§ 1.5. Extension operators (cont'd)

Thm: Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, and of class C^k, $k \geq 1$. For all $l \in [0,k]$, there exists an extension operator

$$E_{\Omega \to \mathbb{R}^d} : W^{l,2}(\Omega) \to W^{l,2}(\mathbb{R}^d).$$

Proof: *Step 1: (Localization)* We aim to apply our finite covering lemma from Lecture 4 to $K = \overline{\Omega}$ and choose to this end for each $x \in \partial\Omega$ a sequence of open nbhds. $\overline{V_x} \subset\subset V_x'$, $\overline{V_x'} \subset\subset U_x$, $U_x \subset\subset \mathbb{R}^d$ such that there exists a "boundary flattening C^k-diffeo"

$\Phi_x : Q_1^d \to U_x$ and $\Phi_x^{-1}(V_x') = Q^d_{\frac{1}{k+1}}$ as well as $\Phi_x^{-1}(V_x) = Q^d_{\frac{1}{2}\frac{1}{k+1}}$, and for each $x \in \Omega$ a constant $\delta_x \in (0,1)$ s.t. $U_x := B_{\delta_x}(x)$, $V_x' := B_{\delta_x/2}(x)$, $V_x := B_{\delta_x/4}(x)$ and $\overline{U_x} \subset\subset \Omega$. Denote then by $V_1, \dots, V_K$ (boundary nbhds.), $V_{K+1}, \dots, V_{K+N}$ (interior nbhds.) a finite cover of $\overline{\Omega}$ with associated finite partition of unity $(\chi_1, \dots, \chi_K, \chi_{K+1}, \dots, \chi_{K+N})$.

We will construct for each $j = 1, \dots, K, K+1, \dots, K+N$ a continuous linear operator $T_j : W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^d)$ such that $E_{\Omega \to \mathbb{R}^d} := \sum_{j=1}^{K+N} T_j$ is an extension operator $W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^d)$.

Step 2: (Interior nbhds.) Fix $j \in \{K+1, \dots, K+N\}$.

We define T_j via

$$W^{k,2}(\Omega) \xrightarrow[\substack{\text{localization} \\ \text{via partition} \\ \text{of unity multiplier}}]{\alpha_j \cdot} W_0^{k,2}(U_j) \xrightarrow[\substack{\text{trivial} \\ \text{extension} \\ \text{by zero}}]{} W^{k,2}(\mathbb{R}^d).$$

Step 3: (Boundary nbhds.) Fix $j \in \{1, \dots, K\}$.

We define T_j via

$$W^{k,2}(\Omega) \xrightarrow[\substack{\text{localization} \\ \text{via partition} \\ \text{of unity multiplier}}]{\alpha_j \cdot} W^{k,2}(\Omega) \cap \{ \operatorname{supp} \varphi \subset\subset V_j' \}$$

$$F_{\Phi_j} \Big\downarrow \quad \text{change of coordinates by boundary flattening diffeomorphism } \Phi_j; \text{ Lecture 6}$$

$$W^{k,2}(Q_{1,+}^d) \cap \{ \operatorname{supp} \varphi \subset\subset Q_{1-\frac{1}{k+1},+}^d \}$$

$$\widetilde{E}_{Q_{1,+}^d \to Q_1^d} \Big\downarrow \quad \text{extension operator for model problem; Lecture 8}$$

$$W_0^{k,2}(Q_1^d)$$

$\Gamma_{\Phi_j}^{-1}$ ↓ change of coordinates back to original curved coordinates; Lecture 6

$$W^{l,2}(\mathbb{R}^d) \xleftarrow{\text{trivial extension by zero}} W_0^{l,2}(U_j)$$

Step 4: $\sum_{j=1}^{K+N} \Gamma_j : W^{l,2}(\Omega) \to W^{l,2}(\mathbb{R}^d)$

is indeed an extension operator due to the previous steps. ∎

§ 1.6. Continuous embeddings

Thm: (Sobolev's lemma) Let $l > d/2$. Then, there exists a continuous, linear and injective operator $\Gamma : W^{l,2}(\mathbb{R}^d) \to C_0(\mathbb{R}^d) := \overline{\mathcal{D}(\mathbb{R}^d)}^{C(\mathbb{R}^d)}$ such that $\Gamma(\varphi) = \varphi$, $\forall \varphi \in \mathcal{D}(\mathbb{R}^d)$.

We write $W^{l,2}(\mathbb{R}^d) \overset{\text{cont.}}{\hookrightarrow} C_0(\mathbb{R}^d)$

Proof: By density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{l,2}(\mathbb{R}^d)$ wrt. $\|\cdot\|_{W^{l,2}(\mathbb{R}^d)}$ and by the equivalence of $H^l(\mathbb{R}^d) \simeq W^{l,2}(\mathbb{R}^d)$, it suffices to show for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$:

$$\sup_{x \in \mathbb{R}^d} |\varphi(x)| \leq C \|\varphi\|_{H^l(\mathbb{R}^d)}.$$

By Fourier inversion:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, \hat{\varphi}(\xi)\, d\xi ,$$

so that

$$|\varphi(x)| \le \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^{l/2}} (1+|\xi|^2)^{l/2} |\hat{\varphi}(\xi)|\, d\xi$$

$$\overset{\text{Cauchy-Schwartz (or Hölder)}}{\le} \frac{1}{\sqrt{2\pi}^d} \underbrace{\left(\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^l} \right)^{1/2}}_{< \infty \text{ iff } 2l > d} \underbrace{\left(\int_{\mathbb{R}^d} (1+|\xi|^2)^l |\hat{\varphi}(\xi)|^2 \right)^{1/2}}_{= \|\varphi\|_{H^l(\mathbb{R}^d)}}$$

$$\le C(l,d)\, \|\varphi\|_{H^l(\mathbb{R}^d)} . \qquad \blacksquare$$

Thm: Let $l > d/2$, $\Omega \subseteq \mathbb{R}^d$ open, bounded and of class C^k, $k \ge l$. Then

$$W^{l,2}(\Omega) \overset{\text{cont.}}{\hookrightarrow} C(\overline{\Omega}) .$$

Proof: An operator realizing the desired continuous embedding is given by

$$W^{k,2}(\Omega) \xrightarrow{E_{\Omega\to\mathbb{R}^d}} W^{k,2}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d) \xrightarrow{\text{restriction}} \overline{\mathcal{D}(\mathbb{R}^d)|_\Omega}^{C(\bar\Omega)} = C(\bar\Omega).$$

$E_{\Omega\to\mathbb{R}^d}$: extension operator; $\hookrightarrow$: cont. embedding

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§ 1.7. Compact embeddings

Def: Let X, Y be two Banach spaces. A linear map $T: X \to Y$ is called *compact*, if for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ in Y admits a converging subsequence, or equivalently, if $T(B_X)$ is pre-compact in Y, where $B_X := \{x \in X : \|x\|_X \le 1\}$.

Thm: (Kolmogorov-Riesz) [without proof]

Let $M \subseteq L^2(\Omega)^*$. The subset M is pre-compact in $L^2(\Omega)$ iff

(i) $\sup_{\varphi\in M} \|\varphi\|^2_{L^2(\Omega)} < \infty$ ("M is bounded in $L^2(\Omega)$")

(ii) $\lim_{h\to 0} \sup_{\varphi\in M} \|\varphi(\cdot+h) - \varphi\|^2_{L^2(\Omega)} = 0$ ("uniform continuity of translations in $L^2(\Omega)$")

* and let $\Omega \subseteq \mathbb{R}^d$ be bounded

Thm: (Rellich-Kondrachov)

Let $\Omega \subset \mathbb{R}^d$ be open & bounded, $k_2 > k_1$, $k_2, k_1 \in \mathbb{N}$.

Then
$$W_0^{k_2,2}(\Omega) \xrightarrow{\text{compact}} W_0^{k_1,2}(\Omega),$$

i.e., there exists a linear, injective and compact map $T: W_0^{k_2,2}(\Omega) \to W_0^{k_1,2}(\Omega)$.

Proof: We obviously try to realize the required map T by $\varphi \mapsto \varphi$. Hence, we aim to show that the unit ball in $W_0^{k_2,2}(\Omega)$ is pre-compact in $W_0^{k_1,2}(\Omega)$. ~~By the Kolmogorov-Riesz Compactness Criterion for $L^2(\Omega)$~~ It suffices to show that

$$\mathcal{M} := \{ D^\alpha \varphi : \varphi \in W_0^{k_2,2}(\Omega),\ \|\varphi\|_{W_0^{k_2,2}(\Omega)} \leq 1,\ |\alpha| \leq k_2 - 1 \}$$

is pre-compact in $L^2(\Omega)$. By definition, $\mathcal{M}$ satisfies (i) of the Kolmogorov-Riesz Criterion. To show (ii), fix $\varphi \in \mathcal{D}(\Omega)$ s.t. $\|\varphi\|_{W_0^{k_2,2}(\Omega)} \leq 2$ and fix $|\alpha| \leq k_2 - 1$. Then

$$D^\alpha\varphi(x+h) - D^\alpha\varphi(x) \overset{\text{fundamental theorem of calculus}}{=} \int_0^1 \frac{d}{dt}\left(D^\alpha\varphi(x+th)\right) dt \quad L^2$$

$$\overset{\text{chain rule}}{=} \int_0^1 \sum_{i=1}^d (D_i D^\alpha\varphi)(x+th)\, h_i \, dt,$$

hence

$$\int_{\mathbb{R}^d} |(D^\alpha\varphi)(x+h) - D^\alpha\varphi(x)|^2 dx$$

$$\leq |h|^2 \int_{\mathbb{R}^d}\int_0^1 |(\nabla D^\alpha\varphi)(x+th)|^2 dt\, dx$$

$|\alpha|\leq k_2-1$

$$\leq |h|^2 \|\varphi\|_{W_0^{k_2,2}(\Omega)} \leq 2|h|^2.$$

By density of $\mathcal{D}(\Omega)$ in $W_0^{k,2}(\Omega)$:

$$\lim_{h\to 0} \sup_{f\in M} \|f(x+h) - f(x)\|^2_{L^2(\Omega;dx)} = 0. \qquad \blacksquare$$

Thm: Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, and of class C^k, $k\geq 1$. Let $k_2 > k_1$, $k_1, k_2 \in \mathbb{N}$ s.t. $k_2 \leq k$. Then

$$W^{k_2,2}(\Omega) \overset{\text{compact}}{\hookrightarrow} W^{k_1,2}(\Omega).$$

Proof: Since Ω is bounded, there exists $R\geq 1$ s.t. $\overline{\Omega} \subset\subset B_R(0) =: \Omega'$. You will show in the exercises that there exists an extension operator

$E_{\Omega\to\Omega'}: W^{k,2}(\Omega) \to W^{k,2}_0(\Omega')$. Given that, 28

we construct the embedding via

$$W^{k,2}(\Omega) \xrightarrow[\text{extension}]{E_{\Omega\to\Omega'}} W^{k,2}_0(\Omega') \xrightarrow[\text{embedding}]{\text{Compact}} W^{k,1,2}_0(\Omega')$$

$$\downarrow \text{restriction}$$

$$W^{k,1,2}(\Omega).$$

Thm: (Poincaré inequality) Let $\Omega \subset \mathbb{R}^d$ be open & bounded, $k \in \mathbb{N}$, $k \geq 1$. Then, there exists a constant $c = c(\Omega)$ such that for all $\varphi \in W^{k,2}_0(\Omega)$:

$$\|\varphi\|^2_{W^{k,2}(\Omega)} \leq C \sum_{|\alpha|=k} \|D^\alpha \varphi\|^2_{L^2(\Omega)}.$$

Proof: Let $k=1$; the higher-order case follows by induction. By density, it suffices to show for all $\varphi \in \mathcal{D}(\Omega)$ that

$$\|\varphi\|^2_{L^2(\Omega)} \leq C \sum_{j=1}^d \|D_j \varphi\|^2_{L^2(\Omega)}.$$

Wlog, we may assume $\varphi \not\equiv 0$. Then

$$\|\varphi\|^2_{L^2(\Omega)} = \frac{1}{d} \sum_{i=1}^d \int_\Omega |\varphi|^2 \, dx = \frac{1}{d} \sum_{i=1}^d \int_\Omega (D_i x_i) |\varphi|^2 \, dx$$

$$\overset{\text{integration by parts}}{=} -\frac{1}{d} \sum_{i=1}^d \int_\Omega x_i \left(\varphi \overline{D_i \varphi} + D_i \varphi \, \overline{\varphi}\right)$$

$$\leq C \sup_{x \in \Omega} |x| \left(\int_\Omega |\varphi|^2\right)^{1/2} \left(\int_\Omega |D\varphi|^2\right)^{1/2}.$$

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