Lecture 4, Advanced PDE, WS 2026/27

§ 1.2. Density theorems (cont'd)

A clarification for function space notations:

Let $\Omega \subseteq \mathbb{R}^d$ be open and $l \in \mathbb{N}$.

- The set $\mathcal{E}^l(\Omega)$ is the set of all $f: \Omega \to \mathbb{C}$ which are continuously differentiable up to order l.
- We define $\mathcal{E}^\infty(\Omega) := C^\infty(\Omega) := \bigcap_{l \geq 0} \mathcal{E}^l(\Omega)$.
- The set $C^l(\Omega)$ is the subset consisting of those $f \in \mathcal{E}^l(\Omega)$ such that for all $|\alpha| \leq l$ the function is bounded. We norm this space by $\sup_{|\alpha| \leq l} \sup_{x \in \Omega} |D^\alpha f(x)|$.
- The set $C^l(\overline{\Omega})$ is the subspace of $f \in C^l(\Omega)$ such that for all $|\alpha| \leq l$ the function $D^\alpha f$ admits a continuous extension to $\overline{\Omega}$.
- $\mathcal{D}(\Omega) := \{ f \in C^\infty(\Omega) : \operatorname{supp} f \subset\subset \Omega \}$.

Thm: (Global approximation by smooth functions)

$$W^{k,2}(\Omega) = \overline{\{ \varphi \in W^{k,2}(\Omega) : \varphi \in C^\infty(\Omega) \}}.$$

Proof: ($l \in \mathbb{N}$ only) Let $\varphi \in W^{l,2}(\Omega)$ be given.

For each $\varepsilon > 0$, we need to show that there exists $\varphi_\varepsilon \in C^\infty(\Omega) \cap W^{l,2}(\Omega)$ such that $\|\varphi - \varphi_\varepsilon\|_{W^{l,2}(\Omega)} \leq 2\varepsilon$.

Since Sobolev functions with bounded support are dense in $W^{l,2}(\Omega)$,* we may find $\tilde{\varphi} \in W^{l,2}(\Omega)$ with $K := \operatorname{supp} \varphi \subset \Omega$ bounded such that $\|\varphi - \tilde{\varphi}\|_{W^{l,2}(\Omega)} \leq \varepsilon$.

Hence, to prove the claim it suffices to approximate $\tilde{\varphi}$ by functions from $C^\infty(\Omega) \cap W^{l,2}(\Omega)$.

*Lecture 3

To this end, let $K \subset \bigcup_{j \geq 0} B_{r_j}(x_j)$ be a locally finite cover of K with $B_{r_j}(x_j) \subset\subset \Omega$, and let $(\alpha_i)_{i \in \mathbb{N}}$ be a partition of unity subordinate to it. The existence of these is guaranteed by Lecture 3.

Now, decompose $\tilde{\varphi} = \sum_{i=0}^{\infty} \alpha_i \tilde{\varphi}$ so that by the multiplier property from Lecture 3, $\varphi_i := \alpha_i \tilde{\varphi} \in W_0^{l,2}(B_{r_{j(i)}}(x_{j(i)}))$.

By definition of the Sobolev space $W_0^{l,2}(\dots)$, we may find for each $i \in \mathbb{N}$ and $n \geq 1$ an element

$$\varphi_i^n \in \mathcal{D}(B_{r_{j(i)}}(x_{j(i)}))$$

such that

$$\|\varphi_i^n - \varphi_i\|_{W^{l,2}(\Omega)} \leq \frac{1}{n 2^i}.$$

Define $\varphi^n := \sum_{i=0}^{\infty} \varphi_i^n \in C^\infty(\Omega)$.

↑ because the sum is locally finite

Note that

$$\sum_{i=N}^{N+m} \| \varphi_i^n - \varphi_i \|_{W^{k,2}(\Omega)} \leq \frac{1}{n} \frac{1}{2^N} 2 .$$

Hence, the series $\varphi^n - \tilde{\varphi} = \sum_{i=0}^{\infty} (\varphi_i^n - \varphi_i)$ converges absolutely in $W^{k,2}(\Omega)$ with norm

$$\| \varphi^n - \tilde{\varphi} \|_{W^{k,2}(\Omega)} \leq \frac{2}{n} .$$

In particular, $\varphi^n \in W^{k,2}(\Omega) \cap C^\infty(\Omega)$ and

$$\varphi^n \xrightarrow{n \to \infty} \tilde{\varphi} \text{ in } W^{k,2}(\Omega) . \quad \square$$

Next goal: For "sufficiently nice" $\Omega \subseteq \mathbb{R}^d$, prove that $\{ \varphi \in W^{k,2}(\Omega) : \exists \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^d) \text{ s.t. } \tilde{\varphi}|_\Omega = \varphi \}$ is dense in $W^{k,2}(\Omega)$. To this end, we first increase our toolbox.

Lemma: (Localization & partition of unity; closure version)

Let $\Omega \subseteq \mathbb{R}^d$ be open, and for each $x \in \overline{\Omega}$ let V_x, V_x', U_x be open neighborhoods of $x \in \overline{\Omega}$ such that

- $\overline{V_x} \subset\subset V_x'$, $\overline{V_x'} \subset\subset U_x$, $U_x \subset\subset \mathbb{R}^d$,
- and for $x \in \Omega$: $U_x \subset\subset \Omega$.

Let $K \subset \Omega$ bounded & relatively closed in Ω. Then there exists a finite selection of $(x_i)_{i=1,\dots,N}$ in $\overline{\Omega}$ s.t. for $V_i := V_{x_i}$, $V_i' := V_{x_i}'$ it holds

(i) $\overline{K} \subset\subset \bigcup_{i=1}^{N} V_i$,

(ii) $\exists\, (\alpha_i)_{i=1,\dots,N}$:
- $\alpha_i \in \mathcal{D}(V_i')$,
- $\alpha_i \in [0,1]$,
- $\sum_{i=1}^{N} \alpha_i = 1$ on $\overline{K}$.

Proof: (i) Define $\mathcal{F}_0$ as in the proof of the interior localization lemma from Lecture 3,* and cover it by finitely many V_i for $x_i \in \Omega$.

Since $\overline{K}$ is compact and by construction of $\mathcal{F}_0$, we may choose finitely many V_i for $x_i \in \partial\Omega$ such that

$$\overline{K} \subset\subset \bigcup_{i=1}^{N} V_i.$$

* by this, $\mathcal{F}$ mean with respect to the given family $(V_x)_{x\in\partial\Omega}$

(ii) Since (i) holds and $\overline{\bigcup_{i=1}^{N} V_i'} \subset\subset \mathbb{R}^d$, we may choose a finite collection of open sets

$$(V_i^{aux})_{i=N+1}^{N+m}, \quad (V_i^{1,aux})_{i=N+1}^{N+m}$$

such that
- $\bigcup_{i=N+1}^{N+m} V_i^{1,aux} \cap \overline{K} = \emptyset$,
- $\left(\bigcup_{i=1}^{N} V_i\right) \cup \left(\bigcup_{i=N+1}^{N+m} V_i^{aux}\right) \supset\supset \bigcup_{i=1}^{N} V_i'$.

Abbreviate for convenience $\begin{cases} V_i := V_i^{aux}, & i = N+1, \dots, N+m, \\ V_i' := V_i^{1,aux}, & i = N+1, \dots, N+m. \end{cases}$

For every $i = 1, \dots, N, N+1, \dots, N+m$, choose $\tilde{\alpha}_i \in \mathcal{D}(V_i')$ such that $\tilde{\alpha}_i = 1$ in V_i,

define $\gamma := \sum_{i=1}^{N+m} \tilde{\alpha}_i$ and finally

$$\alpha_i := \frac{\tilde{\alpha}_i}{\gamma} \quad \text{for } i = 1, \dots, N,$$

which fulfil the required properties. ∎

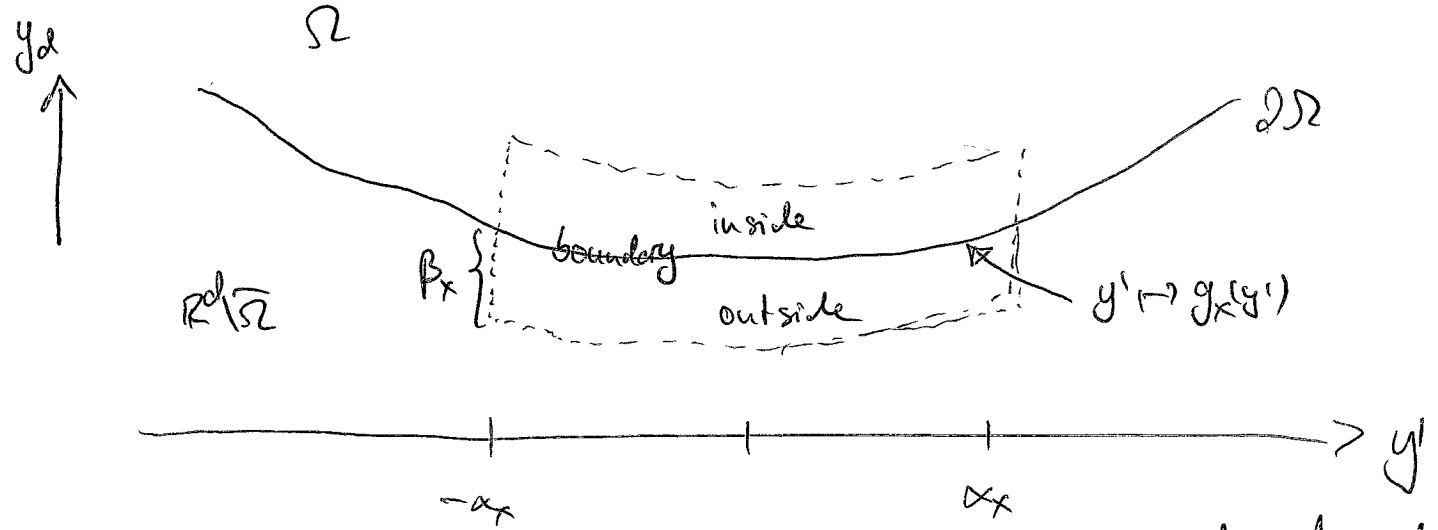
To formalize what we mean by "sufficiently nice" $\Omega \subseteq \mathbb{R}^d$, we introduce the notions of C^k-domains and C^k-diffeomorphisms.

Def: Let $\Omega \subseteq \mathbb{R}^d$ be open and $k \in \mathbb{N}$, $k \geq 1$. We say that $\underline{\Omega \in C^k}$, if for each $x \in \partial\Omega$, possibly after a rotation and translation, there exist numbers $\alpha_x, \beta_x \in (0,1)$, a map $g_x : \underbrace{(-\alpha_x, \alpha_x)^{d-1}}_{=: Q'_{\alpha_x}} \to \mathbb{R} \in C^k$

and an open neighborhood U_x of x such that

- $\partial\Omega \cap U_x = \{(y', y_d) : y_d = g_x(y'), y' \in Q^{d-1}_{\alpha_x}\}$,
- $\Omega \cap U_x = \{(y', y_d) : g_x(y') < y_d < g_x(y') + \beta_x\}$,
- $(\mathbb{R}^d \setminus \overline{\Omega}) \cap U_x = \{(y', y_d) : g_x(y') - \beta_x < y_d < g_x(y')\}$.

Picture:



Def: Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be two open sets, $k \geq 1$.

A map $\Phi : \Omega \to \Omega'$ is called a C^k-diffeomorphism

if
- $\Phi : \Omega \to \Omega'$ is bijective and of class $C^k(\overline{\Omega})$,
- $\Phi^{-1} : \Omega' \to \Omega \in C^k(\overline{\Omega'})$
- there exist $0 < c < C < \infty$ such that $\forall x \in \overline{\Omega}: \; c \leq |\det D\Phi(x)| \leq C$.

Thm. [without proof]

$\Omega \in C^k$ iff for each $x \in \partial\Omega$ there exists a C^k-diffeo $\Phi_x : Q_1^d \to U_x$, where U_x is an open nbhd. of $x \in \partial\Omega$, such that

- $\Phi_x\left(Q_1^d \cap \{y_d = 0\}\right) = U_x \cap \partial\Omega$,
- $\Phi_x\left(Q_1^d \cap \{y_d > 0\}\right) = U_x \cap \Omega$,
- $\Phi_x\left(Q_1^d \cap \{y_d < 0\}\right) = U_x \cap (\mathbb{R}^d \setminus \overline{\Omega})$.

Picture:

