§ 1.3. Change of coordinates (cont'd)

Thm: Let $m \in \mathbb{N}$, $\lambda \in (0,1)$, $k \in \mathbb{N}$ s.t. $l := m + \lambda \leq k$.
Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be two open sets of class C^1, and let $\Phi : \Omega \to \Omega'$ be a C^k- diffeomorphism.

Then
- $W^{l,2}(\Omega) \simeq W^{l,2}(\Omega')$
- $W^{l,2}_0(\Omega) \simeq W^{l,2}_0(\Omega')$

by means of the isomorphism (a continuous and linear operator, which is bijective with continuous inverse)

$$T_\Phi : W^{l,2}_{(0)}(\Omega') \to W^{l,2}_{(0)}(\Omega)$$
$$\varphi' \mapsto \varphi' \circ \Phi .$$

Proof: For C^1 open sets, we proved in Lecture 5 that $\overline{\mathcal{D}(\mathbb{R}^d)|_{\Omega'}}^{W^{l,2}(\Omega')} = W^{l,2}(\Omega')$. However, the proof actually yields the slight improvement that

$$U := \mathcal{D}(\mathbb{R}^d)|_{\Omega'} \cap \{\varphi \in W^{l,2}(\Omega') : K := \operatorname{supp}\varphi \text{ is bounded}\}$$

is a dense linear subspace of $W^{l,2}(\Omega')$. Hence, it suffices to prove

$$\|T_\Phi(\varphi')\|^2_{W^{l,2}(\Omega)} \leq C \|\varphi'\|^2_{W^{l,2}(\Omega')}, \quad \forall \varphi' \in U.$$

We already know from last lecture that

$$\|F_\Phi(\varphi')\|^2_{W^{m,2}(\Omega)} \leq C \|\varphi'\|^2_{W^{m,2}(\Omega')}.$$

Hence, it remains to show for all $|\alpha| \leq m$ that

$$\underline{I}_\lambda\left(D^\alpha F_\Phi(\varphi')\right) = \iint_{\Omega\Omega} \frac{|D^\alpha F_\Phi(\varphi')(x) - D^\alpha F_\Phi(\varphi')(y)|^2}{|x-y|^{d+2\lambda}}\, dx\, dy \quad (*)$$

$$\leq C \|\varphi'\|_{W^{m,2}(\Omega')}.$$

We will again only consider the case $|\alpha|=1$; the higher-order case can be established analogously by the Faà di Bruno formula.

Proof of (*) for $|\alpha|=1$: By the finite covering lemma applied to $K := \operatorname{supp}\varphi'$, $\varphi' \in \mathcal{U}$ fixed, and an associated finite partition of unity, we may decompose $\varphi' = \sum_{n=1}^N \varphi'_n$, $\varphi'_n = \alpha_n \varphi'$, on $\overline{K}$, such that
- $\varphi'_n \in W^{m,2}(V'_n \cap \Omega')$, $\operatorname{supp}\varphi'_n \subseteq V'_n \cap \Omega'$
- $V'_n \cap \Omega'$ path-connected.

By the linearity of F_Φ and the triangle inequality for L^2-spaces, it suffices to show (*) for each φ'_n, $n = 1, \dots, N$. → and the multiplies property

Hence, fix $n \in \{1, \dots, N\}$ and abbreviate $V_n'' := V_n' \cap \Omega'$.

Decomposing $\Omega = \underset{\text{disjointly}}{\cup} \Phi^{-1}(V_n'') \cup \Phi^{-1}(\Omega' \setminus V_n'')$,

we may further subdivide the estimation task (*) as

$$\begin{aligned} & V_\lambda\left(D_i F_\Phi(\varphi_n')\right) \\ &= \int\int_{\Phi^{-1}(V_n'')\,\Phi^{-1}(V_n'')} \frac{|D_i F_\Phi(\varphi_n')(x) - D_i F_\Phi(\varphi_n')(y)|^2}{|x-y|^{d+2\lambda}}\, dy\, dx \\ &\quad + 2\int\int_{\Phi^{-1}(V_n'')\,\Phi^{-1}(\Omega' \setminus V_n'')} \frac{|D_i F_\Phi(\varphi_n')(x)|^2}{|x-y|^{d+2\lambda}}\, dy\, dx. \\ &=: V_1 + 2V_2. \end{aligned}$$

We proceed estimating term by term, starting with V_1.

By the chain rule

$$D_i F_\Phi(\varphi_n') = \left((D\varphi_n') \circ \Phi\right) D_i \Phi,$$

and by the triangle inequality

$$\begin{aligned} & \left| (D\varphi_n')(\Phi(x))\, D_i\Phi(x) - (D\varphi_n')(\Phi(y))\, D_i\Phi(y) \right|^2 \\ &\le 2^2 \Big(|(D\varphi_n')(\Phi(x))|^2 \, |D_i\Phi(x) - D_i\Phi(y)|^2 \\ &\qquad + |D_i\Phi(y)|^2 \, |(D\varphi_n')(\Phi(x)) - D\varphi_n'(\Phi(y))|^2 \Big), \end{aligned}$$

So that we subdivide the estimation task for Σ_1 into $\Sigma_1 \leq 2^2 (\Sigma_1' + \Sigma_1'')$, 4

where $$\Sigma_1' = \int_{\Phi^{-1}(V_n'')} |(D\varphi_n')(\Phi(x))|^2 \int_{\Phi^{-1}(V_n'')} \frac{|D_i\Phi(x) - D_i\Phi(y)|^2}{|x-y|^{d+2\lambda}} dy\, dx$$

~~$\Sigma_1'' = \int_{\Phi^{-1}(V_n'')} |D_i\Phi(y)|^2 \int_{\Phi^{-1}(V_n'')} |(D\varphi_n')(\Phi(x)) - (D\varphi_n')(\Phi$~~

$$\Sigma_1'' = \int_{\Phi^{-1}(V_n'')} |D_i\Phi(y)|^2 \int_{\Phi^{-1}(V_n'')} \frac{|(D\varphi_n')(\Phi(x)) - (D\varphi_n')(\Phi(y))|^2}{|x-y|^{d+2\lambda}} dy\, dx.$$

Since $\Phi \in C^2(\bar{\Omega})$ and since $\Phi^{-1}(V_n'')$ is path-connected, we may apply the mean value inequality in the form of

$$|D_i\Phi(x) - D_i\Phi(y)| \leq C|x-y|$$

for $x, y \in \Phi^{-1}(V_n'')$ and obtain

$$\Sigma_1' \leq C \int_{\Phi^{-1}(V_n'')} |(D\varphi_n')(\Phi(x))|^2 \underbrace{\left(\int_{\Phi^{-1}(V_n'')} \frac{|x-y|^2}{|x-y|^{d+2\lambda}} dy \right)}_{\leq C \text{ independent of } x,y} dx$$

$$\leq C \int_{\Phi^{-1}(V_n'')} |(D\varphi_n')(\Phi(x))|^2 dx$$

$$\overset{\text{transformation formula}}{\leq} C \|\varphi_n'\|^2_{W^{1,2}(\Omega')}.$$

~~$\leq$ multiplies properly $C \|\varphi_n'\|$~~

This bounds Σ_1' of required type.

To estimate I_4'', we would like to apply the transformation formula directly to the double integral defining I_1'' [effect estimating $|D_i \Phi(y)|^2 \le \|D_i \Phi\|^2_{L^\infty(\Omega)}$] in order to relate I_1'' directly to $I_1(D\varphi_n') \le \|\varphi_n'\|^2_{W^{1,2}(\Omega')}$; however, for this we first have to replace $\frac{1}{|x-y|^{d+2\lambda}}$ by $\frac{1}{|\Phi(x)-\Phi(y)|^{d+2\lambda}}$. This is possible by the mean value inequality applied to Φ^{-1} in the form of

$$|x-y| \ge \frac{1}{C} |\Phi(x)-\Phi(y)|, \quad x,y \in \Phi^{-1}(V_n'').$$

Hence

$$I_4'' \le C \int_{\Phi^{-1}(V_n'')} \int_{\Phi^{-1}(V_n'')} \frac{|D\varphi_n'(\Phi(x)) - D\varphi_n'(\Phi(y))|^2}{|\Phi(x)-\Phi(y)|^{d+2\lambda}} dy\,dx$$

$$\overset{\text{transformation formula}}{\le} C\, I_1(D\varphi_n') \le C \|\varphi_n''\|^2_{W^{1,2}(\Omega')}.$$

In summary, $I_1 \le C \|\varphi_n'\|^2_{W^{1,2}(\Omega')}$.

It remains to estimate

$$I_2 = \int_{\Phi^{-1}(V_n'')} \int_{\Phi^{-1}(\Omega' \setminus V_n'')} \frac{|D_i F_{\Phi}(\varphi_n')(x)|^2}{|x-y|^{d+2\lambda}} dy\,dx.$$

The idea is that a pair $(x,y) \in \Phi^{-1}(\operatorname{supp}\varphi_n') \times \Phi^{-1}(\Omega' \setminus V_n'')$ avoids the diagonal $\{(x,y) : x=y\}$ with a fixed distance.

Indeed, by the localization property of the partition of unity function α_n, there exists $C \geq 1$ s.t.

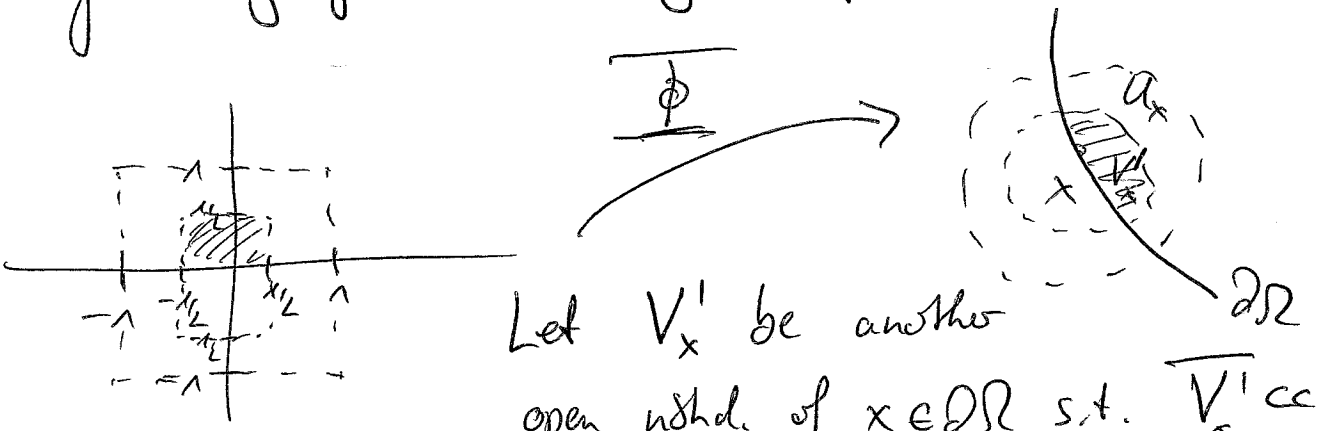
$$|x-y| \geq 1/C \quad , \forall (x,y) \in \overline{\Phi}^{-1}(\operatorname{supp} \varphi_n') \times \overline{\Phi}^{-1}(\Omega' \setminus V_n').$$

Hence, the chain rule and the transformation formula directly imply

$$\overline{\tau}_2 \leq C \|D\varphi_n'\|^2_{L^2(\Omega')} \leq C \|\varphi_n'\|^2_{W^{1,2}(\Omega')}. \quad \square$$

Cor: Let $\Omega \subseteq \mathbb{R}^d$ be of class C^1, $x \in \partial\Omega$,

$\overline{\Phi}: Q_1^d := (-1,1)^d \longrightarrow U_x$ nbhd. of x a "flattening of the boundary" diffeomorphism of class C^k:



Let V_x' be another open nbhd. of $x \in \partial\Omega$ s.t. $\overline{V_x'} \subset\subset U_x$.

Then, for all $l \in [0,k]$:

$$\{\varphi \in W^{l,2}(\Omega) : \operatorname{supp}\varphi \subset \overline{V_x'}\} \qquad (\text{by } \overline{\Phi})$$

$$\cong \{\varphi' \in W^{l,2}(Q_1^d) : \operatorname{supp}\varphi' \subset \overline{\overline{\Phi}^{-1}(V_x')}\}.$$

Sobolev-Slobodeckij spaces on the submanifold $\partial\Omega$

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, and of class C^k, $k \geq 1$. In particular, $\partial\Omega$ is a compact, $(d-1)$-dimensional, C^k-submanifold of $\mathbb{R}^d$. Let $(U_i)_{i=1}^N$ be a finite collection of open sets s.t. $\partial\Omega \subset \bigcup_{i=1}^N U_i$ *, and let $(\alpha_i)_{i=1}^N$ an associated partition of unity satisfying

- $\alpha_i \in \mathcal{D}(U_i)$,
- $\alpha_i \in [0,1]$,
- $\sum_{i=1}^N \alpha_i = 1$ on $\partial\Omega$.

[Exercise: Why does such data exist?]

* and such that there exists a C^k-diffeomorphism $\Phi_i : Q_1^d \to U_i$ flattening the boundary.

Def: $\varphi \in W^{k,2}(\Omega)$, $k \leq k$

$$:\Leftrightarrow \quad \alpha_i \varphi \circ \Phi_i\big|_{Q_1^{d-1}} : Q_1^{d-1} \longrightarrow \mathbb{C} \in W_0^{k,2}(Q_1^{d-1})$$

for all $i = 1, \dots, N$.

We introduce a scalar product

$$(\varphi, \psi)_k := \sum_{i=1}^N \left(\alpha_i \varphi \circ \Phi_i , \alpha_i \psi \circ \Phi_i \right)_{W^{k,2}(Q_1^{d-1})}.$$

Thm: The set $W^{k,2}(\partial\Omega)$ is independent of the choice of data $\left((U_i)_{i=1}^N, (\alpha_i)_{i=1}^N\right)$ and the associated norms are equivalent. **Proof:** Exercise.

Thm: $\overline{C^k(\partial\Omega)}^{W^{k,2}(\partial\Omega)} = W^{k,2}(\partial\Omega)$.

Proof: Let $\varphi \in W^{k,2}(\partial\Omega)$ and decompose

$$\varphi = \sum_{i=1}^{N} \varphi_i, \quad \varphi_i := \chi_i \varphi.$$

By assumption, $\varphi_i \circ \underline{\Phi}_i|_{Q_1^{d-1}} \in W_0^{k,2}(Q_1^{d-1})$. Hence, for each $\varepsilon > 0$ $\exists\, \tilde{\varphi}_i \in \mathcal{D}(Q_1^{d-1})$ s.t.

$$\| \varphi_i \circ \underline{\Phi}_i|_{Q_1^{d-1}} - \tilde{\varphi}_i \|_{W^{k,2}(Q_1^{d-1})} \leq \varepsilon.$$

Lifting $\tilde{\varphi}_i$ to $\varphi_i \in C^k(\partial\Omega)$ by pushforward under $\underline{\Phi}_i$, we obtain from transformation rule for Sobolev spaces

$$\| \varphi_i - \varphi_i \|_{W^{k,2}(\partial\Omega)} \leq C\varepsilon. \quad \square$$