§ 1.4. Fourier transform characterization

Def: Let $\ell \in \mathbb{R}$. We define

$$H^\ell(\mathbb{R}^d) := \left\{ \varphi \in L^2(\mathbb{R}^d) : \|\varphi\|^2_{H^\ell(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^2 (1+|\xi|^2)^\ell \, d\xi < \infty \right\},$$

where $L^2(\mathbb{R}^d) \ni \varphi \mapsto \hat{\varphi} = \mathcal{F}(\varphi) \in L^2(\mathbb{R}^d)$ is the Fourier transform, defined on the dense set of Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\hat{\varphi}(\xi) := \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) \, dx.$$

[**Recall:** $\varphi \in \mathcal{S}(\mathbb{R}^d)$ are all those $\varphi : \mathbb{R}^d \to \mathbb{C}$ in $C^\infty(\mathbb{R}^d)$ s.t. for all $\alpha, \beta \in \mathbb{N}_0^d$:

$$[\varphi]_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha| |D^\beta \varphi(x)| < \infty.]$$

Thm: $\overline{\mathcal{D}(\mathbb{R}^d)}^{H^\ell(\mathbb{R}^d)} = H^\ell(\mathbb{R}^d)$.

Proof: **Step 1:** Define

$$L^2_\ell(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \|f\|^2_{L^2_\ell(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f|^2 (1+|\xi|^2)^\ell \, d\xi < \infty \right\},$$

and we first show that $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^2_\ell(\mathbb{R}^d)$.

Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, and for $f \in L^2_\ell(\mathbb{R}^d)$ it holds $f(1+|\xi|^2)^{\ell/2} \in L^2(\mathbb{R}^d)$ there exists for each $\varepsilon > 0$ some $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\| \tilde{\varphi} - f(1+|\xi|^2)^{\ell/2} \|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon.$$

Since $\varphi := \tilde{\varphi}(1+|\xi|^2)^{-\ell/2} \in \mathcal{D}(\mathbb{R}^d)$ and

$$\|\varphi - f\|^2_{L^2_\ell(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| \frac{\tilde{\varphi}}{(1+|\xi|^2)^{\ell/2}} - f \right|^2 (1+|\xi|^2)^\ell \, d\xi$$

$$= \| \tilde{\varphi} - f(1+|\xi|^2)^{\ell/2} \|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon,$$

the claim follows.

Step 2: Since $\varphi \in H^\ell(\mathbb{R}^d) \Leftrightarrow \hat{\varphi} \in L^2_\ell(\mathbb{R}^d)$, we have to show that $\overline{\{\hat{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^d)\}}^{L^2_\ell(\mathbb{R}^d)} = L^2_\ell(\mathbb{R}^d)$.

In the exercises, you will prove:

- We say $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ iff $\forall \alpha, \beta \in \mathbb{N}_0^d : [f_n - f]_{\alpha,\beta} \xrightarrow{n \to \infty} 0$.
- $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ with respect to this convergence.
- $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous in this convergence.

Hence,

(1) $\{\hat{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^d)\}$ is dense in $\mathcal{S}(\mathbb{R}^d)$ with respect to convergence in $\mathcal{S}(\mathbb{R}^d)$.

Furthermore, you will prove in the exercises

- $$\|f\|^2_{L^2(\mathbb{R}^d)} \leq C \sum_{|\alpha| \leq d+1} [f]_{\alpha,0}, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

and it is a combinatorial exercise from the multinomial theorem that

- $$(1+|\xi|^2)^\ell \sim \sum_{|\alpha| \leq \ell} |\xi^\alpha|^2.$$

Hence, for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $\hat{\psi}, \psi \in \mathcal{D}(\mathbb{R}^d)$:

$$\begin{aligned}
\|f - \hat{\psi}\|^2_{L^2_\ell(\mathbb{R}^d)} &\leq C \sum_{|\alpha| \leq \ell} \int_{\mathbb{R}^d} |f - \hat{\psi}|^2 |\xi^\alpha|^2 \, d\xi \\
&= C \sum_{|\alpha| \leq \ell} \| \underbrace{(f - \hat{\psi}) \xi^\alpha}_{\in \mathcal{S}(\mathbb{R}^d)} \|^2_{L^2(\mathbb{R}^d)} \\
&\leq C \sum_{|\alpha| \leq \ell} \sum_{|\tilde{\alpha}| \leq d+1} [(f - \hat{\psi}) \xi^\alpha]_{\tilde{\alpha},0} \\
&\leq C \sum_{|\tilde{\alpha}| \leq \ell + d + 1} [f - \hat{\psi}]_{\tilde{\alpha}}.
\end{aligned}$$

$$\Rightarrow \quad \mathcal{S}(\mathbb{R}^d) \subseteq \overline{\{\hat{\psi} : \psi \in \mathcal{D}(\mathbb{R}^d)\}}^{L^2_\ell(\mathbb{R}^d)}.$$

so that Step 1 implies

$$L^2_\ell(\mathbb{R}^d) = \overline{\mathcal{D}(\mathbb{R}^d)}^{L^2_\ell(\mathbb{R}^d)} \subseteq \overline{\mathcal{S}(\mathbb{R}^d)}^{L^2_\ell(\mathbb{R}^d)} \subseteq \overline{\{\hat{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^d)\}}^{L^2_\ell(\mathbb{R}^d)} \subseteq L^2_\ell(\mathbb{R}^d). \quad \square$$

Thm: Let $\ell \geq 0$. Then

$$\mathcal{F} : W^{\ell,2}(\mathbb{R}^d) \to L^2_\ell(\mathbb{R}^d) \quad \text{is an isomorphism.}$$

In particular: $H^\ell(\mathbb{R}^d) \simeq W^{\ell,2}(\mathbb{R}^d)$.

Proof: **Step 1:** $\ell = m \in \mathbb{N}$.

By Plancherel,

$$\|\varphi\|^2_{W^{\ell,2}(\mathbb{R}^d)} = \sum_{|\alpha| \leq \ell} \int_{\mathbb{R}^d} |\widehat{D^\alpha \varphi}|^2 \, d\xi$$

$\widehat{D^\alpha \varphi} = \xi^\alpha \hat{\varphi}$ [recall: $D_j := \frac{1}{i}\frac{\partial}{\partial x_j}$]

$$\to \int_{\mathbb{R}^d} \left(\sum_{|\alpha| \leq \ell} |\xi^\alpha|^2 \right) |\hat{\varphi}|^2 \, d\xi,$$

but since $\sum_{|\alpha| \leq \ell} |\xi^\alpha|^2 \sim (1 + |\xi|^2)^\ell$, the claim follows.

Step 2: $\ell = m + \lambda$, $m \in \mathbb{N}$, $\lambda \in (0,1)$. By density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{\ell,2}(\mathbb{R}^d)$ and $\{\hat{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^d)\}$ in $L^2_\ell(\mathbb{R}^d)$,

It suffices to prove for $\varphi \in \mathcal{D}(\mathbb{R}^d)$:

$$\|\varphi\|^2_{W^{\lambda,2}(\mathbb{R}^d)} \sim \|\hat{\varphi}\|^2_{L^2_\ell(\mathbb{R}^d)}.$$

To this end, we need to control

$$\sum_{|\alpha|\le m} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x) - D^\alpha\varphi(y)|^2}{|x-y|^{d+2\lambda}}\,dx\,dy$$

$$\overset{y=x+z}{=} \sum_{|\alpha|\le m} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x+z) - D^\alpha\varphi(x)|^2}{|z|^{d+2\lambda}}\,dx\,dz.$$

Note that for fixed $z \in \mathbb{R}^d$, $\widehat{\varphi_z}(\xi) = e^{iz\cdot\xi}\hat{\varphi}(\xi)$, where $\varphi_z(x) := \varphi(x+z)$. Hence, by Plancherel

$$\int_{\mathbb{R}^d} |D^\alpha\varphi(x+z) - D^\alpha\varphi(x)|^2\,dx = \int_{\mathbb{R}^d} |e^{iz\cdot\xi} - 1|^2\, |\xi^\alpha|^2\, |\hat{\varphi}(\xi)|^2\,d\xi.$$

Hence

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x+z) - D^\alpha\varphi(x)|^2}{|z|^{d+2\lambda}}\,dx\,dz$$

$$= \int_{\mathbb{R}^d} |\xi^\alpha|^2\, |\hat{\varphi}(\xi)|^2 \left(\int_{\mathbb{R}^d} \frac{|e^{iz\cdot\xi} - 1|^2}{|z|^{d+2\lambda}}\,dz \right) d\xi$$

and furthermore

$$\int_{\mathbb{R}^d} \frac{|e^{iz\cdot\xi}-1|^2}{|z|^{d+2\lambda}}\,dz = \int_0^\infty \int_{\partial B_1} |e^{ir\cdot\xi}-1|^2 \,\dots$$

$$= \int_0^\infty \left(\int_{\partial B_1} \frac{|e^{irz\cdot\xi}-1|^2}{r^{1+2\lambda}}\, d\sigma(z) \right) dr$$

$$= \int_0^\infty \frac{1}{r^{1+2\lambda}} \left(\int_{\partial B_1} |e^{i(|\xi|r)z\cdot\frac{\xi}{|\xi|}}-1|^2\, d\sigma(z) \right) dr$$

$\tilde{r} = |\xi| r$, $dr = \frac{1}{|\xi|} d\tilde{r}$

$$= |\xi|^{2\lambda} \int_0^\infty \frac{1}{r^{1+2\lambda}} \underbrace{\left(\int_{\partial B_1} |e^{irz\cdot e_1}-1|^2\, d\sigma(z) \right)}_{=: 0 < C_{d,\lambda} < \infty} dr$$

$$= C_{d,\lambda} |\xi|^{2\lambda}.$$

In summary:

$$\sum_{|\alpha|\le m} \mathcal{F}_\lambda(D^\alpha \varphi) = \sum_{|\alpha|\le m} C_{d,\lambda} \int_{\mathbb{R}^d} |\xi^\alpha|^2 |\xi|^{2\lambda} |\hat{\varphi}(\xi)|^2,$$

which proves the claim. ∎