§ 1.1. Sobolev & Sobolev-Slobodeckij spaces (cont'd)

Lemma (multiplier property): Let $l \geq 0$, $a \in C^{\lceil l \rceil}(\mathbb{R}^d)$, $\varphi \in W^{l,2}(\Omega)$. Then $a\varphi \in W^{l,2}(\Omega)$ and

$$\|a\varphi\|_{W^{l,2}(\Omega)} \leq C \|a\|_{C^{\lceil l \rceil}(\mathbb{R}^d)} \|\varphi\|_{W^{l,2}(\Omega)}.$$

[I.e., $\varphi \mapsto a\varphi$ is a continuous linear operator $W^{l,2}(\Omega) \to W^{l,2}(\Omega)$.]

Proof: (only the case $l \in \mathbb{N}$) Let $l=1$, the other cases $l>1$ can be deduced inductively [try it!].

Let $\varphi \in \mathcal{D}(\Omega)$ and compute by Leibniz rule for $C^1(\Omega)$-functions and $\varphi \in W^{1,2}(\Omega)$: $\forall j = 1, \dots, d$

$$\int_\Omega (D_j \varphi)\, a\varphi \, dx = -\int_\Omega \varphi \, D_j(a\varphi)\, dx$$

$$= -\int_\Omega \varphi \,(a D_j \varphi + (D_j a)\varphi)\, dx.$$

[This step actually needs an approximation argument since $a\varphi \notin \mathcal{D}(\Omega)$ in general. (Try to modify a.)] ← [will be an exercise]

Hence, by rearranging terms

$$\int_\Omega \left((D_j \varphi) a + \varphi D_j a \right) \psi \, dx = - \int a \varphi D_j \psi \qquad (1)$$

for all $j = 1, \dots, d$ and all $\psi \in \mathcal{D}(\Omega)$. Since

$$f_j := (D_j \varphi) a + \varphi D_j a \in L^2(\Omega),$$

it follows $a\varphi \in W^{1,2}(\Omega)$. The estimate for $\|a\varphi\|_{W^{1,2}(\Omega)}$ follows from squaring and integrating the pointwise estimates

- $|a\varphi| \leq \|a\|_{L^\infty} |\varphi|$
- $|f_j| \leq \|a\|_{L^\infty} |D_j \varphi| + \|D_j a\|_{L^\infty} |\varphi|$. ∎

§ 1.2. Density theorems

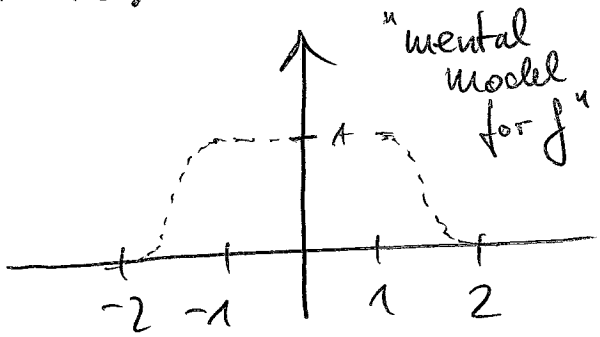
Recall: If $\varphi \in L^1_{loc}(\Omega)$, we define

$$\operatorname{supp} \varphi := \Omega \setminus \{ x \in \Omega : \exists \varepsilon > 0 \text{ s.t. } \varphi \equiv 0 \text{ a.e. in } B_\varepsilon(x) \}.$$

Thm: $\{ \varphi \in W^{1,2}(\Omega) : \operatorname{supp} \varphi \subseteq \Omega \text{ is bounded} \}$ is dense in $W^{1,2}(\Omega)$.

Proof (only $k \in \mathbb{N}$): Choose $\eta \in \mathcal{D}(\mathbb{R}^d)$ such that

- $f(x) = 1$ for $x \in B_1(0)$,
- $f(x) = 0$ outside $B_2(0)$,
- $|D^\alpha f| \le M$, $\forall |\alpha| \le \ell$.



and define $f_\varepsilon(x) = f(\varepsilon x)$.

Let $\varphi \in W^{k,2}(\Omega)$ and define

$\varphi_\varepsilon := f_\varepsilon \varphi$, i.e., $\varphi_\varepsilon - \varphi = (f_\varepsilon - 1)\varphi$

φ_ε: supported inside $B_{2/\varepsilon}(0)$

$(f_\varepsilon - 1)\varphi$: supported outside $B_{1/\varepsilon}(0)$.

Hence

$$\|\varphi_\varepsilon - \varphi\|_{W^{k,2}(\Omega)} = \|(f_\varepsilon - 1)\varphi\|_{W^{k,2}(\Omega \setminus \overline{B_{1/\varepsilon}(0)})}$$

multiplier property

$$\le C \, \|f_\varepsilon - 1\|_{C^\ell(\mathbb{R}^d)} \, \|\varphi\|_{W^{k,2}(\Omega \setminus B_{\frac{1}{\varepsilon}}(0))}$$

$\|f_\varepsilon - 1\|_{C^\ell(\mathbb{R}^d)}$: bounded independent of $\varepsilon \in (0,1)$

$\|\varphi\|_{W^{k,2}(\Omega \setminus B_{\frac{1}{\varepsilon}}(0))} \to 0$ as $\varepsilon \searrow 0$.

$$\xrightarrow{\varepsilon \searrow 0} 0. \qquad \blacksquare$$

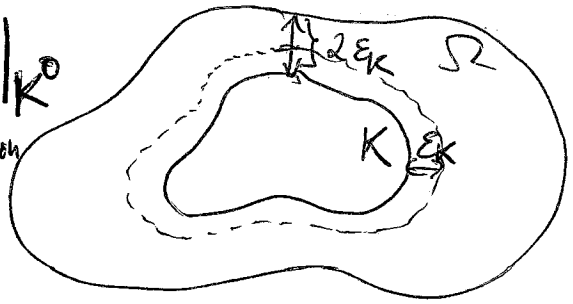
Thm. (Interior approximation by smooth functions)

Let $\varphi \in W^{k,2}(\Omega)$, $K \subset\subset \Omega$, $\varepsilon_K := \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$.

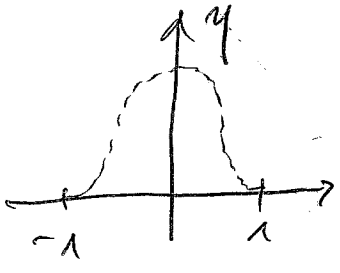
For $\varepsilon \in (0, \varepsilon_K)$ define $\varphi_\varepsilon := \eta_\varepsilon * \varphi|_K$ (convolution)

for $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta\left(\frac{|x|}{\varepsilon}\right)$, $\eta : \mathbb{R} \to [0,1]$

such that $\operatorname{supp}\eta \subseteq [-1,1]$, $\eta \in C^\infty(\mathbb{R})$, $\int_{-1}^{1} \eta = 1$.



[η_ε is called a sequence of mollifiers and $\varphi_\varepsilon := \eta_\varepsilon * \varphi$ the mollification of φ at scale ε.]



Then:
- $\operatorname{supp} \varphi_\varepsilon \subseteq K_\varepsilon := \{x : \operatorname{dist}(x,K) \le \varepsilon\} \subset\subset \Omega$, ↑ "$\varepsilon$-fattening of K"
- $\varphi_\varepsilon \in \mathcal{D}(\Omega) \cap W^{k,2}(K^\circ)$,
- $\varphi_\varepsilon \overset{\varepsilon \downarrow 0}{\longrightarrow} \varphi$ in $W^{k,2}(K^\circ)$,

where K° is the interior of K.

Proof: (only $k \in \mathbb{N}$): ∎ Immediate consequence of standard properties of mollification (see, e.g., the exercises) in the form of

$$\forall |\alpha| \le k: \quad D^\alpha \varphi_\varepsilon = D^\alpha \eta_\varepsilon * \varphi = \eta_\varepsilon * D^\alpha \varphi \overset{\varepsilon \downarrow 0}{\longrightarrow} D^\alpha \varphi \text{ in } L^2(K^\circ)$$

∎

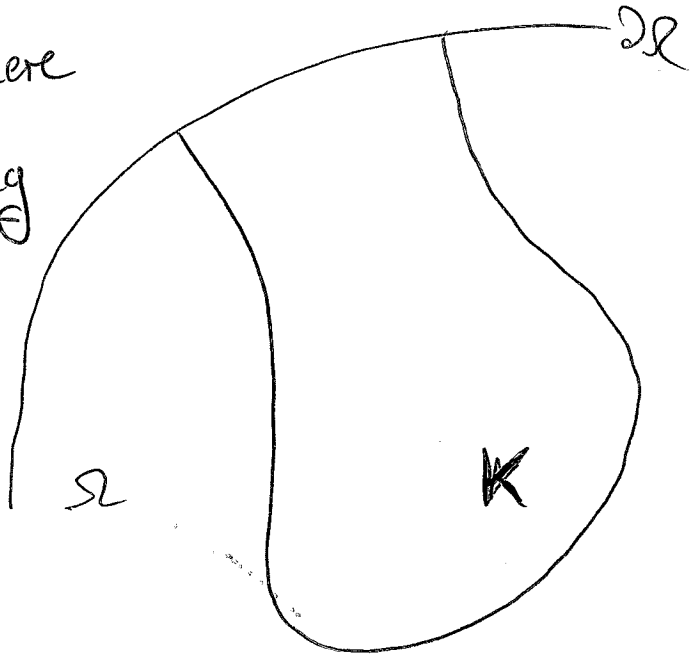
Def: $W_0^{k,2}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{k,2}(\Omega)}$.

Thm: $W_0^{k,2}(\Omega) = \overline{\{\varphi \in W^{k,2}(\Omega) : \operatorname{supp}\varphi \subset\subset \Omega\}}^{W^{k,2}(\Omega)}$.

Proof: $W_0^{k,2}(\Omega) \overset{\text{definition}}{=} \overline{\mathcal{D}(\Omega)}^{W^{k,2}(\Omega)} \overset{\text{interior}}{\subseteq} \text{r.h.s.} \overset{\text{approximation}}{\subseteq} \overline{\mathcal{D}(\Omega)}^{W^{k,2}(\Omega)} = W_0^{k,2}(\Omega)$. ∎

Lemma: (Localization, interior version)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $K \subset \Omega$ bounded and relatively closed in Ω. There exists a locally finite covering $(B_{r_j}(x_j))_{j\in\mathbb{N}}$ of K, i.e.,



- $K \subset \bigcup_{j=0}^{\infty} B_{r_j}(x_j)$,
- $\forall x \in K$, the set $\{j \in \mathbb{N} : x \in B_{r_j}(x_j)\}$ is finite,

such that $\overline{B_{r_j}(x_j)} \subset\subset \Omega$, $\forall j \in \mathbb{N}$.

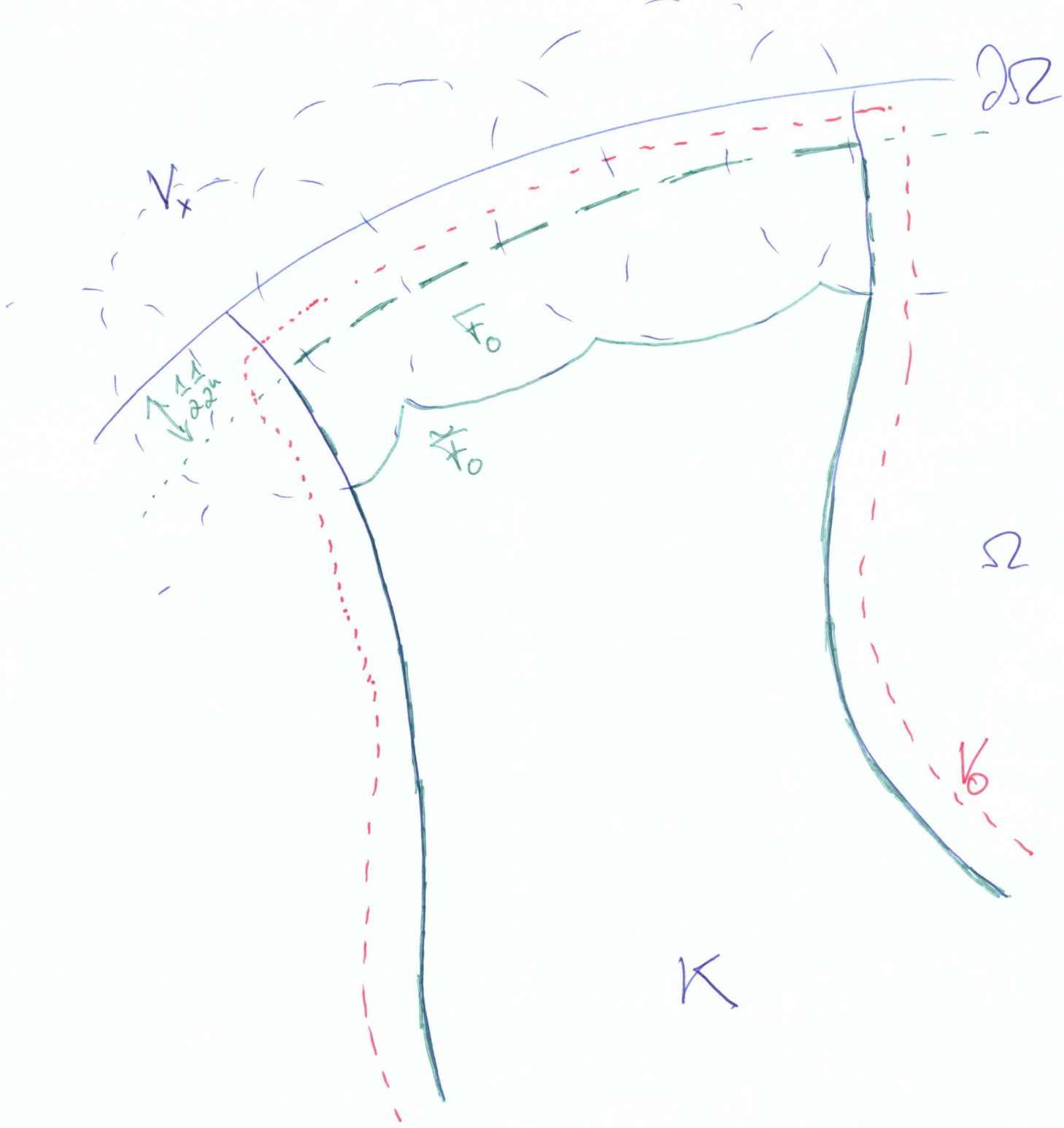
Proof: Consider an open cover $(V_x)_{x\in\partial\Omega}$ of $\partial\Omega$, i.e., $\partial\Omega \subset \bigcup_{x\in\partial\Omega} V_x$, $V_x \subseteq \mathbb{R}^d$ open and bounded. Define the compact set

$$\tilde{F}_0 := \overline{K} \setminus \bigcup_{x\in\partial\Omega} V_x \subset \Omega.$$

Since continuous functions attain their minimum on compact sets, let $n \in \mathbb{N}$ be minimal such that $2^{-n} \le \delta := \min_{x\in\tilde{F}_0} \operatorname{dist}(x, \partial\Omega)$, where $\delta > 0$. Define the compact set $F_0 := \overline{K} \setminus \{x : \operatorname{dist}(x,\partial\Omega) < \frac{1}{2}\frac{1}{2^n}\}$

$$\supseteq \tilde{F}_0,$$

and let $V_0 \subset\subset \Omega$ be an open neighborhood of F_0 given by $V_0 := \{x : \operatorname{dist}(x, F_0) < \frac{1}{4}\frac{1}{2^n}\}$.



For $n \geq 1$, we define the compact sets

$$F_n := \bar{K} \cap \left\{ \frac{1}{2^{n+n}} \leq \operatorname{dist}(\cdot, \partial\Omega) \leq \frac{1}{2^{n+(n+1)}} \right\}$$

$$\subseteq \Omega .$$

Since F_n is compact and $\left(B_{\frac{1}{4}\frac{1}{2^{n+n}}}(x)\right)_{x \in F_n}$ covers F_n, one may select a finite subcover from $\left(B_{\frac{1}{4}\frac{1}{2^{n+n}}}(x)\right)_{x \in F_n}$, $n \in \mathbb{N}$. The union of these finite subcovers has the desired property. ∎

Lemma: (Partition of unity, interior version)

In the situation of the previous lemma, there exists a partition of unity $(\alpha_i)_{i \in \mathbb{N}}$ subordinate to the cover $(B_{r_j}(x_j))_{j \in \mathbb{N}}$, i.e.,

(i) $(\operatorname{supp} \alpha_i)_{i \in \mathbb{N}}$ is locally finite,

(ii) $\forall i \in \mathbb{N} \; \exists \, j(i) \in \mathbb{N}$ s.t. $\alpha_i \in \mathcal{D}(B_{r_j}(x_j))$,

(iii) $\alpha_i \in [0,1]$

(iv) $\sum_{i=1}^{\infty} \alpha_i = 1$ in $\bigcup_{j \in \mathbb{N}} B_{r_j}(x_j)$.

Proof: **Step 1:** Let $r>0$, $x\in\mathbb{R}^d$ be fixed, and consider $B := B_r(x)$. By the previous covering lemma applied to $K=\overline{\Omega}=\overline{B}$, there exists $(B_{r_i}(x_i))_{i\in\mathbb{N}}$ s.t.

- $B_{r_i}(x_i) \subset\subset B = \bigcup_{i\in\mathbb{N}} B_{r_i}(x_i)$.

Let $\gamma(|s|) := \begin{cases} \exp\left(-\frac{1}{1-s^2}\right), & |s|\le 1, \\ 0, & \text{else}, \end{cases}$

and define $\tilde{\alpha}_i(x) := \gamma\left(\frac{|x-x_i|}{r_i}\right)$ having the properties

- $\operatorname{supp}\tilde{\alpha}_i = \overline{B_{r_i}(x_i)}$,
- $\tilde{\alpha}_i > 0$ in $B_{r_i}(x_i)$,
- $\tilde{\alpha}_i \in \mathcal{D}(B)$.

Step 2: Applying **Step 1** to each $B = B_{r_j}(x_j)$, $j\in\mathbb{N}$, we get a collection of $(\tilde{\alpha}_i)_{i\in\mathbb{N}}$ with the properties (i) & (ii).

Step 3: Define $\gamma := \sum_{i=1}^{\infty}\tilde{\alpha}_i$ and $\alpha_i := \frac{\tilde{\alpha}_i}{\gamma}$.

Then, $(\alpha_i)_{i\in\mathbb{N}}$ has the claimed properties. ∎