§2.3 Riesz-Schauder theory & spectrum of compact operators (cont'd)

Thm: (Fredholm alternative) Let $T: X \to X$ be compact, $\lambda \in \mathbb{K}$, $\lambda \neq 0$, $y \in X$, $y' \in X'$. Then:

$$\exists x \in X: \lambda x - Tx = y \iff y \in \ker(\bar{\lambda} - T')_\perp,$$

$$\exists x' \in X': \bar{\lambda} x' - T'x' = y' \iff y' \in \ker(\lambda - T)_\perp.$$

Furthermore:

$$n := \dim \ker(\bar{\lambda} - T') = \dim \ker \lambda - T < \infty.$$

In particular, one has the dichotomy:

(A1) $\forall y \in X \; \exists! \, x \in X: \lambda x - Tx = y.$

(A2) $\exists x \in X \setminus \{0\}: \lambda x - Tx = 0.$

Proof: $\lambda - T = \lambda(\mathrm{Id} - T/\lambda) = \lambda S_{T/\lambda}$, with $S_{T/\lambda}$ a Riesz-Schauder operator.

Hence, $\operatorname{im}\lambda-T$ is closed and

Therefore $\operatorname{im}(\lambda-T) = \overline{\operatorname{im}(\lambda-T)} = \ker(\bar{\lambda}-T')_{\perp}$.

Furthermore, since $S'_{T/\lambda} = S_{T'/\bar{\lambda}}$ is again a Riesz-Schauder operator, it follows from closed range theorem that

$$\operatorname{im}\bar{\lambda}-T' = (\ker(\lambda-T))_{\perp}. \quad \blacksquare$$

Thm: (Spectral theorem of compact operators)

Let $T: X \to X$ be compact. Then

(a) $\dim X = \infty \Rightarrow 0 \in \sigma(T)$,

(b) Every $\lambda \in \sigma(T)\setminus\{0\}$ is necessarily an eigenvalue and induces a block diagonal decomposition:

- $X = N(\lambda) \oplus R(\lambda)$
- $T(N(\lambda)) \subseteq N(\lambda)$, $T(R(\lambda)) \subseteq R(\lambda)$,
- $\dim N(\lambda) < \infty$, $\ker(\lambda-T) \subseteq N(\lambda)$,
- $\lambda-T|_{R(\lambda)} : R(\lambda) \xrightarrow{\sim} R(\lambda)$.

(c) $\sigma(T)\setminus\{0\}$ is at most countable.

(d) No $\lambda \in \sigma(T)\setminus\{0\}$ can be an accumulation point of $\sigma(T)$.

Rmk: ⚠ It may happen that $\sigma(T) = \{0\}$ and that zero either belongs to the continuous or residual spectrum (in particular, no eigenvalue).

Proof: (a) $0 \notin \sigma(T) \Rightarrow T$ is an isomorphism

$\Rightarrow Id = TT^{-1}$ is compact

$\Rightarrow \dim X < \infty$.

(b) Fredholm alternative implies $\lambda \in \sigma(T)\setminus\{0\}$ is an eigenvalue, and the block-diagonal structure from last lecture.

(c) & (d) It suffices to show that

$\forall \varepsilon > 0 : \#\{\lambda \in \sigma(T) : |\lambda| \geq \varepsilon\} < \infty$.

[If not, there exists $\varepsilon > 0$ and $(\lambda_n)_n \subseteq \mathbb{K}$, $(x_n)_n \subseteq X$ s.t. $|\lambda_n| \geq \varepsilon$, $x_n \neq 0$, $Tx_n = \lambda_n x_n$, $\lambda_n \neq \lambda_m$, $n \neq m$. Then, $\{x_n : n \in \mathbb{N}\}$ is a linear independent set (Linear Algebra!).

Let $E_n := \langle x_{1}, \dots, x_n \rangle$, hence

$$E_1 \subsetneq E_2 \subsetneq E_3 \subsetneq \dots .$$

Note that $S_n := \lambda_n - T$ satisfies the property $S_n(E_n) \subseteq E_{n-1}$:

let $y = \sum_{i=1}^{n} \alpha_i^n x_i \in E_n$, then

$$\lambda_n y - Ty = \sum_{i=1}^{n} (\lambda_n - \lambda_i) \alpha_i^n x_i \in E_{n-1}.$$

Hence, similar to the previous lecture, one may select $y_n \in E_n$, $\|y_n\| = 1$ s.t. for all $n > m \geq 1$:

$$\|Ty_n - Ty_m\| \geq |\lambda_n| \operatorname{dist}(y_n, E_{n-1}) \geq \varepsilon/2.$$ ↯?

[Exercise: use again Riesz' lemma!] ∎

From now on, let $X = H$ be a Hilbert space.

Lemma: Let $T \in L(H)$.

(a) $\lambda \in \sigma(T) \Leftrightarrow \overline{\lambda} \in \sigma(T')$.

(b) If T is self-adjoint (i.e. $T' = T$), then eigenvectors for distinct eigenvalues are orthogonal.

If T is self-adjoint and compact, then

(c) $\sigma(T) \subseteq \mathbb{R}$,

(d) $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

Proof: (a) This follows from the general principle: T isomorphism $\Leftrightarrow T'$ isomorphism

[Indeed: $\overline{\operatorname{im} T} = (\ker T')^{\perp}$, $\ker T = (\operatorname{im} T')^{\perp}$.]

(b) If S is normal, then $\ker S = \ker S'$ since $\|Sx\| = \|S'x\|$ for all $x \in X$.

Applied to $S := \lambda - T$:

$$\ker(\lambda - T) = \ker(\lambda - T)^* = \ker(\overline{\lambda} - T).$$

Hence, for $\lambda \in \sigma_p(T)$ an eigenvalue:

$$\lambda \langle x, x \rangle = \langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\lambda} \langle x, x \rangle$$

$$\Rightarrow \lambda = \overline{\lambda} \in \mathbb{R}.$$

Hence, for $x \in \ker(\lambda - T)$, $y \in \ker(\mu - T)$, $\lambda, \mu \in \sigma_p(T)$, $\lambda \neq \mu$:

$$\lambda \langle x, y \rangle = \langle Tx, y \rangle = \langle x, Ty \rangle = \langle x, \mu y \rangle = \mu \langle x, y \rangle$$

$$\Rightarrow \langle x, y \rangle = 0.$$

(c) $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$.

(d) For self-adjoint T: $\|T\| = \sup_{\|x\| \leq 1} |\langle Tx, x \rangle|$.

Assume wlog $\|T\| \neq 0$.

Hence, $\exists x_n \in B_X$: $|\langle Tx_n, x_n\rangle| \to \|T\| =: \lambda$.

Since T is compact, $\exists (n_k)_k$: $Tx_{n_k} \to y$.

Then $\|Tx_{n_k} - \lambda x_{n_k}\|^2 = \langle Tx_{n_k} - \lambda x_{n_k}, Tx_{n_k} - \lambda x_{n_k}\rangle$

$$= \|Tx_{n_k}\|^2 + \lambda^2 \|x_{n_k}\|^2 - 2\lambda \langle Tx_{n_k}, x_{n_k}\rangle$$

$$\leq 2\lambda^2 - 2\lambda \langle Tx_{n_k}, x_{n_k}\rangle \to 0 \text{ as } k \to \infty,$$

hence $\lim_{k\to\infty} \lambda x_{n_k} = y$, i.e. $Ty = \lambda y$.

Then, $y \neq 0$: if $0 = y = \lim_{k\to\infty} Tx_{n_k}$, then $\lambda |\langle Tx_{n_k}, x_{n_k}\rangle| \to \lambda^2 = 0$. ↯ ∎

Thm: (Spectral theorem for compact self-adjoint operators)

Let $T: H \to H$ be compact & self-adjoint. There exists an at most countable ONS

of eigenvectors $e_1, e_2, \ldots \in H$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \in \mathbb{R}\setminus\{0\}$ s.t.

(a) if $\#\{e_1, e_2, \ldots\} = \infty \Rightarrow \lambda_n \to 0$,

(b) $H = \ker T \oplus \overline{\langle e_1, e_2, \ldots \rangle}$,

(c) $Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k$.

Proof: The $(e_k)_k$ and $(\lambda_k)_k$ are obtained from the set of eigenvalues $\mu \in \sigma(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\} \subseteq \mathbb{R}\setminus\{0\}$ by choosing for each such μ an ONB of the finite-dimensional eigenspace $\ker(\mu - T)$ with dimension d_μ, and repeating each μ d_μ-times to obtain $(\lambda_k)_k$. Noting that for all $x \in \ker T$ and all $\mu \in \sigma(T)\setminus\{0\}$ it holds for $y \in \ker(\mu - T)$

$$\mu \langle x, y \rangle = \langle x, Ty \rangle = \langle Tx, y \rangle = 0,$$

hence $\ker T \perp \overline{\langle e_1, e_2, \ldots \rangle}$. Hence, we define $H_1^{\text{closed}} := \ker T \oplus \overline{\langle e_1, e_2, \ldots \rangle}$ and want to show that $H_2 := H_1^{\perp} = \{0\}$.

Note that $T(H_2) \subseteq H_2$:

Let y s.t. $y \perp e_k$. Then

$$\langle Ty, e_k \rangle = \langle y, \lambda_k e_k \rangle = 0.$$

Also, for $y \perp x$, $x \in \ker T$:

$$\langle Ty, x \rangle = \langle y, Tx \rangle = 0.$$

This shows $T(H_2) = H_2$.

Now, $T|_{H_2} : H_2 \to H_2$ is itself compact & self-adjoint.

If $T|_{H_2} \neq 0$, there exists $\lambda \neq 0$, $x_2 \in H_2$, $x \neq 0$ s.t. $T|_{H_2} x = \lambda x$; more precisely, $\lambda \in \{\pm \|T|_{H_2}\|\}$ works. But then

$$x \in \overline{\langle e_1, e_2, \dots \rangle} \subseteq H_2^{\perp},$$

so $x = 0$. ↯

Hence $T|_{H_2} = 0 \Rightarrow H_2 \subseteq \ker T \subseteq H_1$

$\Rightarrow H_2 = 0$. ∎