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Exercise sheet 2 to Advanced Partial Differential Equations

Exercise 1. Show that $f(x) := 1 - |x|$, $x \in (-1, 1)$, is an element of $W^{1,2}((-1,1))$ but not of $W^{2,2}((-1,1))$.

Exercise 2. (Convolution) Let $\Omega \subset \mathbb{R}^d$ be open, $K \subset\subset \Omega$, $\varepsilon_K := \frac{1}{2}\mathrm{dist}(K, \partial\Omega)$, $\varphi \in L^2(\Omega)$, $\eta \in L^1(\mathbb{R}^d)$ such that $\operatorname{supp}\eta \subset \overline{B_{\varepsilon_K}(0)}$. Define $\eta * \varphi \colon \mathrm{int}(K) \to \mathbb{C}$, $x \mapsto \int_\Omega \eta(y-x)\varphi(y)\,dy$, where $\mathrm{int}(K)$ denotes the interior of K.

i) Prove that $\eta * \varphi \in L^2(\mathrm{int}(K))$, and if in addition $\operatorname{supp}\varphi \subset K$ then $\operatorname{supp}\eta * \varphi \subset \{x \colon \mathrm{dist}(x, K) \leq \varepsilon_K\}$. Furthermore, show $\eta_\varepsilon * \varphi \to \varphi$ in $L^2(\mathrm{int}(K))$ as $\varepsilon \to 0$, where $(\eta_\varepsilon)_\varepsilon$ is a standard sequence of mollifiers.

ii) If in addition $\eta \in C^\infty(\mathbb{R}^d)$ then $\eta * \varphi \in C^\infty(\mathrm{int}(K))$, and for all multi-indices $\alpha \in \mathbb{N}^d$ it holds $D^\alpha(\eta * \varphi) = (D^\alpha \eta) * \varphi$.

iii) If in addition $\eta \in C^\infty(\mathbb{R}^d)$ as well as $\varphi \in W^{l,2}(\Omega)$, $l \in \mathbb{N}$, then $\eta * \varphi \in W^{l,2}(\mathrm{int}(K))$ and for all $|\alpha| \leq l$ it holds $D^\alpha(\eta * \varphi) = \eta * D^\alpha \varphi$.

Exercise 3. Fill the gap in the proof of the multiplier property from lecture: let $\Omega \subset \mathbb{R}^d$ be open, $\varphi \in W^{1,2}(\Omega)$, $a \in C^1(\mathbb{R}^d)$, $\psi \in \mathcal{D}(\Omega)$, and justify

$$\int_\Omega a\psi D_j \varphi \, dx = -\int_\Omega \varphi D_j(a\psi)\, dx.$$