§2.3 Riesz-Schauder theory & spectrum of compact operators

Thm. (Riesz-Schauder) Let X be a Banach space, $T: X \to X$ be compact, $S := \mathrm{Id} - T$. Then

(a) $\dim \ker S < \infty$,

(b) $\mathrm{im}\, S$ is closed, $\dim \underbrace{\mathrm{coker}\, S}_{:= X/\mathrm{im}\, S} < \infty$,

(c) $\dim \ker S$
$= \dim \mathrm{coker}\, S$
$= \dim \mathrm{coker}\, S' = \dim \ker S'$.

["S is a Fredholm operator of index 0."]

Proof of (a) & (b):

(a) Note that $\ker S$ corresponds precisely to fixpoints of T, so that $\ker S \cap B_X$

is the set of fix points of T with length ≤ 1. Since T maps bounded sets to pre-compact sets, $\ker S \cap B_X$ is compact, hence finite dimensional.

(b) We show that $\operatorname{im} S$ is closed.

By (a), $X/\ker S$ is Banach. Consider the commutative diagram from the universal property of quotient spaces (which, btw, canonically realizes the isomorphism $(X/U)' \cong U^\perp$)

$$\begin{array}{ccc} X & \xrightarrow{S} & \operatorname{im} S \\ {\scriptstyle \pi}\downarrow & \nearrow_{\exists! \hat{S} \text{ s.t. } S = \hat{S} \circ \pi} & \\ X/\ker S & & \end{array}$$

continuous isomorphism

The claim follows once we proved continuity

of $\hat{S}^{-1}: \operatorname{im} S \to X/\ker S$. Assume to the contrary that $\hat{S}^{-1}$ is not continuous.

For each $n \in \mathbb{N}$ one may then find $x_n \in X \setminus \{0\}$ s.t. for $\pi(x_n) = \hat{x}_n = x_n + \ker S$

$$\|\hat{S}\hat{x}_n\| < \frac{1}{n}\|\hat{x}_n\| = \frac{1}{n}\operatorname{dist}(x_n, \ker S),$$

i.e., wlog $\|\hat{x}_n\| = 1$ and $\|\hat{S}\hat{x}_n\| < \frac{1}{n}$.

Furthermore, wlog $\|x_n\| \leq 2$. Since by assumption $Sx_n = \hat{S}\hat{x}_n \to 0$ as $n \to \infty$, $x_n = Sx_n + Tx_n$ and T is compact, there exists a subsequence x_{n_k} such that $Tx_{n_k} \to x$, hence $x_{n_k} \to x \in \ker S$.

In particular, $0 = \|\hat{x}\| = \lim_{k\to\infty} \|\hat{x}_{n_k}\| = 1$. ↯

Since $\operatorname{im} S$ is closed, we may deduce

$$(X/\operatorname{im} S)' \cong (\operatorname{im} S)^{\perp} = \ker S'$$

hence $\dim \operatorname{coker} S = \dim \ker S'$,
and the latter is finite dimensional
since $S' = \overline{\lambda} \mathrm{Id} - T'$, and by Schauder
T compact iff T' is compact, so that
we may apply (a) to S'. Furthermore,

$$X'/\operatorname{im} S' = X'/(\ker S)^\perp \cong (\ker S)'$$

$(\ker S)^\perp = \operatorname{im} S'$ by closed range theorem

showing $\dim \operatorname{coker} S' = \dim \ker S$. ∎

For a proof of the Riesz-Schauder theorem,
it remains to show

(*) $\dim \ker S = \dim \operatorname{coker} S$.

In case $\dim X < \infty$, you know this from
your linear algebra course:

- $\dim X = \dim \ker S + \dim \operatorname{im} S$
- $X \cong U \oplus X/U$.

The point of the following lemma is to show that any Riesz-Schauder operator $S = \mathrm{Id} - T$, T compact, can be represented in block-diagonal structure with one block being an isomorphism and the other one being representable by a matrix.

Lemma 1: There exist closed subspaces $\hat{N}, \hat{R} \subseteq X$ such that

(a) $\dim \hat{N} < \infty$,

(b) $X = \hat{N} \oplus \hat{R}$,

(c) $S(\hat{N}) \subseteq \hat{N}$, $S(\hat{R}) \subseteq \hat{R}$,

(d) $S|_{\hat{R}} : \hat{R} \to \hat{R}$ is an isomorphism.

Proof of (*) via Lemma 1

(*) follows from 3 facts:

(a') $X/\operatorname{im} S \cong \hat{N}/\operatorname{im} S|_{\hat{N}}$

via $\overline{\Phi}: x + \operatorname{im} S|_{\hat{N}} \mapsto x + \operatorname{im} S$,

(b') $\dim \hat{N}/\operatorname{im} S|_{\hat{N}} = \dim \ker S|_{\hat{N}}$

(the above linear algebra fact),

(c') $\ker S|_{\hat{N}} = \ker S$.

Proof of (a'): $\overline{\Phi}(x + \operatorname{im} S|_{\hat{N}}) := x + \operatorname{im} S$ is well-defined due to $\operatorname{im} S|_{\hat{N}} \subseteq \operatorname{im} S$.

Let $x + \operatorname{im} S|_{\hat{N}} \in \ker \overline{\Phi}$, i.e. $x \in \operatorname{im} S \cap \hat{N}$.

We find $y \in X$ s.t. $x = Sy$, and decomposing $y = y_1 + y_2 \in \hat{N} \oplus \hat{R} = X$ shows $x = Sy_1$, hence $x + \operatorname{im} S|_{\hat{N}} = 0$ since $y_1 \in \hat{N}$. Hence, $\overline{\Phi}$ is injective. For surjectivity, let

$x = x_1 + x_2 \in X = \hat{N} \oplus \hat{R} = \hat{N} \oplus \operatorname{im} S|_{\hat{R}}$,

then $\overline{\Phi}(x_1 + \operatorname{im} S|_{\hat{N}}) = x_1 + \operatorname{im} S$

$= x + \operatorname{im} S$. ∎

Proof of (c'):

$x \in \ker S \iff x = x_1 + x_2 \in \hat{N} \oplus \hat{R} : Sx = 0$

$\iff Sx_1 = -Sx_2 \in \hat{N} \cap \hat{R}$

$\iff Sx_1 = Sx_2 = 0$

$\iff x_2 = 0, \ Sx_1 = 0$

$\iff x \in \ker S|_{\hat{N}}$. ∎

In order to prove Lemma 1, we employ the following construction. For $m \in \mathbb{N}$, let

$N_m := \ker S^m, \ N_0 = \{0\}$

$R_m := \operatorname{im} S^m, \ R_0 = X$,

so that
- $N_0 \subseteq N_1 \subseteq N_2 \subseteq \dots$
- $R_0 \supseteq R_1 \supseteq R_2 \supseteq \dots$
- $\dim N_m < \infty$
- R_m closed and $\dim \operatorname{coker} S^m < \infty$.

The latter facts follow from applying (a) & (b) to the Riesz-Schauder operator

$$S^m = Id - T_m \,, \quad T_m \text{ compact.}$$

(binomial formula)

Lemma 1 then follows from

Lemma 2:

(a) $\exists$ minimal $p \in \mathbb{N}_0 : N_p = N_{p+1}$,

(b) $N_{p+r} = N_p$, $\forall r > 0$,

(c) $N_p \cap R_p = \{0\}$,

(d) $\exists$ minimal $q \in \mathbb{N}_0 : R_q = R_{q+1}$,

(e) $R_{q+r} = R_q$, $\forall r > 0$,

(f) $N_q + R_q = X$,

(g) $p = q$.

Proof of Lemma 1: Define $\hat{N} := N_p$ and $\hat{R} := R_q = R_p$, so that $\dim \hat{N} < \infty$ and $X = \hat{N} \oplus \hat{R}$.

Furthermore,

$$S(\hat{N}) = S(N_p) \subseteq N_{p-1} \subsetneq N_p = \hat{N},$$

$$S(\hat{R}) = S(R_q) = R_{q+1} = R_q = \hat{R}.$$

Let now $y \in \hat{R} = R_q$ s.t. $S(y) = 0$, i.e.,

$y = S^\alpha x$ for some $x \in X$ and $x \in N_{q+1}$,

i.e., $x \in N_{p+1} = N_p = N_q$, hence $y = 0$. ∎

Proof of Lemma 2:

(a) If not, then $N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots$.

Since all N_m are closed, by Riesz' Lemma,

there exists $x_m \in N_m$ s.t. $\operatorname{dist}(x_m, N_{m-1}) \geq 1/2$.

Note that for all $n > m \geq 1$:

$$Tx_n - Tx_m = x_n - (Sx_n + x_m - Sx_m)$$

$$\in N_{n-1} \quad \in N_m \quad \in N_{m-1} \subseteq N_{n-1}$$

$$\Rightarrow \|Tx_n - Tx_m\| \geq \operatorname{dist}(x_n, N_{n-1}) \geq 1/2 \quad \lightning.$$

(b) We already know $N_p \subseteq N_{p+r}$. Hence,

let $x \in N_{p+r}$. Then $S^{r-1}(x) \in N_{p+1} = N_p$, i.e., $S^{p+r-1}(x) = 0$, i.e., $x \in N_{p+r-1}$. Proceed now inductively.

(c) Let $x \in N_p \cap R_p$, i.e. $S^p x = 0$ and $x = S^p y$ so that $S^{2p} y = 0$, i.e. $y \in N_{2p} = N_p$, hence $x = S^p y = 0$.

(d) Similar to (a)! [Exercise.]

(e) Similar to (b)! [Exercise.]

(f) Let $x \in X$, then $S^q x \in R_q = R_{2q}$ so that $\exists y : S^q x = S^{2q} y$. Decompose

$$x = \underbrace{(x - S^q y)}_{\in N_q} + \underbrace{S^q y}_{\in R_q}.$$

(g) If $p > q$, then $R_p = R_q$ and $N_p \setminus N_q \neq \emptyset$. Let $x \in N_p \setminus N_q$, decompose $x = y + z \in N_q + R_q$, so that $z = x - y \in N_p + N_q \subseteq N_p$, i.e. by (c) and $z \in N_p \cap R_p$: $z = 0 \Rightarrow x = y \in N_q$. ↯ ∎