§ 1.4. Fourier transform characterization (cont'd)

Def: Let $l \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^d$ open. We define

$$H^l(\Omega) := \{ \varphi : \exists\, \varphi^e \in H^l(\mathbb{R}^d) \text{ s.t. } \varphi^e|_\Omega = \varphi \}$$

and $$\|\varphi\|^2_{H^l(\Omega)} := \inf_{\substack{\varphi^e \in H^l(\mathbb{R}^d) \\ \varphi^e|_\Omega = \varphi}} \|\varphi^e\|^2_{H^l(\mathbb{R}^d)},$$

which makes $H^l(\Omega)$ a Banach space.

Lemma: $\overline{\mathcal{D}(\mathbb{R}^d)|_\Omega}^{H^l(\Omega)} = H^l(\Omega)$.

Proof: Let $\varphi \in H^l(\Omega)$ and choose $\varphi^e \in H^l(\mathbb{R}^d)$ s.t. $\varphi^e|_\Omega = \varphi$. For all $\varepsilon > 0$, we may choose $\gamma^e \in \mathcal{D}(\mathbb{R}^d)$ s.t. $\|\varphi^e - \gamma^e\|^2_{H^l(\mathbb{R}^d)} \le \varepsilon$, since $H^l(\mathbb{R}^d) \simeq W^{l,2}(\mathbb{R}^d) = W^{l,2}_0(\mathbb{R}^d)$. But then

$$\|\varphi - \gamma^e|_\Omega\|^2_{H^l(\Omega)} \le \|\varphi^e - \gamma^e\|^2_{H^l(\mathbb{R}^d)} \le \varepsilon. \quad \blacksquare$$

Rmk: $H^l_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^l(\Omega)}$. Then $H^l_0(\Omega) = W^{l,2}_0(\Omega)$.

A simple observation:

Let $\Omega' \subseteq \Omega \subseteq \mathbb{R}^d$ be open sets. Then, the restriction operators

$$R_{\Omega \to \Omega'} : W^{l,2}(\Omega) \to W^{l,2}(\Omega')$$
$$\varphi \mapsto \varphi|_{\Omega'}$$

as well as

$$R_{\Omega \to \Omega'} : H^l(\Omega) \to H^l(\Omega')$$
$$\varphi \mapsto \varphi|_{\Omega'}$$

are both continuous linear operators.

Def: Let $l \geq 0$, $\Omega' \subseteq \Omega \subseteq \mathbb{R}^d$ be open sets. We call a linear and continuous operator

$$E_{\Omega' \to \Omega} : W^{l,2}(\Omega') \to W^{l,2}(\Omega)$$

an extension operator if

$$R_{\Omega \to \Omega'}\left(E_{\Omega' \to \Omega}(f)\right) = f, \quad \forall f \in W^{l,2}(\Omega').$$

Thm: Let $l \geq 0$, $\Omega \subseteq \mathbb{R}^d$ be open, and assume $E_{\Omega \to \mathbb{R}^d}$ exists. Then

$$W^{l,2}(\Omega) \cong H^l(\Omega).$$

Proof:

- Let $\varphi \in W^{k,2}(\Omega)$, and define extension

$$\varphi^e := \mathcal{F}_{\Omega \to \mathbb{R}^d}(\varphi) \in W^{k,2}(\mathbb{R}^d) \simeq H^k(\mathbb{R}^d).$$

Then

$$\|\varphi\|_{H^k(\Omega)} \overset{\text{Def. of } \|\cdot\|_{H^k(\Omega)}}{\leq} \|\varphi^e\|_{H^k(\mathbb{R}^d)}$$

$$\overset{\text{Lecture 7}}{\leq} C \|\varphi^e\|_{W^{k,2}(\mathbb{R}^d)}$$

$$\leq C \|\varphi\|_{W^{k,2}(\Omega)}.$$

(continuity of $\mathcal{F}_{\Omega \to \mathbb{R}^d}$)

- Let $\varphi \in H^k(\Omega)$; and let $\varphi^e \in H^k(\mathbb{R}^d) \simeq W^{k,2}(\mathbb{R}^d)$ s.t. $\varphi^e|_\Omega = \varphi$. In particular, $\varphi \in W^{k,2}(\Omega)$

and

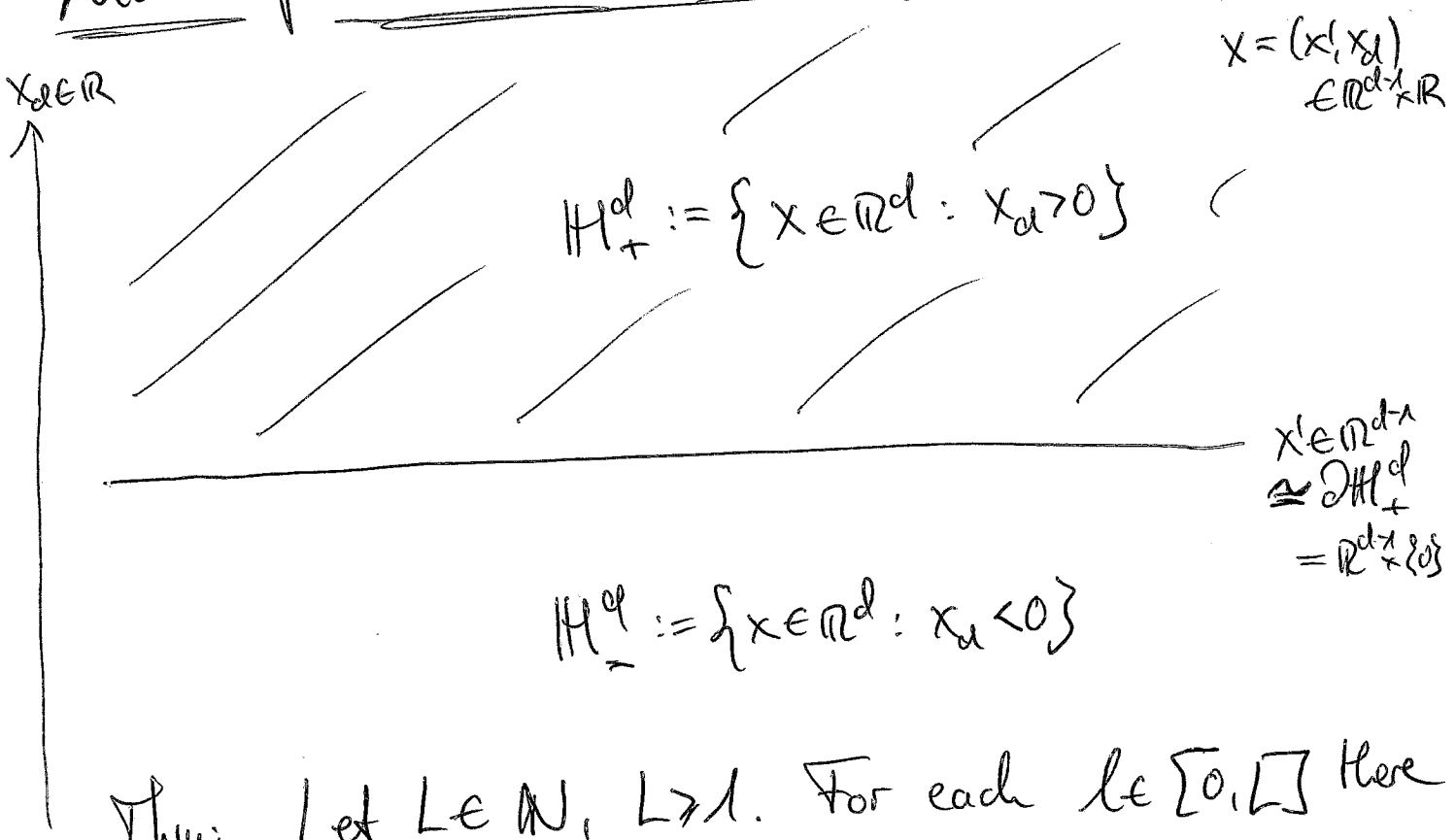
$$\|\varphi\|_{W^{k,2}(\Omega)} \overset{\text{Continuity of } \mathcal{R}_{\mathbb{R}^d \to \Omega}}{\leq} C \|\varphi^e\|_{W^{k,2}(\mathbb{R}^d)}$$

$$\leq C \|\varphi^e\|_{H^k(\mathbb{R}^d)} \quad (\text{Lecture 7})$$

$$\Rightarrow \quad \|\varphi\|_{W^{k,2}(\Omega)} \leq C \|\varphi\|_{H^k(\Omega)}. \qquad \blacksquare$$

§ 1.5. Extension operators

Model problem: Full space, flat boundary



Thm: Let $L \in \mathbb{N}$, $L \geq 1$. For each $\ell \in [0, L]$ there exists an extension operator

$$E_{\mathbb{H}^d_+ \to \mathbb{R}^d} : W^{\ell,2}(\mathbb{H}^d_+) \to W^{\ell,2}(\mathbb{R}^d)$$

independent of $\ell \in [0, L]$.

Proof: We aim for a recipe

$$W^{\ell,2}(\mathbb{H}^d_+) \ni \varphi \mapsto E_{\mathbb{H}^d_+ \to \mathbb{R}^d}(\varphi) \in W^{\ell,2}(\mathbb{R}^d)$$

s.t. $\exists\, C \geq 0$ with

$$\| E_{\mathbb{H}^d_+ \to \mathbb{R}^d}(\varphi) \|^2_{W^{\ell,2}(\mathbb{R}^d)} \leq C \| \varphi \|^2_{W^{\ell,2}(\mathbb{H}^d_+)}.$$

By density $\overline{\mathcal{D}(\mathbb{R}^d)|_{\mathbb{H}^d_+}}^{W^{k,2}(\mathbb{H}^d_+)} = W^{k,2}(\mathbb{H}^d_+)$ it suffices to prove this for $\varphi = \tilde{\varphi}|_{\mathbb{H}^d_+}$, $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^d)$.

Step 1: $k \in \mathbb{N}$, $k \le L$. We choose $\vec{\beta} = (\beta_0, \dots, \beta_L)$ solving the linear system

$$\sum_{m=0}^{L} \left(-\frac{1}{m+1}\right)^q \beta_m = 1, \quad q = 0, \dots, L.$$

This is indeed possible since the Vandermonde matrix

$$\left(\left(-\frac{1}{m+1}\right)^q\right)_{m,q=0,\dots,L} \in \mathbb{R}^{(L+1)\times(L+1)}$$

has non-zero determinant. We then define

$$(*) \qquad \tilde{\varphi}(x', x_d) = \sum_{m=0}^{L} \beta_m \varphi\left(x', -\frac{x_d}{m+1}\right)$$

for $(x', x_d) \in \mathbb{H}^d_-$, and introduce the extension operator by

$$(**) \qquad E_{\mathbb{H}^d_+ \to \mathbb{R}^d}(\varphi)(x) := \begin{cases} \varphi(x) & \text{if } x \in \overline{\mathbb{H}^d_+}, \\ \tilde{\varphi}(x) & \text{if } x \in \overline{\mathbb{H}^d_-}. \end{cases}$$

Now, let $q \in \{0, \dots, k\}$ and α' s.t. $q + |\alpha'| \le k$. Then, for $(x', x_d) \in \mathbb{H}^d_-$

$$D_d^q D_x^{\alpha'} \tilde{\varphi}(x', x_d) = \sum_{m=0}^{L} \beta_m \left(-\frac{1}{m+1}\right)^q D_d^q D_{x'}^{\alpha'} \varphi \Big|_{\left(x', -\frac{x_d}{m+1}\right)^\circ} \quad (***)$$

Hence, all derivatives of desired order are continuous across $\partial \mathbb{H}^q_+ = \{x : x_d = 0\}$.

Furthermore, it is clear that the recipe is linear, leaves φ unchanged in $\mathbb{H}^q_+$ ("extension-restriction consistency") and preserves Sobolev classes (due to linearity & change of coordinates $-x_d \mapsto x_d$) continuously.

Step 2: Let $k = m + \lambda$, $m \in \mathbb{N}$, $\lambda \in (0,1)$. For each $|\alpha| \le m$, it remains to show

$$\Gamma_\lambda\left(D^\alpha E_{\mathbb{H}^q_+ \to \mathbb{R}^d} \varphi\right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| D^\alpha E_{\mathbb{H}^q_+ \to \mathbb{R}^d} \varphi(x) - D^\alpha E_{\mathbb{H}^q_+ \to \mathbb{R}^d} \varphi(y) \right|^2}{|x-y|^{d+2\lambda}} dx dy$$

$$\overset{!}{\le} C \|\varphi\|^2_{W^{k,2}(\mathbb{H}^q_+)}.$$

By the recipe $\overset{(*),(**)}{\wedge}$, it is natural to decompose $\mathbb{R}^d = \mathbb{H}^q_+ \cup \mathbb{H}^q_-$ and split the estimation task into subtasks

$$\Gamma_\lambda\left(E_{\mathbb{H}^q_+ \to \mathbb{R}^d} \varphi\right) =: \Gamma_1 + \Gamma_2 + 2\Gamma_3,$$

where

$$I_1 = \int_{\mathbb{H}^d_+} \int_{\mathbb{H}^d_+} \frac{|D^\alpha \varphi(x) - D^\alpha \varphi(y)|^2}{|x-y|^{d+2\lambda}} \, dx \, dy \,,$$

$$I_2 = \int_{\mathbb{H}^d_-} \int_{\mathbb{H}^d_-} \frac{|D^\alpha \tilde{\varphi}(x) - D^\alpha \tilde{\varphi}(y)|^2}{|x-y|^{d+2\lambda}} \, dx \, dy \,,$$

$$I_3 = 2 \int_{\mathbb{H}^d_+} \int_{\mathbb{H}^d_-} \frac{|D^\alpha \tilde{\varphi}(x) - D^\alpha \varphi(y)|^2}{|x-y|^{d+2\lambda}} \, dx \, dy \,.$$

Nothing to do for I_1: $I_1 \leq \|\varphi\|^2_{W^{\lambda,2}(\mathbb{H}^d_+)}$.

Almost nothing to do for I_2: because of the formula (***) for $D^q_d D^{\alpha'}_{x'} \tilde{\varphi}$, $\alpha = (\alpha', q)$, and a change of coordinates $(x', x_d) \to (x', -x_d)$, it follows $I_2 \leq C \|\varphi\|^2_{W^{\lambda,2}(\mathbb{H}^d_+)}$.

A bit of work for I_3: since $\sum_{m=0}^{L} \beta_m \left(-\frac{1}{m+1}\right)^q = 1$, we compute by (***) for $x \in \mathbb{H}^d_-$, $y \in \mathbb{H}^d_+$ (again, $\alpha = (\alpha', q)$)

$$D^\alpha \tilde{\varphi}(x) - D^\alpha \varphi(y) = \sum_{m=0}^{L} \beta_m \left(-\frac{1}{m+1}\right)^q \left(D^q_d D^{\alpha'}_{x'} \varphi\Big|_{(x', -\frac{x_d}{m+1})} - D^q_d D^{\alpha'}_{x'} \varphi\Big|_{(y', y_d)} \right)$$

So that

$$I_3 \le C \iint_{H^q_+ \, H^q_-} \frac{\left| D_d^q D_{x'}^{\alpha'} \varphi\left(x' - \frac{x_d}{m+1}\right) - D_d^q D_{x'}^{\alpha'} \varphi(y', y_d) \right|^2}{|x-y|^{d+2\lambda}} \, dx\,dy$$

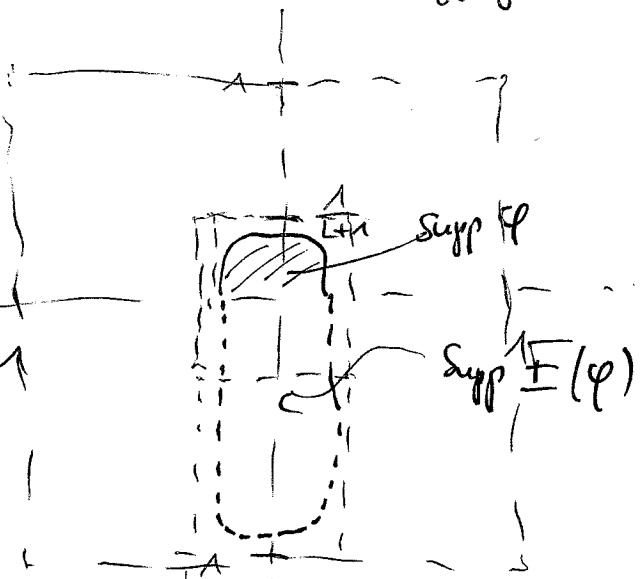
$$\le \text{change of coordinates } \left(x' - \frac{x_d}{m+1}\right) \to (x', x_d) \; C \iint_{H^q_+ \, H^q_-} \frac{\left| D_d^q D_{x'}^{\alpha'} \varphi(x) - D_d^q D_{x'}^{\alpha'} \varphi(y) \right|^2}{\left(|x'-y'|^2 + |(m+1)x_d + y_d|^2 \right)^{\frac{d+2\lambda}{2}}} \, dx\,dy.$$

Since $|x'-y'|^2 + |(m+1)x_d + y_d|^2 = |x'-y'|^2 + (m+1)^2 x_d^2 + 2(m+1)x_d y_d + y_d^2$

$$\ge x_d^2 - 2x_d y_d + y_d^2 = (x_d - y_d)^2$$

we obtain $I_3 \le C \|\varphi\|^2_{W^{l,2}(H^q_\tau)}$. ∎

Cor: The extension operator from the previous theorem restricts to a continuous linear extension operator



$$\tilde{F}_{Q^d_{1,+}} : \left\{ \varphi \in W^{l,2}(Q^d_{1,+}) : \operatorname{supp} \varphi \subseteq Q^d_{\frac{1}{L+1}, 1+} \right\}$$

$$\downarrow$$

$$\left\{ \varphi \in W^{l,2}(Q^d_1) : \operatorname{supp} \varphi \subseteq \left(-\tfrac{1}{L+1}, \tfrac{1}{L+1}\right)^{d-1} \times \left(-1, \tfrac{1}{L+1}\right) \right\}$$

$$\cap$$

$$W^{l,2}_0(Q^d_1).$$