§ 1.8. Trace operator (cont'd)

Thm: (Trace theorem for Ω)

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, and of class C^k, $k \geq 1$.

Let $l \geq m \geq 1$, $m \in \mathbb{N}$, s.t. $l - (m-1) > 1/2$ and $l+1 \leq k$.

Then: (a) There exists a continuous & linear operator

$$T_m : W^{l,2}(\Omega) \longrightarrow \bigtimes_{k=0}^{m-1} W^{l-k-1/2,2}(\partial\Omega)$$

s.t. $T_m \varphi = \left(T_1 \varphi, T_1 \frac{\partial}{\partial \vec{n}_\Omega} \varphi, \dots, T_1 \frac{\partial^{m-1}}{\partial \vec{n}_\Omega^{m-1}} \varphi \right)$, (*)

for all $\varphi \in C^{m-1}(\overline{\Omega}) \cap W^{l,2}(\Omega)$.

(b) There exists a continuous & linear operator

$$Z_m : \bigtimes_{k=0}^{m-1} W^{l-k-1/2,2}(\partial\Omega) \longrightarrow W^{l,2}(\Omega)$$

such that $T_m Z_m (\varphi_1, \dots, \varphi_m) = (\varphi_1, \dots, \varphi_m)$. (**)

(c) For $T_m : W^{m,2}(\Omega) \to \bigtimes_{k=0}^{m-1} W^{m-k-1/2,2}(\partial\Omega)$,

it holds $\ker T_m = W_0^{m,2}(\Omega)$.

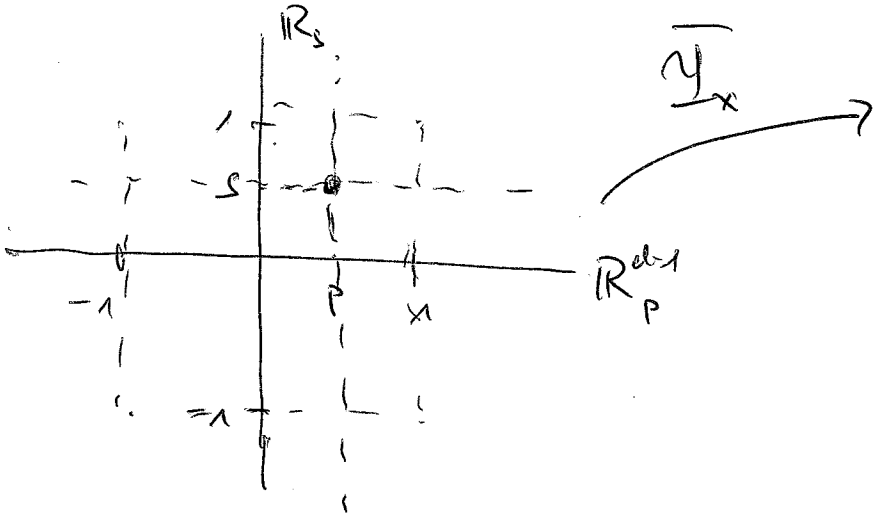
Proof: Analogous to the proof of the trace theorem for $m=1$; however, to recover (*1) and (**), we need more care in designing appropriate flattening of the boundary diffeomorphisms. This also explains the higher regularity assumption on the domain.

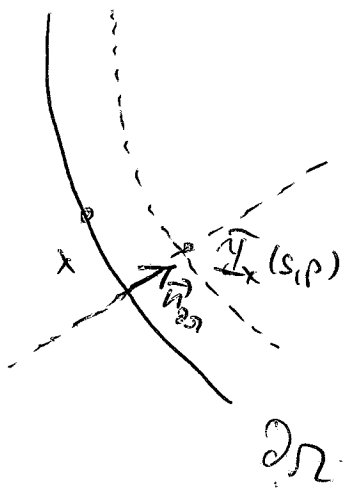
In order for (*) to hold, we need the boundary flattening diffeomorphisms to map tangent spaces onto tangent spaces as well as normal vectors onto normal vectors; otherwise, the corresponding (*)-property for the flat model geometry does not correspond to (*) for curved coordinates after transforming coordinates.

The admissible set of boundary flattening diffeomorphisms can be constructed as follows:

Fix $x \in \partial\Omega$ and $\underline{\Phi}_x : Q^d_1 \xrightarrow{\sim} U_x$ a corresponding C^k-diffeo. Define a map

$$\underline{\Psi}_x : Q^{d-1}_1 \times [-1,1] \to \mathbb{R}^d$$
$$(p, s) \mapsto \underline{\Phi}_x(p) + s\, \vec{n}_{\partial\Omega}(\underline{\Phi}_x(p)).$$





Then,

$$D\overline{\Psi}_x(p,s) = \sum_{i=1}^{d-1} \Big(\underbrace{D_i \underline{\Phi}_x(p)}_{=: \vec{t}^{\,i}_{\partial\Omega}(\underline{\Phi}_x(p))} + s\, D_i \underline{\Phi}_x(p) \big(D\vec{n}_{\partial\Omega} \circ \underline{\Phi}_x\big)(p) \Big) \otimes \vec{e}_i$$

tangent vector

$$+ \vec{n}_{\partial\Omega}(\underline{\Phi}_x(p)) \otimes \vec{e}_d \,;$$

in particular, the differential at $(p,s) = (0,0)$ maps

$$\begin{cases} D\overline{\Psi}_x(0,0)\,\vec{e}_i = \vec{t}^{\,i}_{\partial\Omega}(x) & , \quad i = 1, \dots, d-1, \\ D\overline{\Psi}_x(0,0)\,\vec{e}_d = \vec{n}_{\partial\Omega}(x) & . \end{cases}$$

Hence, $D\overline{\Psi}_x(0,0)$ is invertible, and the inverse mapping theorem guarantees the existence of $\varepsilon_x \in (0,1)$ and an open neighborhood $\widetilde{U}_x \subset\subset U_x$ such that

$$\overline{\Psi}_x : Q^d_{\varepsilon_x} \to \widetilde{U}_x \text{ is a } C^k\text{-diffeo}$$

such that

- $\underline{\Psi}_x(Q^d_{\varepsilon_x,+}) = \tilde{U}_x \cap \Omega$,
- $\underline{\Psi}_x(Q^d_{\varepsilon_x,-}) = \tilde{U}_x \cap (\mathbb{R}^d \setminus \bar{\Omega})$,
- $\underline{\Psi}_x(Q^{d-1}_{\varepsilon_x}) = \tilde{U}_x \cap \partial\Omega$,
- $D\underline{\Psi}_x(p,0)$ maps tangent space $\mathbb{R}^{d-1}_p = \langle \vec{e}_1, \dots, \vec{e}_{d-1} \rangle$ onto tangent space $\langle \vec{t}^1_{\partial\Omega}(\underline{\Phi}_x(p,0)), \dots, \vec{t}^{d-1}_{\partial\Omega}(\underline{\Phi}_x(p,0)) \rangle = \mathrm{Tan}_{\underline{\Phi}_x(p,0)} \partial\Omega$

and normal space $\mathbb{R}_s = \langle \vec{e}_d \rangle$ onto normal space $\langle \vec{n}_{\partial\Omega}(\underline{\Phi}_x(p,0)) \rangle = \mathrm{Nor}_{\underline{\Phi}_x(p,0)} \partial\Omega$ for all $p \in Q^{d-1}_{\varepsilon_x}$.

The proof then proceeds as in the case $m=1$, now using the "tubular neighborhood" diffeomorphisms $\underline{\Psi}_x$ instead of the generic boundary flattening diffeomorphisms $\underline{\Phi}_x$.

In our application to PDE theory,

$$\gamma_m : W^{l,2}(\Omega) \to \bigtimes_{k=0}^{m-1} W^{l-k-1/2,2}(\partial\Omega)$$

models what we will call the Dirichlet problem for an order $2m$ differential operator $A(x,D)$.

§ 1.9. Difference quotients & weak derivatives

Let $i = 1, \dots, d$, $h \in \mathbb{R}$, $(\vec{e}_i)_{i=1,\dots,d}$ standard basis of $\mathbb{R}^d$,

$$\Delta_h^i \varphi(x) := \frac{1}{h}\left(\tau_h^i \varphi(x) - \varphi(x)\right), \quad \tau_h^i \varphi(x) := \varphi(x + h\vec{e}_i).$$

Thm: Let $\Omega \subseteq \mathbb{R}^d$ be open, $l \in \mathbb{N}$, $\varphi \in W_0^{l,2}(\Omega)$.

Then

$$\|\Delta_h^i \varphi\|_{W^{l-1,2}(\Omega)} \leq \|\varphi\|_{W^{l,2}(\Omega)}$$

and

$$\|\Delta_h^i \varphi - \frac{\partial}{\partial x_i}\varphi\|_{W^{l-1,2}(\Omega)} \xrightarrow{h \to 0} 0.$$

Proof: We only consider $l = 1$; $l \geq 2$ follows by induction. Let $\varphi \in \mathcal{D}(\Omega)$, then

$$\varphi(x + h\vec{e}_i) - \varphi(x) = \int_0^1 \left(\frac{\partial}{\partial x_i}\varphi\right)(x + th\vec{e}_i)\, h\, dt$$

hence

$$\|\Delta_h^i \varphi\|_{L^2(\Omega)}^2 \overset{\text{Jensen}}{\leq} \int_\Omega \int_0^1 \left|\frac{\partial}{\partial x_i}\varphi(x + th\vec{e}_i)\right|^2 dt\, dx$$

$$\leq \left\|\frac{\partial}{\partial x_i}\varphi\right\|_{L^2(\Omega)}^2 \leq \|\varphi\|_{W^{1,2}(\Omega)},$$

and

$$\|\Delta_h^i \varphi - \frac{\partial}{\partial x_i}\varphi\|_{L^2(\Omega)}^2 \overset{\text{Jensen and Fubini}}{\leq} \int_0^1 \int_\Omega \left|\frac{\partial}{\partial x_i}\varphi(x + th\vec{e}_i) - \frac{\partial}{\partial x_i}\varphi(x)\right|^2 dx\, dt$$

By the Kolmogorov-Riesz compact ness criteria in L^2, for all $\varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$|\tilde{h}| \le \delta \Rightarrow \int_\Omega \left| \frac{\partial}{\partial x_i}\varphi(x+\tilde{h}e_i) - \frac{\partial}{\partial x_i}\varphi(x) \right|^2 dx \le \varepsilon_\varphi$$

Hence, for $|h| \le \delta$:

$$\| \Delta_h^i \varphi - \frac{\partial}{\partial x_i}\varphi \|^2_{L^2(\Omega)} \le \varepsilon. \qquad \square$$

Thm: Let $l \ge 1$, $\varphi \in W^{l-1,2}(\Omega)$. Assume there exists $M > 0$ s.t. for all $\Omega' \subset\subset \Omega$ and all $|h| \le \frac{\operatorname{dist}(\Omega', \partial\Omega)}{2}$ it holds

$$\| \Delta_h^i \varphi \|^2_{W^{l-1,2}(\Omega')} \le M.$$

Then $\frac{\partial}{\partial x_i}\varphi \in L^2(\Omega)$ and $\| \frac{\partial}{\partial x_i}\varphi \|^2_{W^{l-1,2}(\Omega)} \le CM$.

Proof: Again, we only consider $l = 1$. By reflexivity of the Hilbert space $H = L^2(\Omega')$, there exists $w \in L^2(\Omega')$ s.t. $\|w\| \le M$, and for a subsequence $(h_n)_n$:

$$\Delta_{h_n}^i \varphi \xrightarrow{n\to\infty} w \text{ in } L^2(\Omega'),$$

i.e., $(\Delta_{h_n}^i \varphi, h)_{L^2(\Omega')} \xrightarrow{n\to\infty} (w, h)_{L^2(\Omega')}$ for all $h \in L^2(\Omega')$. In particular, for $h \in D(\Omega')$:

$$\begin{array}{ccc} (\Delta_{h_n}^i \varphi, h)_{L^2(\Omega')} & = & -(\varphi, \Delta_{h_n}^i h)_{L^2(\Omega')} \\ \downarrow n\to\infty & & \downarrow n\to\infty \\ (w, h)_{L^2(\Omega')} & & -(\varphi, \frac{\partial}{\partial x_i} h)_{L^2(\Omega')} \end{array}$$

hence $\frac{\partial}{\partial x_i}\varphi \in \mathcal{D}'(\Omega')$ is realized by L^2
$w \in L^2(\Omega')$. Since $\Omega' \subset\subset \Omega$ was arbitrary,
$\frac{\partial}{\partial x_i}\varphi$ is measurable on Ω and

$$\| \frac{\partial}{\partial x_i}\varphi \|^2_{L^2(\Omega)} = \lim_{\Omega' \uparrow \Omega} \| \frac{\partial}{\partial x_i}\varphi \|^2_{L^2(\Omega')} \leq M. \quad \square$$

§ 1.10. Dual spaces

Thm: Let $k \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open, $V \subseteq W^{k,2}(\Omega)$ a closed subspace.

(a) $F \in V' \iff \forall |\alpha| \leq k \; \exists f_\alpha \in L^2(\Omega)$ s.t.

$$\langle F, \varphi \rangle_{V' \times V} = \sum_{|\alpha| \leq k} (f_\alpha, D^\alpha \varphi)_{L^2(\Omega)} \quad (*)$$

for all $\varphi \in V$.

Furthermore

$$\|F\|_{V'} = \inf \left(\sum_{|\alpha| \leq k} \|f_\alpha\|^2_{L^2(\Omega)} \right)^{1/2},$$

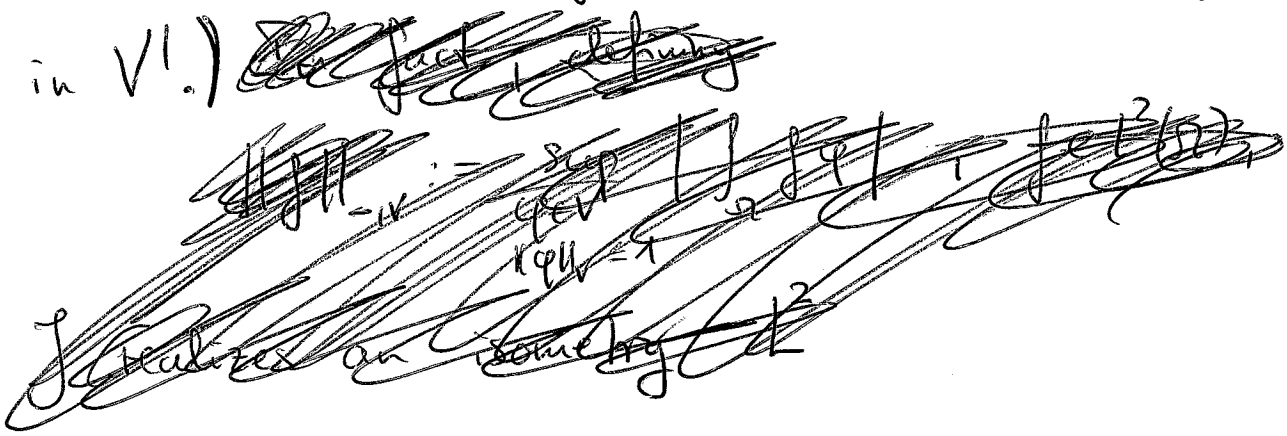
where the inf runs over all representations $(*)$.

(b) If V is dense in $L^2(\Omega)$, then

$$J : L^2(\Omega) \longrightarrow V'$$
$$f \mapsto \{ Jf : V \to \mathbb{C}, \; \varphi \mapsto \langle Jf, \varphi \rangle := (f, \varphi)_{L^2(\Omega)} \}$$

is injective and

has dense image in V'. (I.e., for all $F \in V'$ there exists $(f_n)_n \in L^2(\Omega)$ s.t. $Jf_n \to F$ in V'.)



Rmk: The situation of (b) is what is called in the literature a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$, for $H := L^2(\Omega)$, and it is customary to write by an abuse of notation $\langle F, \varphi \rangle_{V' \times V} = (F, \varphi)_{L^2(\Omega)}$. [See, e.g., Wloka]

Proof: (a) "$\Leftarrow$" By Cauchy-Schwarz,

$$|\langle F, \varphi \rangle| = \left| \sum_{|\alpha| \le \ell} \int_\Omega f_\alpha \overline{D^\alpha \varphi} \, dx \right|$$

$$\le \left(\sum_{|\alpha| \le \ell} \|f_\alpha\|^2_{L^2(\Omega)} \right)^{1/2} \left(\sum_{|\alpha| \le \ell} \|D^\alpha \varphi\|^2_{L^2(\Omega)} \right)^{1/2}$$

$$= \left(\sum_{|\alpha| \le \ell} \|f_\alpha\|^2_{L^2(\Omega)} \right)^{1/2} \|\varphi\|_{W^{\ell,2}(\Omega)}.$$

Thus, $F \in V'$ and $\|F\|_{V'} \le \left(\sum_{|\alpha| \le \ell} \|f_\alpha\|^2_{L^2(\Omega)} \right)^{1/2}$.

"$\Rightarrow$" By Riesz representation theorem (V is a Hilbert space w.r.t. the $W^{\ell,2}(\Omega)$ scalar product), there exists a unique $u_F \in V$ s.t.

$$\langle F, \varphi \rangle = (u_F, \varphi)_{W^{\ell,2}(\Omega)} = \sum_{|\alpha| \le \ell} \int_\Omega D^\alpha u_F \overline{D^\alpha \varphi} \, dx$$

and $\|F\|_{V'} = \|u_F\|_V$. Hence, the representation (*) holds with $f_\alpha := D^\alpha u_F$ and

$$\|F\|_{V'} = \|u_F\|_V = \left(\sum_{|\alpha| \le l} \|f_\alpha\|^2_{L^2(\Omega)} \right)^{1/2}.$$

This proves (a).

(b) Injectivity: assume $\langle Jf, \varphi \rangle = 0$ for all $\varphi \in V$, hence

$$\int_\Omega f\varphi = 0, \quad \forall \varphi \in V.$$

$\Rightarrow$ V dense in $L^2(\Omega)$ implies

$$\int_\Omega f\varphi = 0, \quad \forall \varphi \in L^2(\Omega)$$

$\Rightarrow$ $\varphi = f$ implies $f = 0$.

Dense image: Define $U := \overline{J(L^2(\Omega))}^{V'}$ and decompose orthogonally $V' = U \oplus U^\perp$. Let $F \in U^\perp$, i.e., for all $f \in L^2(\Omega)$: $(F, Jf)_{V'} = 0$.

If $R: V' \to V$ denotes the Riesz isomorphism,

$$0 = (F, Jf)_{V'} = (RF, RJf)_V$$
$$= \langle Jf, RF \rangle_{V' \times V} = \int_\Omega f \, RF \, dx$$

for all $f \in L^2(\Omega)$. Taking $f = RF$ shows $RF = 0$ $\Leftrightarrow$ $F = 0$. Hence $U^\perp = \{0\}$. ∎