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Exercise sheet 6 to Advanced Partial Differential Equations

Exercise 1 (Poincaré inequality)**.** Let $\Omega \subset \mathbb{R}^d$ be open, bounded, connected, and of class C^1. Prove that there exists $C = C(\Omega) > 0$ such that for all $\varphi \in W^{1,2}(\Omega)$ it holds

$$\|\varphi - (\varphi)_\Omega\|^2_{L^2(\Omega)} \leq C \sum_{j=1}^{d} \int_\Omega |D_j \varphi|^2 \, dx,$$

where $(\varphi)_\Omega = \frac{1}{|\Omega|} \int_\Omega \varphi \, dx$ is the average of φ over Ω.

Hint: If not, there exists for all $n \in \mathbb{N}$ some $\varphi_n \in W^{1,2}(\Omega)$ such that $(\varphi_n)_\Omega = 0$, $\|\varphi_n\|^2_{L^2(\Omega)} = 1$, and $1 > n \sum_{j=1}^d \int_\Omega |D_j \varphi_n|^2 \, dx$ (why?). Derive a contradiction.

Exercise 2 (Ehrling inequality)**.** Let $\Omega \subset \mathbb{R}^d$ be open, bounded, and of class C^k. Let $0 \leq l < m \leq k$ for $l, m \in \mathbb{N}$. Then, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $\varphi \in W^{m,2}(\Omega)$ it holds

$$\|\varphi\|_{W^{l,2}(\Omega)} \leq \varepsilon \|\varphi\|_{W^{m,2}(\Omega)} + C_\varepsilon \|\varphi\|_{L^2(\Omega)}.$$

Hint: If not, there exists $\varepsilon > 0$ and, for each $n \in \mathbb{N}$, a function $\varphi_n \in W^{m,2}(\Omega)$ such that $\|\varphi_n\|_{W^{m,2}(\Omega)} = 1$ and $\|\varphi_n\|_{W^{l,2}(\Omega)} > \varepsilon + n \|\varphi_n\|_{L^2(\Omega)}$ (why?). Derive a contradiction.

Exercise 3 (Kernel of trace operator)**.**

i) Let $T_1^1 \colon W^{1,2}(\Omega) \to W^{\frac{1}{2},2}(\partial\Omega)$ be the trace operator from lecture for $\Omega \subset \mathbb{R}^d$ open, bounded, and of class C^1. Prove $\ker T_1^1 = W_0^{1,2}(\Omega)$.

ii) Let $T_m^m \colon W^{m,2}(\mathbb{H}^d_+) \to \prod_{k=0}^{m-1} W^{m-k-\frac{1}{2},2}(\partial \mathbb{H}^d_+)$ be the upper half space higher-order trace operator from lecture. Prove $\ker T_m^m = W_0^{m,2}(\mathbb{H}^d_+)$.

Exercise 4 (Trace operator for anisotropic Sobolev space)**.** Define the anisotropic Sobolev space $W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R}) := \{f \in L^1_{loc}(\mathbb{R}^d \times \mathbb{R}) \colon D_{d+1} f \in L^2(\mathbb{R}^d \times \mathbb{R}) \text{ and } D^{(\alpha,0)} f \in L^2(\mathbb{R}^d \times \mathbb{R}) \text{ for all } \alpha \in \mathbb{N}^d, |\alpha| \leq 2\}$. For example, for $d = 1$, this means f, $D_1 f, D_1^2 f$, and $D_2 f$ are in $L^2(\mathbb{R} \times \mathbb{R})$. Equip $W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R})$ with the obvious Sobolev norm.

i) Try to prove yourself or find a proof in the literature for the following Fourier characterization: $f \in W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R})$ if, and only if, $f \in L^2(\mathbb{R}^d \times \mathbb{R})$

$$\sqrt{a(\xi)^2 + \xi_{d+1}^2}\, \hat{f}(\xi, \xi_{d+1}) \in L^2(\mathbb{R}^d \times \mathbb{R}),$$

where $a(\xi) = 1 + |\xi|^2$.

ii) Show that there exists a continuous linear trace operator

$$T_{\mathrm{Cauchy}} \colon W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R}) \to W^{1,2}(\mathbb{R}^d)$$

such that $T_{\mathrm{Cauchy}} \varphi = \varphi(\cdot, 0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d \times \mathbb{R})$. You may use without proof that the latter space is dense in $W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R})$.

iii) Show that there exists a continuous linear reconstruction operator

$$Z_{\mathrm{Cauchy}} \colon W^{1,2}(\mathbb{R}^d) \to W_{\mathrm{par}}(\mathbb{R}^d \times \mathbb{R})$$

such that it is a right-inverse for T_{Cauchy}.