'+o(1)(\bar{H}'_{\bar{I}^{z,T}}\circ P_{\bar{I}^{z,T}})\mathfrak{h}+o(1)\bar{H}^2_{\bar{I}^{z,T}}\mathfrak{h}\big)\bar{\tau}_{\bar{I}^{z,T}}
\end{aligned} \tag{188}$$

and

$$B^{z,T}\cdot\xi^{z,T}-V = -\bar{H}^2_{\bar{I}^{z,T}}\mathfrak{h}-\mathfrak{h}''+o(1)\mathfrak{h}''+o(1)\mathfrak{h}'\bar{H}_{\bar{I}^{z,T}}+o(1)\mathfrak{h}\bar{H}'_{\bar{I}^{z,T}}+o(1)\mathfrak{h}\bar{H}^2_{\bar{I}^{z,T}}. \tag{189}$$

Next, we exploit the information gathered so far to express the terms originating from the stability estimate of E_{int} in terms of the height function h (and its derivatives). First, we get from combining (183), (176), (168)–(169), (171), (173) and (185),

$$
\begin{aligned}
&-\int_I \big(\partial_t\xi^{z,T}+(B^{z,T}\cdot\nabla)\xi^{z,T}+(\nabla B^{z,T})^{\mathsf{T}}\xi^{z,T}\big)\cdot(\mathrm{n}-\xi^{z,T})\,d\mathcal{H}^1\\
&\leq \int_{\bar I^{z,T}} H_{\bar I^{z,T}}h'(\tau_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1-\int_{\bar I^{z,T}} H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\,d\mathcal{H}^1\\
&\quad+\int_{\bar I^{z,T}}|o(1)|\Big(\big|H_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h'\big|+\big|H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\big|\Big)\,d\mathcal{H}^1\\
&=-\int_{\bar I^{z,T}} H^2_{\bar I^{z,T}}h(\mathrm{n}_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1\\
&\quad-\int_{\bar I^{z,T}} H'_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h\,d\mathcal{H}^1-\int_{\bar I^{z,T}} H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\,d\mathcal{H}^1\\
&\quad+\int_{\bar I^{z,T}}|o(1)|\Big(\big|H_{\bar I^{z,T}}(\tau_{\bar I^{z,T}}\cdot\dot z)h'\big|+\big|H'_{\bar I^{z,T}}\dot{\mathfrak{T}}h'\big|\Big)\,d\mathcal{H}^1,
\end{aligned}
\tag{190}
$$

where in the last step we also integrated by parts. Next, it directly follows from (183), (168)–(169) and (172)

$$
\int_I \frac{1}{2}\big|\nabla\cdot\xi^{z,T}+B^{z,T}\cdot\xi^{z,T}\big|^2\,d\mathcal{H}^1\leq\int_{\bar I^{z,T}}\big(1+|o(1)|\big)\frac{1}{2}H^4_{\bar I^{z,T}}h^2\,d\mathcal{H}^1,
\tag{191}
$$

and exploiting in addition (186)

$$
\begin{aligned}
&-\int_I(1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot\xi^{z,T}(B^{z,T}\cdot\xi^{z,T})\,d\mathcal{H}^1\\
&\leq\int_{\bar I^{z,T}}\big(1+|o(1)|\big)\frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1.
\end{aligned}
\tag{192}
$$

Analogously, recalling also (173) and (174),

$$
\int_I(1-\mathrm{n}\cdot\xi^{z,T})\nabla\cdot B^{z,T}\,d\mathcal{H}^1\leq-\int_{\bar I^{z,T}}\big(1-|o(1)|\big)\frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1
\tag{193}
$$

as well as

$$
\begin{aligned}
&-\int_I(\mathrm{n}-\xi^{z,T})\otimes(\mathrm{n}-\xi^{z,T}):\nabla B^{z,T}\,d\mathcal{H}^1\\
&\leq\int_{\bar I^{z,T}}\big(1+|o(1)|\big)H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1+\int_{\bar I^{z,T}}|o(1)|\big|H'_{\bar I^{z,T}}\big|(h')^2\,d\mathcal{H}^1.
\end{aligned}
\tag{194}
$$

Just plugging in (187) and estimating by Young's inequality yields

$$
\begin{aligned}
&-\int_I\frac{1}{2}\big|V+\nabla\cdot\xi^{z,T}\big|^2\,d\mathcal{H}^1\\
&\leq-\int_{\bar I^{z,T}}\big(1-|o(1)|\big)\frac{1}{2}(h'')^2\,d\mathcal{H}^1\\
&\quad+\int_{\bar I^{z,T}}|o(1)|\Big(H^2_{\bar I^{z,T}}(h')^2+\big((H'_{\bar I^{z,T}})^2+H^4_{\bar I^{z,T}}\big)h^2\Big)\,d\mathcal{H}^1,
\end{aligned}
\tag{195}
$$

and analogously based on (188)

$$
\begin{aligned}
&-\int_I \frac{1}{2}\big|V - (B^{z,T}\cdot\xi^{z,T})\xi^{z,T}\big|^2\,d\mathcal{H}^1\\
&\le -\int_{\bar I^{z,T}} \big(1-|o(1)|\big)\frac{1}{2}\Big((h'')^2 + H^2_{\bar I^{z,T}}(h')^2 + H^4_{\bar I^{z,T}}h^2\Big)\,d\mathcal{H}^1\\
&\quad -\int_{\bar I^{z,T}} H^2_{\bar I^{z,T}}hh''\,d\mathcal{H}^1\\
&\quad +\int_{\bar I^{z,T}} |o(1)|(H'_{\bar I^{z,T}})^2h^2\,d\mathcal{H}^1\\
&\le -\int_{\bar I^{z,T}} \big(1-|o(1)|\big)\frac{1}{2}\Big((h'')^2 + H^4_{\bar I^{z,T}}h^2\Big)\,d\mathcal{H}^1\\
&\quad +\int_{\bar I^{z,T}} \frac{1}{2}H^2_{\bar I^{z,T}}(h')^2\,d\mathcal{H}^1\\
&\quad +\int_{\bar I^{z,T}} 2H_{\bar I^{z,T}}H'_{\bar I^{z,T}}hh'\,d\mathcal{H}^1\\
&\quad +\int_{\bar I^{z,T}} |o(1)|\Big(H^2_{\bar I^{z,T}}(h')^2 + (H'_{\bar I^{z,T}})^2h^2\Big)\,d\mathcal{H}^1,
\end{aligned}
\tag{196}
$$

where in the last step we also performed an integration by parts and estimated by Young's inequality.

We continue with the terms originating from the stability estimate of E_{bulk}. First, by means of (183), (170), (189) and integration by parts we obtain

$$
\begin{aligned}
&\int_I \vartheta^{z,T}(B^{z,T}\cdot\xi^{z,T} - V)\,d\mathcal{H}^1\\
&\le \int_{\bar I^{z,T}} \frac{H^2_{\bar I^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} \frac{1}{r_T^2}(h')^2\,d\mathcal{H}^1\\
&\quad + \int_{\bar I^{z,T}} |o(1)|\bigg(\Big(\frac{H^2_{\bar I^{z,T}}}{r_T^2} + \frac{1}{r_T^4} + (H'_{\bar I^{z,T}})^2\Big)h^2 + H^2_{\bar I^{z,T}}(h')^2 + (h'')^2\bigg)\,d\mathcal{H}^1.
\end{aligned}
\tag{197}
$$

Next, just plugging in (169)–(170) and (185) into (183) and applying Young's inequality entails

$$
\int_I \vartheta^{z,T}B^{z,T}\cdot(\mathrm{n} - \xi^{z,T})\,d\mathcal{H}^1 \le \int_{\bar I^{z,T}} |o(1)|\Big(\frac{1}{r_T^4}h^2 + H^2_{\bar I^{z,T}}(h')^2\Big)\,d\mathcal{H}^1.
\tag{198}
$$

In addition, based on (184), (170) and (174), we may infer

$$
\int_{\mathbb{R}^2} (\chi - \bar\chi^{z,T})\vartheta^{z,T}\nabla\cdot B^{z,T}\,dx \le -\int_{\bar I^{z,T}} \big(1-|o(1)|\big)\frac{1}{2}\frac{H^2_{\bar I^{z,T}}}{r_T^2}h^2\,d\mathcal{H}^1,
\tag{199}
$$

whereas it finally follows from (184), (177), (169) and (175)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (\chi - \bar\chi^{z,T})\big(\partial_t\vartheta^{z,T} + (B^{z,T}\cdot\nabla)\vartheta^{z,T}\big)\,dx\\
&\le \int_{\bar I^{z,T}} \frac{1}{r_T^4}h^2\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} \frac{H_{\bar I^{z,T}}}{r_T^2}h\dot{\mathfrak{T}}\,d\mathcal{H}^1 - \int_{\bar I^{z,T}} \frac{1}{r_T^2}h(\mathrm{n}_{\bar I^{z,T}}\cdot\dot z)\,d\mathcal{H}^1\\
&\quad + \int_{\bar I^{z,T}} |o(1)|\bigg(\frac{1}{r_T^3}\big|\dot{\mathfrak{T}}h\big| + \frac{1}{r_T^2}\big|H_{\bar I^{z,T}}\dot{\mathfrak{T}}h\big| + \frac{1}{r_T^2}\big|(\mathrm{n}_{\bar I^{z,T}}\cdot\dot z)h\big| + \frac{1}{r_T^4}h^2\bigg)\,d\mathcal{H}^1.
\end{aligned}
\tag{200}
$$

Step 5: Conclusion. Due to (168) and (169), the identities (155), (157) and (159) hold true for trivial reasons. The remaining estimates follow from (190)–(200) and $|H_{\bar{I}^{z,T}}| \leq 2/r_T$. □

Appendix A. Auxiliary results from geometric measure theory

In this appendix, we recall the two main ingredients from geometric measure theory which we use to prove Proposition 4, namely the Allard's regularity theory for integer rectifiable varifolds [22, Chapter 5, Theorem 23.1 and Remark 23.2(a)] and the decoposition of a reduced boundary of a set of finite perimeter in $\mathbb{R}^2$ into rectifiable Jordan-Lipschitz curves [2, Section 6, Theorem 4].

Using the notation from Definition 3 and omitting the dependence in time, we recall the usual definition of tilt-excess E_{tilt} for an integer rectifiable 1-varifold $\mathcal{V}$ with associated mass measure μ, namely

$$E_{\mathrm{tilt}}[x_0,\varrho,G] = \varrho^{-1}\int_{B_\varrho(x_o)} |P_{\mathrm{Tan}_{x_0}(\operatorname{supp}\mu)} - P_G|^2 \,\mathrm{d}\mu\,,$$

where $x_0 \in \operatorname{supp}\mu$, $\varrho > 0$, and G is a one dimensional subspace of $\mathbb{R}^2$, whreas P denotes the projection onto the approximate tangent space $\mathrm{Tan}_{x_0}(\operatorname{supp}\mu)$ and onto the given subspace G, respectively. In our setting, Allard's regularity theorem [22, Chapter 5, Theorem 23.1 and Remark 23.2(a)] reads as follows.

Theorem 23 (Allard's regularity theory)**.** *Fix $\varrho > 0$, $p > 1$, $x_0 \in \operatorname{supp}\mu$, and a one-dimensional subspace G of $\mathbb{R}^2$. There exist $\varepsilon = \varepsilon(p)$, $\gamma = \gamma(p) \in (0,1)$ and $C_{Allard} = C_{Allard}(p) > 0$ such that: if*

$$\frac{\mu(B_\varrho(x_0))}{\mathrm{Vol}_1\,\varrho} \leq 1+\varepsilon, \tag{201}$$

where Vol_1 denotes the one-dimensional volume of the unit ball, and

$$\max\left\{E_{\mathrm{tilt}}[x_0,\varrho,G], \varepsilon^{-1}\Big(\int_{B_\varrho(x_0)} |H_\mu|^p\,dx\Big)^{\frac{2}{p}}\varrho^{2(1-\frac{1}{p})}\right\} \leq \varepsilon, \tag{202}$$

then there exists a $C^{1,1-\frac{1}{p}}$ function $u\colon (x_0+G)\cap B_{\gamma\varrho}(x_0) \to \mathbb{R}$ satisfying:

i) $u(x_0) = 0$;
ii) $\operatorname{supp}\mu \cap B_{\gamma\varrho}(x_0) = B_{\gamma\varrho}(x_0) \cap \big\{y + u(y)\mathrm{n}_G(y) : y \in (x_0+G)\cap B_{\gamma\varrho}(x_0)\big\}$, *where n_G is the normal vector field to the affine space $x_0 + G$;*
iii) it holds

$$\begin{aligned}&\varrho^{-1}\sup_x |u(x)| + \sup_x |\nabla u(x)| + \varrho^{1-\frac{1}{p}} \sup_{x\neq y}\big\{|x-y|^{-(1-\frac{1}{p})}|\nabla u(x)-\nabla u(y)|\big\}\\ &\leq C_{Allard}\bigg[E_{\mathrm{tilt}}^{\frac{1}{2}}[x_0,\varrho,G] + \Big(\int_{B_\varrho(x_0)} |H_\mu|^p\,dx\Big)^{\frac{1}{p}}\varrho^{1-\frac{1}{p}}\bigg].\end{aligned} \tag{203}$$

The second result makes use of the notion of Jordan-Lipschitz curves, for which we refer the reader to [2, Section 8], and states a decomposition for the reduced boundary of a planar set of finite perimeter (see Figures 8 and 9 for an example) as given by [2, Section 8, Corollary 1].

Theorem 24 (Decomposition result for planar sets of finite perimeters)**.** *Let $\chi \in BV(\mathbb{R}^2;\{0,1\})$ be the indicator function associated to a set of finite perimeter. Then, $\operatorname{supp}|\nabla\chi|$ can be uniquely decomposed into a countable family of Jordan-Lipschitz curves $\{J_i^+, J_k^- : i,k \in \mathbb{N}\}$ such that the following properties hold:*

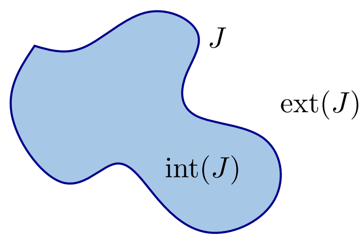

FIGURE 8. Jordan curve J

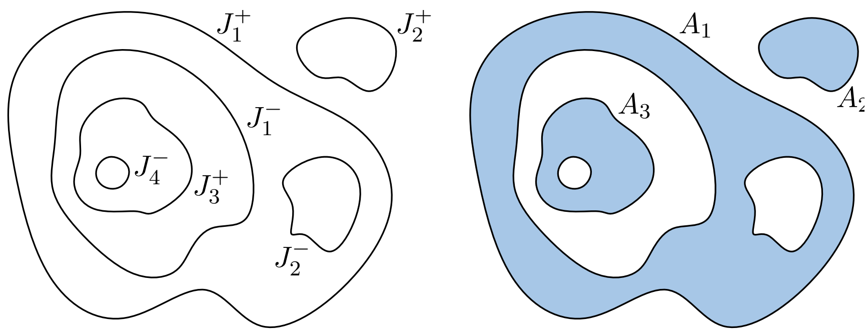

FIGURE 9. Decomposition of the reduced boundary of a planar set of finite perimeter into Jordan-Lipschitz curves $\{J_i^+, J_k^- : i, k \in \mathbb{N}\}$.

i) *Let* $i \neq k$, *then either* $\mathrm{int}(J_i^+) \cap \mathrm{int}(J_k^+) = \emptyset$ *or* $\mathrm{int}(J_i^+) \subseteq \mathrm{int}(J_k^+)$ *and analogously for* $\{J_i^- : i \in \mathbb{N}\}$. *Furthermore, for each* $i \in \mathbb{N}$ *there exists* $k = k(i) \in \mathbb{N}$ *such that* $\mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_k^+)$.

ii) $\mathcal{H}^1(\mathrm{supp}\,|\nabla\chi|) = \sum_i \mathcal{H}^1(J_i^+) + \sum_k \mathcal{H}^1(J_k^-)$.

iii) *If* $\mathrm{int}(J_i^+) \subseteq \mathrm{int}(J_k^+)$, *there exists* $j \in \mathbb{N}$ *such that* $\mathrm{int}(J_i^+) \subseteq \mathrm{int}(J_j^-) \subseteq \mathrm{int}(J_k^+)$, *and analogously for the roles of* $\{J_i^+ : i \in \mathbb{N}\}$ *and* $\{J_k^- : i \in \mathbb{N}\}$ *switched.*

iv) *Setting* $L_k := \{i \in \mathbb{N} : \mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_k^+)\}$, *then the sets* $A_k := \mathrm{int}(J_k^+) \setminus \cup_{i \in L_k} \mathrm{int}(J_i^-)$ *are piecewise disjoint, indecomposable, and* $\chi = \sum_k \chi_{A_k}$.

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