§1.1 Sobolev & Sobolev-Slobodeckij spaces

Distributions: Let $\Omega \subseteq \mathbb{R}^d$ be open. We define

$$\mathcal{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : \underbrace{\operatorname{supp}\varphi \subset\subset \Omega}\}$$

i.e., $\operatorname{supp}\varphi$ is compact and $\operatorname{supp}\varphi \subset \Omega$

A linear functional $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is called a **distribution**, if for each $K \subset\subset \Omega$ there exist constants C and k such that for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subseteq K$ it holds

$$|T(\varphi)| \le C \sum_{|\alpha| \le k} \sup_K |D^\alpha \varphi|.$$

We denote by $\mathcal{D}'(\Omega)$ the space of distributions and also write $\langle T, \varphi \rangle := T(\varphi)$, for $T \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$.

Examples: (i) Let $f \in L^1_{loc}(\Omega) := \{f : f \in L^1(K) \text{ for all } K \subset\subset \Omega\}$ and define $\langle T_f, \varphi \rangle := \int_\Omega f \varphi \, dx$, $\varphi \in \mathcal{D}(\Omega)$.

$\Rightarrow T_f \in \mathcal{D}'(\Omega)$, and it is called a **regular distribution**.

Indeed, for all $K \subset\subset \Omega$ and all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subseteq K$ it holds

$$|\langle T_f, \varphi\rangle| \le \int_\Omega |f(x)|\,|\varphi(x)|\,dx$$
$$\le \sup_K |\varphi| \int_K |f(x)|$$
$$= \|f\|_{L^1(K)} \sup_K |\varphi|.$$

Remark: The map $L^1_{loc}(\Omega) \ni f \mapsto T_f \in \mathcal{D}'(\Omega)$ is injective, so in what follows we will usually identify $f \leftrightsquigarrow T_f$.

(ii) The Dirac δ-distribution is defined by

$$\langle \delta_0, \varphi\rangle := \varphi(0), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

(iii) Let $T \in \mathcal{D}'(\Omega)$ and define for $\alpha \in \mathbb{N}^d$,

$$\langle D^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle.$$

Then $D^\alpha T \in \mathcal{D}'(\Omega)$, and $D^\alpha T$ is called the α-th distributional derivative of T.

Definition of Sobolev & Sobolev-Slobodeckij spaces

Def: Let $\Omega \subseteq \mathbb{R}^d$ be open and let $l \in \mathbb{N}$.
We define the Sobolev space

$$W^{l,2}(\Omega) := \left\{ \varphi \in L^1_{loc}(\Omega) : \forall |\alpha| \le l \text{ the distributional derivative } D^\alpha \varphi \in L^2(\Omega) \right\},$$

introduce a sesquilinear form on $W^{l,2}(\Omega)$ by

$$(\varphi, \psi)_l := \sum_{|\alpha| \le l} \int_\Omega D^\alpha \varphi \, \overline{D^\alpha \psi} \, dx$$

with associated norm

$$\|\varphi\|^2_{W^{l,2}(\Omega)} := \|\varphi\|^2_l := (\varphi, \varphi)_l .$$

Lemma: $\varphi \in W^{l,2}(\Omega) \iff \varphi \in L^1_{loc}(\Omega)$ and for all $|\alpha| \le l$ there exists $\varphi_\alpha \in L^2(\Omega)$ such that for all $\psi \in \mathcal{D}(\Omega)$ it holds

$$\int_\Omega \varphi \, (-1)^{|\alpha|} D^\alpha \psi \, dx = \int_\Omega \varphi_\alpha \psi \, dx .$$

[φ_α is called the α-th weak derivative of φ.]

Proof: "$\Leftarrow$"

$$\int_\Omega \varphi \, (-1)^{|\alpha|} D^\alpha \psi \, dx = \langle T_\varphi, (-1)^{|\alpha|} D^\alpha \psi \rangle$$
$$= \langle D^\alpha T_\varphi, \psi \rangle ,$$

hence $D^\alpha T_\varphi = T_{\varphi_\alpha}$, i.e., $\varphi \in W^{l,2}(\Omega)$.

"$\Rightarrow$" Fix $|\alpha| \le l$. There exists $\varphi_\alpha \in L^2(\Omega)$ such that $D^\alpha T_\varphi = T_{\varphi_\alpha}$, i.e.,

$$\int_\Omega \varphi\, (-1)^{|\alpha|} D^\alpha \psi \, dx = \langle T_\varphi, (-1)^{|\alpha|} D^\alpha \psi \rangle$$

$$= \langle D^\alpha T_\varphi, \psi \rangle$$

$$= \langle T_{\varphi_\alpha}, \psi \rangle$$

$$= \int_\Omega \varphi_\alpha \psi \, dx. \quad \blacksquare$$

Example: Let $\alpha \in \mathbb{R}$, $f_\alpha(x) := |x|^{-\alpha}$.

We aim to find a condition on α such that

$$f_\alpha \in W^{1,2}(B_1(0)).$$

$\uparrow$ open unit ball in $\mathbb{R}^d$

Note that $\nabla f_\alpha(x) = |x|^{-\alpha-1} \frac{x}{|x|}$ for $x \neq 0$.

Heuristics: Decompose $B_1(0)$ into dyadic annuli

$$A_k := \left\{ \frac{1}{2^{k+1}} \le |x| \le \frac{1}{2^k} \right\}, \quad k \in \mathbb{N}.$$

By heuristically understanding

$H_k(f_\alpha)$ "height" ~ typical size of f_α on dominant component of support of f_α on A_k

$W_k(f_\alpha)$ "width" ~ measure of dominant component of support of f_α on A_k,

so that $\|f_\alpha\|^2_{L^2(A_k)} \overset{\text{heuristically}}{\sim} H_k^2(f_\alpha)\, W_k(f_\alpha)$

we have

- $\|f_\alpha\|^2_{L^2(A_k)} \sim 2^{2k\alpha}\, 2^{-kd}$
- $\|D_j f_\alpha\|^2_{L^2(A_k)} \sim 2^{2k(\alpha+1)}\, 2^{-kd}$

and thus

$$\|f_\alpha\|^2_{L^2(B_1(0))} \sim \sum_k \left(\frac{1}{2^{d-2\alpha}}\right)^k$$

$$\|D_j f_\alpha\|^2_{L^2(B_1(0))} \sim \sum_k \left(\frac{1}{2^{d-2(\alpha+1)}}\right)^k,$$

so that both are finite if $d > 2(\alpha+1)$

$$\Leftrightarrow \frac{d}{2} - 1 > \alpha.$$

Hence, for the rigorous argument, assume from now on $\frac{d}{2} - 1 > \alpha$.

We aim to show that $D_j f_\alpha = |x|^{-\alpha-1} \frac{x_j}{|x|}$ are the first order weak derivatives of f_α.

To this end, fix $\varepsilon \in (0,1)$ and $\varphi \in \mathcal{D}(B_1(0))$.

Then, by classical integration by parts

$$\int_{B_1(0)\setminus B_\varepsilon(0)} D_j f_\alpha\, \varphi\, dx = -\int_{B_1(0)\setminus B_\varepsilon(0)} f_\alpha\, D_j \varphi\, dx + \int_{\partial B_\varepsilon(0)} \vec{n}_{\partial B_\varepsilon(0)}\, e_j\, f_\alpha\, \varphi\, dS.$$

By the above, $f_\alpha \cdot D_j f_\alpha \in L^2(B_1(0))$,

hence

- $\int_{B_1(0)} D_j f_\alpha \, \varphi \, dx = \lim_{\varepsilon \downarrow 0} \int_{B_1(0) \setminus B_\varepsilon(0)} D_j f_\alpha \, \varphi \, dx$,
- $\int_{B_1(0)} (-D_j \varphi) f_\alpha \, dx = \lim_{\varepsilon \downarrow 0} \int_{B_1(0) \setminus B_\varepsilon(0)} (-D_j \varphi) f_\alpha \, dx$.

Furthermore,

$$\left| \int_{\partial B_\varepsilon(0)} \vec{n}_{\partial B_\varepsilon(0)} \cdot e_j \, f_\alpha \, \varphi \, dS \right| \leq \|\varphi\|_{L^\infty(B_1(0))} \, \varepsilon^{-\alpha} \, \varepsilon^{d-1} \xrightarrow{\varepsilon \downarrow 0}$$

Since $\alpha < d-1$.

Def: Let $\Omega \subseteq \mathbb{R}^d$ be open, let $l \in \mathbb{N}$ and $\lambda \in (0,1)$.

Define $I_\lambda(\varphi) := \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{d+2\lambda}} \, dx \, dy$.

We define the Sobolev-Slobodeckij space

$$W^{l+\lambda,2}(\Omega) := \left\{ \varphi \in W^{l,2}(\Omega) : \forall |\alpha| \leq l : I_\lambda(D^\alpha \varphi) < \infty \right\}$$

with sesquilinear form

$$(\varphi, \psi)_{l+\lambda} := (\varphi, \psi)_l + \sum_{|\alpha| \leq l} \int_\Omega \int_\Omega \frac{(D^\alpha \varphi(x) - D^\alpha \varphi(y)) \overline{(D^\alpha \psi(x) - D^\alpha \psi(y))}}{|x-y|^{d+2\lambda}}.$$

Thm: $W^{l,2}(\Omega)$ is a Hilbert space, $l \geq 0$.

Proof: Completeness is the only non-trivial part.

Step 1: Let $l \in \mathbb{N}$, and let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{l,2}(\Omega)$.

$\Rightarrow \forall |\alpha| \leq l : (D^\alpha \varphi_n)_{n\in\mathbb{N}}$ Cauchy in $L^2(\Omega)$

$\Rightarrow$ by completeness of $L^2(\Omega)$, $\forall |\alpha| \leq l$

$\exists \varphi_\alpha \in L^2(\Omega)$ s.t. $D^\alpha \varphi_n \xrightarrow{n\to\infty} \varphi_\alpha \in L^2(\Omega)$

$\Rightarrow$ Let $\varphi := \varphi_0 \in L^2(\Omega)$. We claim that $\varphi_\alpha = D^\alpha \varphi$. Indeed: $\forall \psi \in \mathcal{D}(\Omega)$

$$\int_\Omega D^\alpha \varphi_n \, \psi \, dx = \int_\Omega \varphi_n (-1)^{|\alpha|} D^\alpha \psi \, dx$$

$$\downarrow n\to\infty \qquad\qquad \downarrow n\to\infty$$

$$\int_\Omega \varphi_\alpha \, \psi \, dx \qquad\qquad \int_\Omega \varphi \, (-1)^{|\alpha|} D^\alpha \psi \, dx$$

$\Rightarrow \varphi_n \xrightarrow{n\to\infty} \varphi$ in $W^{l,2}(\Omega)$.

Step 2: Let $l \in \mathbb{N}$, $\lambda \in (0,1)$, $(\varphi_n)_{n\in\mathbb{N}}$ Cauchy sequence in $W^{l+\lambda,2}(\Omega)$. Since this implies that $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy in $W^{l,2}(\Omega)$, we know by Step 1 that $\exists \varphi \in W^{l,2}(\Omega)$ s.t. $\varphi_n \xrightarrow{n\to\infty} \varphi$ in $W^{l,2}(\Omega)$.

It remains to show: $\forall |\alpha| \le l$:

$$I_\lambda(D^\alpha \varphi_n - D^\alpha \varphi) \xrightarrow{n\to\infty} 0.$$

Fix $|\alpha| \le l$. Since $D^\alpha \varphi_n \to D^\alpha \varphi$ in $L^2(\Omega)$, there exists a subsequence $(n_m)_{m\in\mathbb{N}}$ s.t.

$$D^\alpha \varphi_{n_m} \xrightarrow{m\to\infty} D^\alpha \varphi \quad \text{a.e. in } \Omega.$$

$$\Rightarrow \iint_{\Omega\Omega} \frac{|D^\alpha\varphi(x) - D^\alpha\varphi(y)|^2}{|x-y|^{d+2\lambda}} = \iint_{\Omega\Omega} \lim_{m\to\infty} \frac{|D^\alpha\varphi_{n_m}(x) - D^\alpha\varphi_{n_m}(y)|^2}{|x-y|^{d+2\lambda}}$$

Fatou's lemma

$$\le \liminf_{m\to\infty} I_\lambda(D^\alpha \varphi_{n_m})$$

$$\le \sup_{n\in\mathbb{N}} I_\lambda(D^\alpha \varphi_n) < \infty,$$

since Cauchy $\Rightarrow$ bounded

hence $\varphi \in W^{l+\lambda,2}(\Omega)$.

$\Rightarrow$ Again by Fatou's lemma, for each $m \in \mathbb{N}$,

$$I_\lambda(D^\alpha \varphi_{n_m} - D^\alpha \varphi) \le \liminf_{\tilde{m}\to\infty} I_\lambda(D^\alpha \varphi_{n_m} - D^\alpha \varphi_{n_{\tilde{m}}})$$

So that at least for the subsequence $(n_m)_{m\in\mathbb{N}}$:

$$\varphi_{n_m} \xrightarrow{m\to\infty} \varphi \text{ in } W^{l+\lambda,2}(\Omega).$$

$\Rightarrow \varphi_n \to \varphi$ in $W^{l+\lambda,2}(\Omega)$ by Cauchy property of $(\varphi_n)_{n\in\mathbb{N}}$. ∎