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Exercise sheet 5 to Advanced Partial Differential Equations

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be open and bounded, and let $M \subset L^2(\Omega)$. Assume

i) $\sup_{\varphi \in M} \|\varphi\|^2_{L^2(\Omega)} < \infty$,

ii) $\lim_{h \to 0} \sup_{\varphi \in M} \|\varphi(\cdot + h) - \varphi\|^2_{L^2(\Omega)} = 0$.

Prove that M is pre-compact in $L^2(\Omega)$, i.e., each sequence $(f_n)_{n \in \mathbb{N}} \subset M$ admits a convergent subsequence in $L^2(\Omega)$. You may argue as follows:

- Denote by $(\eta_\varepsilon)_{\varepsilon \in (0,1)}$ a sequence of standard mollifiers, and extend each f_n by zero outside of Ω.
- Show that $\lim_{k \to \infty} \sup_{n \in \mathbb{N}} \|\eta_{\frac{1}{k}} * f_n - f_n\|_{L^2(\Omega)} = 0$.
- Show that, for all $k \in \mathbb{N}$, the sequence $(\eta_{\frac{1}{k}} * f_n)_{n \in \mathbb{N}}$ is bounded in $C^1(\mathbb{R}^d)$ with respect to its norm $\| \cdot \|_{C^1(\mathbb{R}^d)}$.
- Conclude by an application of the Arzéla–Ascoli theorem and a diagonal sequence argument.

Exercise 2. Let $d = 2$, $f \colon \partial B_1(0) \to \mathbb{C} \in C^\infty(\partial B_1(0))$, and define $\tilde{f} \colon [0, 2\pi] \to \mathbb{C}$ by $\theta \mapsto f(e^{i\theta})$. I.e., $\tilde{f}$ is smooth and, together with all its derivatives, 2π-periodic. For $k \in \mathbb{Z}$, define the Fourier coefficient $c_k(\tilde{f}) := \int_0^{2\pi} \tilde{f}(\theta) e^{-ik\theta} \, d\theta$. Let $l \in \mathbb{N}$. Show that $f \in W^{l,2}(\partial B_1(0))$ if, and only if,

$$\|f\|^2_{H^l(\partial B_1(0))} := \sum_{k \in \mathbb{Z}} (1 + |k|^2)^l |c_k(\tilde{f})|^2 < \infty.$$

Show also that $\| \cdot \|_{H^l(\partial B_1(0))}$ is an equivalent norm to $\| \cdot \|_{W^{l,2}(\partial B_1(0))}$.

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be open, bounded, and of class C^k, $k \geq 1$. Let $\Omega' \subset \mathbb{R}^d$ be open such that $\overline{\Omega'} \subset\subset \mathbb{R}^d$. Let $l \in [0, k]$. Show that there exists an extension operator $E_{\Omega \to \Omega'} \colon W^{l,2}(\Omega) \to W^{l,2}_0(\Omega')$.

Exercise 4.

i) Let $k \in \mathbb{N}$ and $l \in [0, \infty)$ such that $l - k > \frac{d}{2}$. Show that $W^{l,2}(\mathbb{R}^d) \hookrightarrow C^k_0(\mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d)}^{C^k(\mathbb{R}^d)}$ continuously. Formulate and prove the corresponding statement for $\Omega \subset \mathbb{R}^d$ bounded, open, and of class C^m, $m \geq 1$.

ii) Let $k \in \mathbb{N}$, $l \in [0, \infty)$ such that $l - k > \frac{d-1}{2}$, and let $\Omega \subset \mathbb{R}^d$ be bounded, open, and of class C^m, $m \geq l$. Show that $W^{l,2}(\partial\Omega) \hookrightarrow C^k(\partial\Omega)$ continuously.