Lecture 1, Advanced PDE, WS 2026/27

Topics of the lecture

(I) The L^2-theory of derivatives (or Sobolev spaces)

(II) Strongly elliptic & coercive differential operators: The variational method

(III) Linear parabolic / hyperbolic evolution equations

References:
- Wloka, "Partial differential equations"
- Evans, ——— 〃 ———

We will study linear differential operators $A(x,D)$ of order $2m$, $m \in \mathbb{N}$, in "divergence form", i.e., in multiindex notation

$$u \mapsto A(x,D)u = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha \left(a_{\alpha\beta}(x) D^\beta u(x) \right),$$

where $u : \Omega \to \mathbb{C}$, $a_{\alpha\beta} : \Omega \to \mathbb{C}$ are "sufficiently nice" behaved functions on an open & bounded domain $\Omega \subseteq \mathbb{R}^d$,

$$\left[\alpha, \beta \in \mathbb{N}_0^d,\ \alpha = (\alpha_1, \dots, \alpha_d),\ |\alpha| := \alpha_1 + \dots + \alpha_d,\ D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d},\ D_j := \frac{1}{i} \frac{\partial}{\partial x_j},\ j = 1, \dots, d \right]$$

Supplemented by m boundary differential operators $(b_j(x,D))_{j=1,\dots,m}$ of order $m_j \le 2m-1$ on the boundary $\partial\Omega$:

$$u \mapsto b_j(x,D)u = \sum_{|\alpha| \le m_j} b_{j,\alpha}(x) D^\alpha u \Big|_{\partial\Omega}.$$

and try to solve the associated boundary value problem

$$(\mathcal{A}) \quad \begin{cases} \mathcal{A}(x,D)u = f & \text{in } \Omega, \\ b_1(x,D)u = 0, \dots, b_m(x,D)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Example ($m=1$, i.e., 2nd-order operators)

In tensor notation,

$$\mathcal{A}(x,D)u = \sum_{i,j=1}^{d} D_i \left(a_{ij}(x) D_j u(x) \right) + \sum_{j=1}^{d} c_j(x) D_j u(x) + h(x) u(x),$$

or by introducing the matrix $a = (a_{ij})_{i,j=1,\dots,d}$ and vector $c = \left(\frac{1}{i} c_j \right)_{j=1,\dots,d}$,

$$\mathcal{A}(x,D)u = -\nabla \cdot (a \nabla u) + c \cdot \nabla u + hu.$$

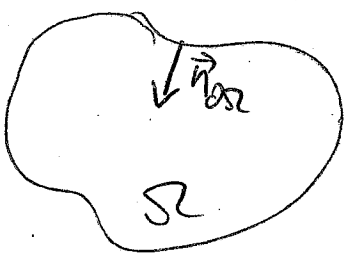
Note, $(a = \mathrm{Id},\ c = 0,\ h = 0) \Rightarrow \mathcal{A}(x,D)u = -\Delta$.

For a general boundary operator of order $\leq 2m-1 = 1$:

$$b(x,D)u = \sum_{j=1}^{d} b_j(x) D_j u\big|_{\partial\Omega} + b_0(x) u\big|_{\partial\Omega}.$$

Eg.,

- $b(x,D)u = u\big|_{\partial\Omega} \rightsquigarrow$ "Dirichlet problem for 2nd-order operator"
- $b(x,D)u = (\vec{n}_{\partial\Omega} \cdot \nabla) u\big|_{\partial\Omega} = \frac{\partial}{\partial \vec{n}_{\partial\Omega}} u\big|_{\partial\Omega}$ $\rightsquigarrow$ "Neumann problem for 2nd-order operator"



Example: ($m=2$, i.e., 4^{th}-order operator)

$$A(x,D)u = \Delta^2 u := \Delta(\Delta u).$$

For the required 2 boundary value operators of order $\leq 2m-1 = 3$ possible choices are:

- $b_1(x,D)u = u|_{\partial\Omega}$, $b_2(x,D)u = \frac{\partial}{\partial \vec{n}_{\partial\Omega}} u|_{\partial\Omega}$
  $\rightsquigarrow$ "Dirichlet problem for 4^{th}-order operator"
- $b_1(x,D)u = u|_{\partial\Omega}$, $b_2(x,D)u = \Delta u|_{\partial\Omega}$
- $b_1(x,D)u = \frac{\partial}{\partial \vec{n}_{\partial\Omega}} u|_{\partial\Omega}$, $b_2(x,D)u = \frac{\partial}{\partial \vec{n}_{\partial\Omega}} \Delta u|_{\partial\Omega}$.

A storyline for Chapter (I) & (II)

Variational method / weak formulation

- Multiply $A(x,D)u(x) = f(x)$ by $\overline{\varphi(x)}$, $\varphi \in \mathcal{D}(\Omega) := C_c^\infty(\Omega)$, integrate over Ω, and integrate by part to obtain

$$\int_\Omega f(x)\overline{\varphi(x)}\,dx = \int_\Omega A(x,D)u(x)\,\overline{\varphi(x)}\,dx \overset{\text{integration by parts}}{=} \int_\Omega \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x) D^\beta u(x)\, \overline{D^\alpha \varphi(x)}$$

$$=: a(u,\varphi)$$

[Examples: • $A = -\Delta$, $a(u,\varphi) = \int_\Omega \nabla u \cdot \overline{\nabla \varphi}$
• $A = \Delta^2$: $a(u,\varphi) = \int_\Omega \Delta u\, \overline{\Delta \varphi}$]

- In Chapter (I), we introduce and study a Hilbert space of functions $W^m := W^{m,2}(\Omega) \hookrightarrow L^2(\Omega)$ ("derivatives up to order m are in $L^2(\Omega)$") so that

$a: W^m \times W^m \to \mathbb{C}$ is a bounded, hence continuous, ~~sesquilinear~~ sesquilinear form on W^m.

- Given $b_1, \dots, b_m$, we define an associated subspace

$$V := V(b_1, \dots, b_m) \subseteq W^m;$$

the "boundary value space".

- Given $f \in V'$, we say that $u \in V$ is a weak solution to the boundary value problem (*) iff

$$\forall \varphi \in V: \quad a(u, \varphi) = \langle f, \varphi \rangle := f(\varphi).$$

- It is convenient to encode the existence of a weak solution by an operator equation:

Functional analysis fact: $\exists$ continuous linear operator

$$\mathcal{L}_a: V \to V', \quad \text{"representation operator of } a \text{ resp. } A\text{"}$$

such that

$$a(u, \varphi) = \langle \mathcal{L}_a u, \varphi \rangle_{V' \times V} = (\mathcal{L}_a u)(\varphi).$$

- Hence, $u \in V$ is a weak solution to (*) iff

$$\mathcal{L}_a u = f \quad \text{in } V'.$$

- 1st - main question: Existence & uniqueness of weak solutions

To this end, we will introduce the following abstract solvability condition:

Def: A (resp. a) is called V-coercive
$\Leftrightarrow \exists k \geq 0, c > 0$ s.t. for all $u \in V$:

$$\mathrm{Re}\, a(u,u) + k\|u\|_{L^2}^2 \geq c\|u\|_{W^m}^2.$$

- With the Lax-Milgram theorem from functional analysis, we will prove: Let A be V-coercive. Then, for all $f \in V'$ $\exists!$ $u \in V$ weak solution of

$$A(x,D)u + ku = f,$$

or equivalently $(\mathcal{L}_a + k): V \to V'$

is an isomorphism.

- 2nd-main question: How to check / satisfy abstract solvability condition?
  - $V \sim$ "Dirichlet problem" $\left(V := \overset{\circ}{W}{}^{m,2}(\Omega)\right)$

    Gårding's theorem: A is V-coercive $\Leftrightarrow$ A is strongly elliptic in Ω
  - generic boundary value space $V \subseteq W^m$:

    Agmon's theorem: A is V-coercive $\Leftrightarrow$ A satisfies Agmon's condition

- Gain: strong ellipticity / Agmon's condition are conditions on the coefficients $a_{\alpha\beta}, b_{j\alpha}$. ☺

• 3rd main question: How about existence and uniqueness of weak solutions for $A(x,D)u=f$?

• We will learn in Chapter (V) that $W^m \overset{\text{comp.}}{\hookrightarrow} L^2(\Omega)$, which will imply that the representation operator

$$\mathcal{L}_a : V \to V'$$

is a Fredholm operator (of index 0).

• This in particular means:

• $\dim \ker \mathcal{L}_a < \infty$ ("finite-dimensional non-uniqueness")

• $V' = \operatorname{im} \mathcal{L}_a \oplus U$, U finite dimensional, so that, for given $f \in V'$,

$$\exists u \in V : \mathcal{L}_a u = f \iff "f \perp U."$$

("finite-dimensional solvability condition")

• Also implies understanding on spectrum of $\mathcal{L}_a$, i.e., the eigenvalue problems $A(x,D)u = \lambda u$.

• 4th main question: The regularity problem

• $f \in L^2$, $\mathcal{L}_a u = f \overset{?}{\Rightarrow} u \in W^{2m}$,

• $f \in C^\infty$, $\mathcal{L}_a u = f \overset{?}{\Rightarrow} u \in C^\infty$.