Thm: (Trace theorem for $\mathbb{H}^d_+$; localized)

Let $l > 1/2$, and let $T_\lambda : W^{l,2}(\mathbb{H}^d_+) \to W^{l-1/2,2}(\partial\mathbb{H}^d_+)$ as well as $Z_\lambda : W^{l-1/2,2}(\partial\mathbb{H}^d_+) \to W^{l,2}(\mathbb{H}^d_+)$ the two continuous and bounded operators from the trace theorem for upper half space $\mathbb{H}^d_+$; see Lecture 10.

Then: (a) T_λ restricts to a continuous and linear operator

$$T_\lambda : W^{l,2}(Q^d_{\lambda,+}) \cap \{\operatorname{supp}\varphi \subseteq Q^d_{\lambda/2,+}\} \longrightarrow W^{l-1/2,2}_0(Q^{d-1}_\lambda)$$

such that

$$T_\lambda \varphi = \varphi(\cdot,0) \text{ in } W^{l-1/2,2}_0(Q^{d-1}_\lambda)$$

for all $\varphi \in C(\overline{Q^d_{\lambda,+}}) \cap W^{l,2}(Q^d_{\lambda,+}) \cap \{\operatorname{supp}\varphi \subseteq Q^d_{\lambda/2,+}\}$.

(b) Z_λ restricts to a continuous and linear operator

$$Z_\lambda : W^{l-1/2,2}_0(Q^{d-1}_{\lambda/2}) \to W^{l,2}(Q^d_{\lambda,+}) \cap \{\operatorname{supp}\varphi \subseteq Q^d_{\frac{3}{4}\lambda,+}\}$$

such that

$$T_\lambda Z_\lambda \varphi = \varphi, \quad \forall \varphi \in W^{l-1/2,2}_0(Q^{d-1}_{\lambda/2}).$$

Proof: (a) Let $\varphi \in W^{k,2}(Q^d_{1,+}) \cap \{\operatorname{supp}\varphi \subseteq Q^d_{1/2,+}\}$. L^2

~~Let $\varphi_\varepsilon := \eta_\varepsilon * \varphi \in \mathcal{D}$~~

Then $\overline{F}_{\mathbb{H}^d_+ \to \mathbb{R}^d} \varphi \in W^{k,2}(\mathbb{R}^d)$ ~~$\cap \{\operatorname{supp}\varphi \subseteq Q^{d-1}_{1/2} \times$~~

$\cap \left\{ \operatorname{supp} \overline{\varphi} \subseteq Q^{d-1}_{1/2} \times \left(-\frac{k+1}{2}, \frac{1}{2}\right] \right\}$.

Consider then the mollification $\varphi_\varepsilon := \eta_\varepsilon * \overline{F}_{\mathbb{H}^d_+ \to \mathbb{R}^d} \varphi$.

It holds $T_\wedge(\varphi_\varepsilon) = \varphi_\varepsilon(\cdot, 0)$, hence

$$\operatorname{supp} T_\wedge(\varphi_\varepsilon) \subseteq \{\operatorname{dist}(\cdot, Q^{d-1}_{1/2}) \le \varepsilon\} \cap \mathbb{R}^{d-1}.$$

Hence, by $\varepsilon \downarrow 0$,

$$\operatorname{supp} T_\wedge \varphi \subseteq \{\operatorname{dist}(\cdot, Q^{d-1}_{1/2}) \le \varepsilon_0\} \cap \mathbb{R}^{d-1},$$

for all $\varepsilon_0 \in (0,1)$. This proves $T_\wedge \varphi \in$ ~~$W^{k-1/2,2}$~~ $W^{k-1/2,2}_0(Q^{d-1}_1)$.

(b) Choose $\alpha \in \mathcal{D}(Q^d_{3/4})$ s.t. $\alpha \equiv 1$ in $Q^{d-1}_{1/2}$.

Define $\mathcal{Z}_\wedge := \alpha \mathcal{Z}^{\mathbb{H}^d_+}_\wedge$, which has the required properties. ∎

Thm: (Trace theorem for Ω)

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded, and of class C^k, $k \ge 1$.

Let $1/2 < \ell \le k$. Then:

(a) There exists a continuous and linear operator

$\gamma_1 : W^{1,2}(\Omega) \to W^{1-1/2,2}(\partial\Omega)$

such that $\gamma_1 \varphi = \varphi(\cdot,0)$ in $W^{1-1/2,2}(\partial\Omega)$ for all $\varphi \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$.

(b) There exists a continuous and linear operator

$$\zeta_1 : W^{1-1/2,2}(\partial\Omega) \to W^{1,2}(\Omega)$$

such that $\gamma_1 \zeta_1 \varphi = \varphi$, $\forall \varphi \in W^{1-1/2,2}(\partial\Omega)$.

(c) For $l = k = 1$, $\gamma_1 : W^{1,2}(\Omega) \to W^{1/2,2}(\partial\Omega)$ it holds $\ker \gamma_1 = W_0^{1,2}(\Omega)$.

Proof: For each $x \in \partial\Omega$, choose an open nbhd. U_x s.t. $\Phi_x : Q_1^d \to U_x$ is a C^k-diffeomorphism flattening the boundary of Ω. Choose $V_x' := \Phi_x(Q_{1/2}^d)$, and $V_x := \Phi_x(Q_{1/4}^d)$. For $x \in \Omega$, choose δ_x s.t. $B_{\delta_x}(x) \subset\subset \Omega$ and define $U_x := B_{\delta_x}(x)$, $V_x' := B_{\delta_x/2}(x)$, $V_x := B_{\delta_x/4}(x)$.

Denote them by $\underbrace{V_1, \dots, V_K}_{\text{boundary nbhds}}$, $\underbrace{V_{K+1}, \dots, V_{K+N}}_{\text{interior nbhds.}}$ a finite cover of $\overline{\Omega}$ with associated partition of unity $(\alpha_1, \dots, \alpha_K, \alpha_{K+1}, \dots, \alpha_{K+N})$; see Lecture 4. We will construct for each $j = 1, \dots, K, K+1, \dots, K+N$ a continuous linear operator

$S_1^j : W^{l,2}(\Omega) \to W^{l-1/2,2}(\partial\Omega)$ such that 24

$T_1 := \sum_{j=1}^{k+N} S_1^j$ is the required operator. For

interior neighborhoods $j \in \{k+1, \dots, k+N\}$, we obviously

define $S_1^0 := 0$. Hence, fix $j \in \{1, \dots, k\}$. We

then define S_1^j by means of

$$W^{l,2}(\Omega) \xrightarrow[\text{localization via partition of unity multipliers}]{\alpha_j \cdot} W^{l,2}(\Omega) \cap \{\operatorname{supp}\varphi \subseteq V_j' \cap \Omega\}$$

$$\downarrow F_{\Phi_j}$$

change of coordinates by boundary flattening diffeomorphism Φ_j; Lecture 6

$$W^{l,2}(Q_{1,+}^d) \cap \{\operatorname{supp}\varphi \subseteq Q_{1/2,+}^d\}$$

$$\downarrow T_1$$

trace operator for upper half space after localization

$$W_0^{l-1/2,2}(Q_1^{d-1})$$

$$\downarrow F_{\Phi_j|_{\partial\Omega}}$$

change of coordinates back to original variables; Lecture 6

$$W_0^{l-1/2,2}(\partial\Omega \cap U_j) \xrightarrow[\text{trivial extension by zero}]{} W^{l-1/2,2}(\partial\Omega).$$

(b) We define $\mathcal{Z}_1 := \sum_{j=1}^{K} \mathcal{Z}_1^j$ by means of

$$\mathcal{Z}_1^j : W^{l-1/2,2}(\partial\Omega) \xrightarrow{\alpha_j \circ} W_0^{l-1/2,2}(\partial\Omega \cap V_j')$$

($\alpha_j \circ$: localization via partition of unity multiplier)

$$\downarrow \mathcal{F}_{\Phi_j}$$

change of coordinates by boundary flattening diffeo Φ_j; Lecture 6

$$W_0^{l-1/2,2}(Q_{1/2}^{d-1})$$

$$\downarrow \mathcal{Z}_1$$

trace reconstruction for upper half space after localization

$$W^{l,2}(Q_{1,+}^d) \cap \{\operatorname{supp}\varphi \subseteq Q_{3/4,+}^d\}$$

$$\downarrow \mathcal{F}_{\Phi_j}^{-1}$$

back to curved coordinates; Lecture 6

$$W^{l,2}(\Omega) \xleftarrow{\text{trivial extension}} W^{l,2}(\Omega) \cap \{\operatorname{supp}\varphi \subseteq \Omega \cap \Phi_j(Q_{3/4,+}^d)\}$$

(c) Exercise.

Def: Let $l \geq m \geq 1$, $m \in \mathbb{N}$, $l-(m-1) > 1/2$. We then define a continuous $\mathbb{Q}$ linear operator by means of

$$T_m : W^{l,2}(\mathbb{R}^d) \longrightarrow \bigtimes_{k=0}^{m-1} W^{l-k-1/2,2}(\mathbb{R}^{d-1})$$

$$\varphi \mapsto (\gamma_1\varphi, \gamma_1 D_d\varphi, \dots, \gamma_1 D_d^{m-1}\varphi).$$

Thm: There exists a continuous and linear operator

$$Z_m : \bigtimes_{k=0}^{m-1} W^{l-k-1/2,2}(\mathbb{R}^{d-1}) \to W^{l,2}(\mathbb{R}^d)$$

such that $\gamma_m Z_m (\varphi_0, \varphi_1, \dots, \varphi_{m-1}) = (\varphi_0, \varphi_1, \dots, \varphi_{m-1})$.

Proof: Analogous to the proof of the existence of $Z_1 : W^{l-1/2,2}(\mathbb{R}^{d-1}) \to W^{l,2}(\mathbb{R}^d)$, we first define continuous & linear operators

$$\tilde{Z}_{k+1} : W^{l-k-1/2,2}(\mathbb{R}^{d-1}) \to W^{l,2}(\mathbb{R}^d)$$

by $$(\tilde{Z}_{k+1}\varphi)(x) := \frac{x_d^k}{C_{l,k}\, k!} \frac{1}{\sqrt{2\pi}^{d-1}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \frac{(1+|\xi'|^2)^{l-k-1/2}}{(1+|\xi|^2)^{l-k}} \hat{\varphi}(\xi')\, d\xi,$$

where the constants $C_{l,k}$ are chosen such that

$$(\tilde{Z}_{k+1}\varphi)(x',0) = 0, \dots, (D_d^k \tilde{Z}_{k+1}\varphi)(x',0) = \varphi(x'),$$

~~$k \in \{1, \dots, m-1\}$~~ $k \in \{0, \dots, m-1\}$. Define now

$$Z_m(\varphi_0, \varphi_1, \dots, \varphi_{m-1}) := \tilde{Z}_1 \varphi_0 + \tilde{Z}_2(\varphi_1 - \gamma_0 D_d \tilde{Z}_1 \varphi_0) + \dots \text{ (inductively)}.$$

∎

Thm: For $T_m : W^{m,2}(\mathbb{R}^d) \to \bigtimes_{k=0}^{m-1} W^{m-k-1/2,2}(\mathbb{R}^{d-1})$ [7]

it holds $\ker T_m = W_0^{m,2}(\mathbb{R}^d)$.

Proof: "$\supseteq$" Trivial.

"$\subseteq$" Exercise. ∎

One can again restrict to half space and localize.

Thm: (without proof) Let $l \geq m \geq 1$, $m \in \mathbb{N}$, $l-(m-1) > 1/2$.

(a) There exists a continuous & linear operator

$$T_m : W^{l,2}(Q_{1,+}^d) \cap \{\operatorname{supp} \varphi \subseteq Q_{1/2,+}^d\} \to \bigtimes_{k=0}^{m-1} W_0^{l-k-1/2,2}(Q_1^{d-1}).$$

(b) There exists a continuous & linear operator

$$Z_m : \bigtimes_{k=0}^{m-1} W_0^{l-k-1/2,2}(Q_{1/2}^{d-1}) \to W^{l,2}(Q_{1,+}^d) \cap \{\operatorname{supp} \varphi \subseteq Q_{3/4,1,+}^d\}$$

such that

$$T_m Z_m (\varphi_0, \varphi_1, \dots, \varphi_{m-1}) = (\varphi_0, \varphi_1, \dots, \varphi_{m-1}).$$