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Exercise sheet 4 to Advanced Partial Differential Equations

Exercise 1. The set $W^{l,2}(\partial\Omega)$ is independent of the choice of data $U = (U_i)_{i=1,\dots,N}$ (the covering open sets), $\Phi = (\Phi_i)_{i=1,\dots,N}$ (the boundary flattening diffeomorphisms), and $\alpha = (\alpha_i)_{i=1,\dots,N}$ (the associated partition of unity). Furthermore, for two such choices of data (U, Φ, α), the associated norms for $W^{l,2}(\partial\Omega)$ are equivalent.

Exercise 2. Let $\mathcal{S}(\mathbb{R}^d)$ be the class of Schwartz functions, i.e., all those $f \in C^\infty(\mathbb{R}^d)$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}^d$:

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha||D^\beta f(x)| < \infty.$$

We furthermore say that $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ if for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ it holds $[f_n - f]_{\alpha,\beta} \to 0$.

1) Show that $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ with respect to the above convergence. Furthermore, for given $f \in \mathcal{S}(\mathbb{R}^d)$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$, it holds $D^\beta f \in \mathcal{S}(\mathbb{R}^d)$ as well as $x^\alpha f \in \mathcal{S}(\mathbb{R}^d)$.
2) A linear map $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous with respect to the above convergence if, and only if, for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ there exist $\ell_{\alpha,\beta}, m_{\alpha,\beta} \in \mathbb{N}$ and a constant $C_{\alpha,\beta} > 0$ such that
$$[Tf]_{\alpha,\beta} \leq C_{\alpha,\beta} \sum_{|\tilde\alpha|\leq\ell_{\alpha,\beta},|\tilde\beta|\leq m_{\alpha,\beta}} [f]_{\tilde\alpha,\tilde\beta}.$$
3) Let $p \in [1,\infty]$, $f \in \mathcal{S}(\mathbb{R}^d)$. Then $f \in L^p(\mathbb{R}^d)$, and the embedding $S(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, $f \mapsto f$, is continuous. In particular, $\hat f(\xi) := \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$ is well-defined for $f \in \mathcal{S}(\mathbb{R}^d)$.
4) Let $f \in \mathcal{S}(\mathbb{R}^d)$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$ be given. Then $\widehat{D^\beta f}(\xi) = \xi^\beta \hat f(\xi)$ and $\widehat{x^\alpha f} = D^\alpha \hat f$.
5) Use the above to show that $\hat f \in \mathcal{S}(\mathbb{R}^d)$ for $f \in \mathcal{S}(\mathbb{R}^d)$, and that $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous.
6) *Reading exercise (if you do not know these statements):* locate the Fourier inversion theorem and Plancherel theorem in your favorite reference and digest the proofs.