## § 1.2. Density Theorems (cont'd)

**Thm:** (Global approximation by smooth functions up to the boundary)

Let $\Omega \subseteq \mathbb{R}^d$ be open and $C^k$, $k \geq 1$. Then

$$W^{k,2}(\Omega) = \overline{\{\varphi \in W^{k,2}(\Omega) : \exists\, \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^d) \text{ s.t. } \tilde{\varphi}|_\Omega = \varphi\}}^{W^{k,2}(\Omega)}.$$

**Proof:** Let $\varphi \in W^{k,2}(\Omega)$; without loss of generality, $K := \operatorname{supp} \varphi$ can be assumed to be bounded by our density theorem from Lecture 3.

**Step 1:** (Localization) For $x \in \partial\Omega$, let $U_x$ be the open neighborhoods of the definition $\Omega \in C^k$ from Lecture 4, and let $V_x'$ resp. $V_x$ be the associated neighborhoods with parameters $(\frac{\alpha_x}{2}, \frac{\beta_x}{2})$ resp. $(\frac{\alpha_x}{4}, \frac{\beta_x}{4})$. For $x \in \Omega$, choose $B_{\varepsilon_x}(x) \subset\subset \Omega$ and define $V_x' := B_{\frac{\varepsilon_x}{2}}(x)$, $V_x := B_{\frac{\varepsilon_x}{4}}(x)$. $=: U_x$ By the finite covering lemma from Lecture 4, we may select finitely many $x_1, \dots, x_k \in \partial\Omega$ and finitely many $x_{k+1}, \dots, x_{k+N} \in \Omega$ such that

$$K \subset \left(\bigcup_{n=1}^{k} V_n\right) \cup \left(\bigcup_{n=1}^{N} V_{k+n}\right),$$

and let $(\alpha_1, \dots, \alpha_L, \alpha_{L+1}, \dots, \alpha_{L+N})$ be an associated partition of unity. We then decompose

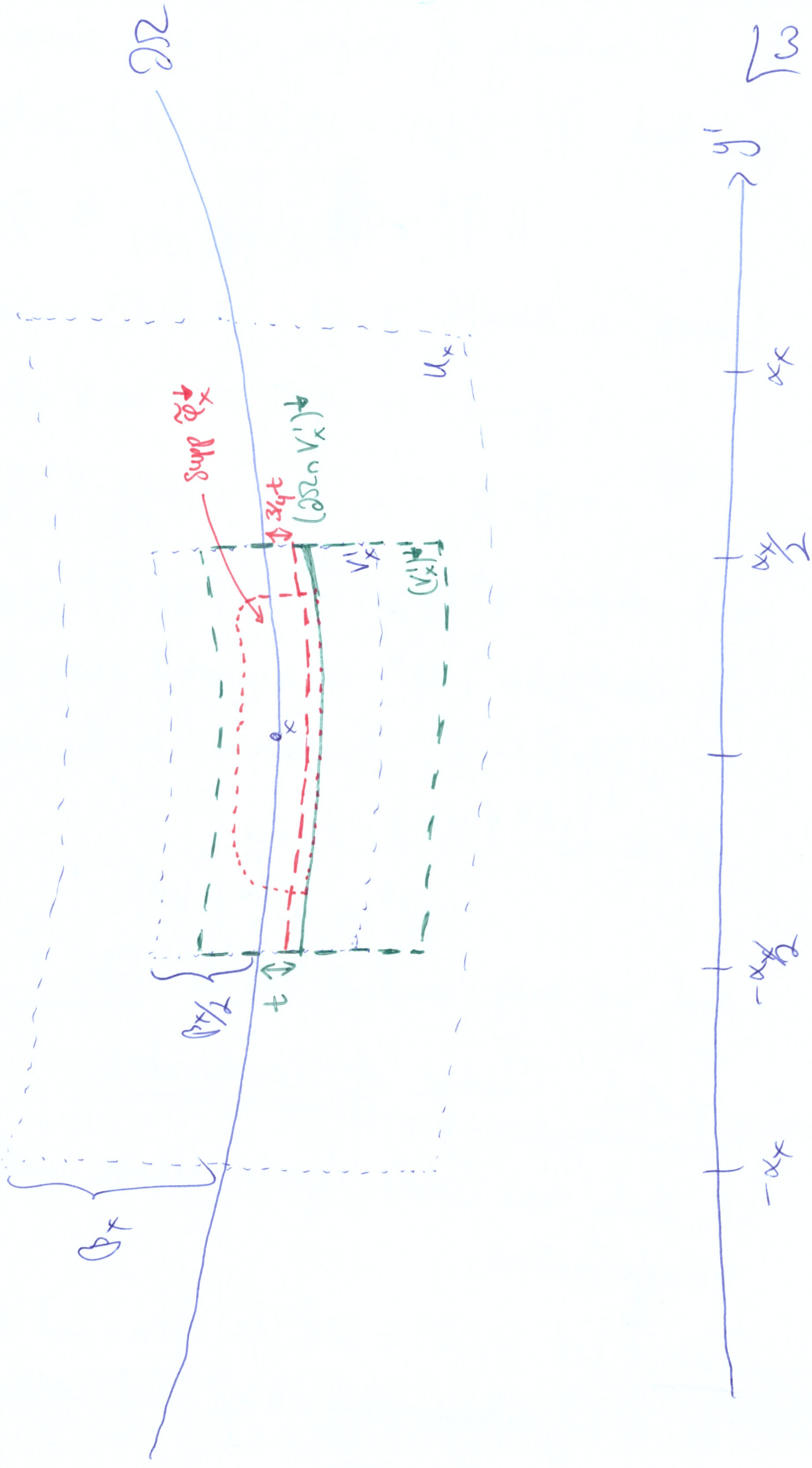
$$\varphi = \sum_{i=1}^{L+N} \varphi_i, \quad \varphi_i := \alpha_i \varphi.$$

Hence, $\operatorname{supp} \varphi_i \subseteq V_i' \cap \Omega$. It suffices to show: for all $\varepsilon > 0$ and all $i = 1, \dots, N+L$ there exists $\gamma_i \in \mathcal{D}(U_i)$ s.t. $\| \varphi_i - \gamma_i \|_{W^{k,2}(U_i \cap \Omega)} \leq \varepsilon$.

Step 2: (Interior) For $i \in \{L+1, \dots, L+N\}$, since $\operatorname{supp} \varphi_i \subset\subset V_i' \subset\subset \Omega$, it holds $\varphi_i \in W_0^{k,2}(V_i')$, and the claim follows by the very definition of the Sobolev spaces $W_0^{k,2}(\dots)$.

Step 3: (Boundary) [cf. the picture on the next page]

The only potentially problematic behaviour of $\varphi_i$, $i = 1, \dots, L$, lies on $\partial\Omega \cap V_i'$. The geometric idea therefore is to consider a shifted version of $\varphi_i$, where the problematic behaviour of this shifted function lies inside $\mathbb{R}^d \setminus \overline{\Omega}$ and with finite positive distance to $\partial\Omega \cap V_i'$. This provides enough "room" to essentially conclude like in Step 2.


Ω2
L3
y'
Δx
Δx/2
-Δx/2
-Δx
$u_{x'}$
$\vec{e}_{x'}$
supp
$\Delta y = v_{x'} t$
$(\partial\Omega_2 \cap V_{x'})^+$
$V_{x'}$
$(\partial\Omega_1 \cap V_{x'})^+$
x
t
Δx/2
Δx

By $L^2$-continuity of shifts, we may choose $t \ll_\varepsilon 1$ [4] such that $\tilde{\varphi}_i^\downarrow(y', y_d) := \varphi_i(y', y_d + t)$ satisfies

$$\| \tilde{\varphi}_i^\downarrow - \varphi_i \|_{W^{k,2}(U_i \cap \Omega)} \leq \varepsilon/2 .$$

Choose a smooth $\gamma$ s.t. $\gamma \in [0,1]$ and

- $\gamma = 1$ in $\{x : \operatorname{dist}(x, V_i' \cap \Omega) \leq t/4\}$,
- $\gamma = 0$ in $\{x : \operatorname{dist}(x, V_i' \cap \Omega) \geq \frac{3}{4} t\}$,

and define $\varphi_i^\downarrow(y', y_d) := \gamma(y', y_d)\, \tilde{\varphi}_i^\downarrow(y', y_d)$.

By construction, $\varphi_i^\downarrow$ is supported away from $(\partial\Omega, \tilde{V}_i)^\downarrow := (\partial\Omega \cap \tilde{V}_i) - t$. Furthermore, by the multiplier property $\varphi_i^\downarrow \in W^{k,2}(V_i')$. Hence

$$\varphi_i^\downarrow \in W^{k,2}_0(V_i')$$

and we may conclude as in Step 2. ∎

## § 1.3. Change of coordinates

We study how Sobolev spaces behave under coordinate transformations.

**Thm:** Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be open, and let $\Phi : \Omega \to \Omega'$ be a $C^k$-diffeomorphism, $k \geq 1$ (cf. Lecture 4). Let $l \leq k$, $l \in \mathbb{N}$. Then the map

$F_\Phi : W^{1,2}(\Omega') \to W^{1,2}(\Omega)$, $\varphi' \mapsto \varphi' \circ \Phi$

is a continuous linear operator with continuous linear inverse

$$F_{\Phi^{-1}} = (F_\Phi)^{-1} : W^{1,2}(\Omega) \to W^{1,2}(\Omega')$$
$$\varphi \mapsto \varphi \circ \Phi^{-1}.$$

In particular: $W^{1,2}(\Omega) \simeq W^{1,2}(\Omega')$. We also obtain via $F_\Phi | W^{1,2}_0(\Omega)$ that $W^{1,2}_0(\Omega) \simeq W^{1,2}_0(\Omega')$.

Proof: Step 1: It suffices to show

(*) $\| F_\Phi(\varphi') \|_{W^{1,2}(\Omega)} \leq C \| \varphi' \|_{W^{1,2}(\Omega')}$

for all $\varphi' \in C^\infty(\Omega') \cap W^{1,2}(\Omega')$. Indeed, by density and trivial functional analytic arguments, (*) then also holds for all $\varphi \in W^{1,2}(\Omega')$.

Since by the same argument (i.e. switching the roles of $\Omega$ and $\Omega'$), one may establish the continuity of $F_{\Phi^{-1}}$ with corresponding estimate, and since obviously $F_\Phi$ and $F_{\Phi^{-1}}$ are inverse to each other, the claim follows.

**Step 2:** (Proof of (*1))

Let $\varphi' \in C^\infty(\Omega') \cap W^{l,2}(\Omega')$, hence

$$\varphi := T_\Phi(\varphi') = \varphi' \circ \Phi \in C^k(\Omega).$$

Note that $\varphi \in L^2(\Omega)$ as a direct consequence of the transformation formula:

$$\int_\Omega |\varphi|^2 = \int_{\Phi(\Omega)} |\varphi \circ \Phi^{-1}|^2 \, |\det D\Phi^{-1}|$$

$$\downarrow \text{ transformation formula}$$

$$= \int_{\Omega'} |\varphi'|^2 \, |\det D\Phi^{-1}|$$

$$\leq \text{ boundedness of Jacobian for } C^k\text{-diffeo's} \quad C \int_{\Omega'} |\varphi'|^2 .$$

Furthermore, by the chain rule

$$D_i \varphi = \left( (D\varphi') \circ \Phi \right) D_i \Phi ,$$

so that we may estimate

$$\int_\Omega |D_i \varphi|^2 \leq \int_\Omega \| D\Phi \|_{L^\infty}^2 \, |(D\varphi') \circ \Phi|^2$$

$$\leq \text{ transformation formula and boundedness of Jacobian as above} \quad \int_{\Omega'} |D\varphi'|^2 .$$

This shows the claim for $l=1$; the higher-order case $l>1$ follows analogously based on the Faà di Bruno formula.