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Exercise sheet 1 to Advanced Partial Differential Equations

Exercise 1. (Distributions)

i) For $f \in L^1_{loc}(\Omega)$, let $T_f \in \mathcal{D}'(\Omega)$ denote the associated regular distribution. Show that the map $f \mapsto T_f$ is injective, i.e., $T_f = 0$ implies $f = 0$. (*Hint:* Show the claim first under the additional assumption that f is continuous. Derive then the claim for general f by testing the equation $T_f = 0$ with test functions of the form $\phi_\varepsilon * \psi$, where $\psi \in \mathcal{D}(\Omega)$, $*$ denotes convolution, and $(\phi_\varepsilon)_{\varepsilon\in(0,1)}$ is a standard sequence of mollifiers, i.e., $\phi_\varepsilon(x) := \frac{1}{\varepsilon^d}\phi(\frac{|x|}{\varepsilon})$ for a non-negative $\phi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \subset [-1,1]$ and $\int_{-1}^{1}\phi = 1$.)

ii) We say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ as $n \to \infty$ if there exists a compact subset $K \subset \Omega$ such that $\operatorname{supp}\varphi_n \subset K$ for all $n \in \mathbb{N}$ and if for all multi-indices $\alpha \in \mathbb{N}^d$ it holds $D^\alpha\varphi_n \to D^\alpha\varphi$ uniformly as $n \to \infty$. Let $T\colon \mathcal{D}(\Omega)$ be a linear map. Show that $T \in \mathcal{D}'(\Omega)$ if and only if T is continuous with respect to this notion of sequential convergence on $\mathcal{D}(\Omega)$.

iii) We say that a sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}'(\Omega)$ if for all $\varphi \in \mathcal{D}(\Omega)$ it holds $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ in $\mathbb{C}$ as $n \to \infty$. Show that $T_n \to T$ in $\mathcal{D}'(\Omega)$ implies for all multi-indices $\alpha \in \mathbb{N}^d$ that $D^\alpha T_n \to D^\alpha T$ in $\mathcal{D}'(\Omega)$.

Exercise 2. (Examples for Sobolev and Sobolev–Slobodeckij functions)

i) Let $\beta \in \mathbb{R}\setminus\{0\}$, $f_\beta(x) := |x|^{-\beta}$, $x \in \mathbb{R}^d \setminus \{0,\}$, and $l \in \mathbb{N}$. For which β does it hold $f_\beta \in W^{l,2}(B_1(0))$?

ii) Analyze for which β it holds $f_\beta \in W^{l,2}(\mathbb{R}^d \setminus \overline{B_1(0)})$.

iii) Let $\lambda \in (0,1)$. Show that $f_\beta \in W^{\lambda,2}(B_1(0))$ if $\frac{d}{2} - \lambda > \beta$. (*Hint:* One has to show $I_\lambda(f_\beta) < \infty$. To this end, first decompose $B_1(0)\times B_1(0)$ disjointly into $A_k := \{(x,y) \in B_1(0)\times B_1(0)\colon 2^{-(k+1)} < \max\{|x|,|y|\} \le 2^{-k}\}$ for $k \in \mathbb{N}$ (and draw a picture for $d = 1$ to visualize it). Show then that $C^k_{\beta,d,\lambda} := \int_{A_k} \frac{|f_\beta(x)-f_\beta(y)|^2}{|x-y|^{d+2\lambda}}\,\mathrm{d}(x,y) = 2^{-k(d-2\beta-2\lambda)}C^0_{\beta,d,\lambda}$. Conclude by showing that $0 < C^0_{\beta,d,\lambda} < \infty$. For the upper bound, it is useful to decompose A_0 into the region where additionally $|x-y| \le \frac{1}{4}$ and its complement.)