§ 1.8 Trace operator

Model problem: Flat boundary

Thm: Let $l > 1/2$.

(a) There exists a continuous and linear operator operator $T_1 : W^{l,2}(\mathbb{R}^d) \to W^{l-1/2,2}(\mathbb{R}^{d-1})$ such that for all $\varphi \in C_0(\mathbb{R}^d) \cap W^{l,2}(\mathbb{R}^d)$

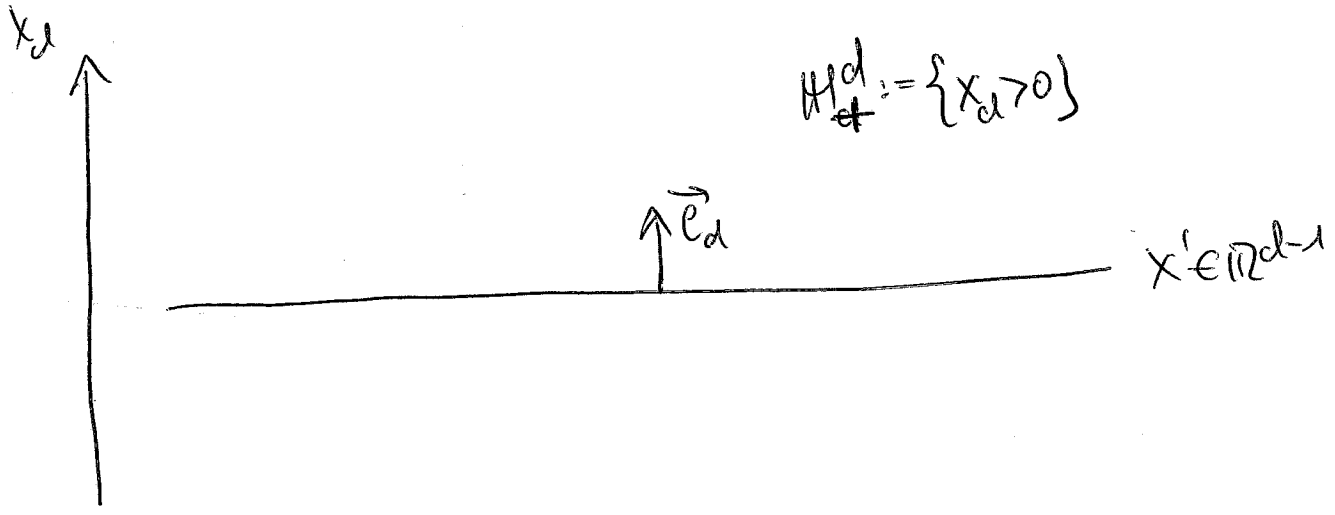
(*) $\quad T_1 \varphi = \varphi(\cdot, 0)$ in $W^{l-1/2,2}(\mathbb{R}^{d-1})$.

(b) There exists a continuous and linear operator

$$Z_1 : W^{l-1/2,2}(\mathbb{R}^{d-1}) \to W^{l,2}(\mathbb{R}^d)$$

such that

$$T_1 Z_1 \varphi = \varphi, \quad \forall \varphi \in W^{l-1/2,2}(\mathbb{R}^{d-1}).$$



Proof: (a) It suffices to show that $\exists C>0$:

$$(**) \qquad \|\varphi(x',0)\|_{W^{l-1/2,2}(\mathbb{R}^{d-1})} \le C \|\varphi\|_{W^{l,2}(\mathbb{R}^d)}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Indeed, by density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{l,2}(\mathbb{R}^d)$, this uniquely determines $T_1: W^{l,2}(\mathbb{R}^d) \to W^{l-1/2,2}(\mathbb{R}^d)$ such that (**) holds for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Approximating $\varphi \in C_0(\mathbb{R}^d) \cap W^{l,2}(\mathbb{R}^d)$ by $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ s.t. $\varphi_n \to \varphi$ in $W^{l,2}(\mathbb{R}^d)$ as well as in $C(\mathbb{R}^d)$, it follows from continuity of T_1 that $T_1\varphi_n = \varphi_n(\cdot,0) \to T_1\varphi$ in $W^{l-1/2,2}(\mathbb{R}^d)$, hence pointwise a.e. for a subsequence. Since $\varphi_n(\cdot,0) \to \varphi(\cdot,0)$ uniformly, (*) follows.

Hence, we need to show (**) for $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

By Fourier inversion,

$$T_1\varphi(x') = \varphi(x',0) = \int_{\mathbb{R}^d} e^{ix'\cdot\xi'} \hat{\varphi}(\xi',\xi_d)\, d\xi'\, d\xi_d$$

$$= \int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'} \Big(\underbrace{\int_{-\infty}^{\infty} \hat{\varphi}(\xi',\xi_d)\, d\xi_d}_{\in S(\mathbb{R}^{d-1})} \Big) d\xi'$$

so that

$$\widehat{T_1\varphi}(\xi') = \int_{-\infty}^{\infty} \hat{\varphi}(\xi',\xi_d)\, d\xi_d, \quad \forall \xi' \in \mathbb{R}^{d-1}.$$

We deduce

$$\int_{\mathbb{R}^{d-1}} (1+|\xi'|^2)^{l-1/2} \, |\widehat{\Gamma_1 \varphi}(\xi')|^2 \, d\xi'$$

$$\leq \int_{\mathbb{R}^{d-1}} (1+|\xi'|^2)^{l-1/2} \left(\int_{-\infty}^{\infty} |\hat{\varphi}(\xi', \xi_d)| \, d\xi_d \right)^2 d\xi' .$$

Defining $m^2(\xi', \xi_d) := \dfrac{(1+|\xi'|^2)^{l-1/2}}{(1+|\xi|^2)^l}$, we get

$$\cdot \int_{-\infty}^{\infty} m^2(\xi', \xi_d) \, d\xi_d = \frac{(1+|\xi'|^2)^{l-1/2}}{(1+|\xi'|^2)^l} \int_{-\infty}^{\infty} \frac{1}{\left(1+\left(\frac{\xi_d}{\sqrt{1+|\xi'|^2}}\right)^2\right)^l} d\xi_d$$

change of variables $\frac{\xi_d}{\sqrt{1+|\xi'|^2}} \rightsquigarrow \xi_d$

$$= \int_{-\infty}^{\infty} \frac{1}{(1+\xi_d^2)^l} \, d\xi_d$$

$$=: C_l \in (0, \infty), \text{ since } l > 1/2. \quad \blacksquare$$

Hence,

$$\|\Gamma_1 \varphi\|^2_{H^{l-1/2}(\mathbb{R}^{d-1})} \leq \int_{\mathbb{R}^{d-1}} (1+|\xi'|^2)^{l-1/2} \left(\int_{-\infty}^{\infty} m(\xi', \xi_d) \frac{|\hat{\varphi}(\xi', \xi_d)|}{m(\xi', \xi_d)} \right)^2 d\xi'$$

Hölder's

$$\leq C_l \int_{\mathbb{R}^d} \underbrace{\frac{(1+|\xi'|^2)^{l-1/2}}{(1+|\xi|^2)^l} m^2(\xi', \xi_d)}_{=1 \text{ by def.}} \underbrace{(1+|\xi|^2)^l |\hat{\varphi}(\xi)|^2 d\xi}_{\text{what we want}}$$

$$= C_l \|\varphi\|^2_{H^l(\mathbb{R}^d)} . \quad \blacksquare$$

(b) Let $\varphi \in \mathcal{D}(\mathbb{R}^{d-1})$. Then, $m^2(\xi', \xi_d) \, \hat{\varphi}(\xi')$ is in $L^2_l(\mathbb{R}^d)$, and in fact

$$\int_{\mathbb{R}^d} (1+|\xi|^2)^l \, |m^2(\xi', \xi_d) \, \hat{\varphi}(\xi')|^2 \, d\xi$$

$$= \int_{\mathbb{R}^d} \underbrace{\frac{(1+|\xi|^2)^\ell}{(1+|\xi'|^2)^{\ell-1/2}}}_{= m^2(\xi',\xi_d)} m^4(\xi',\xi_d)\,(1+|\xi'|^2)^{\ell-1/2}\,|\hat{\varphi}(\xi')|^2\,d\xi$$

$$= \int_{\mathbb{R}^{d-1}} (1+|\xi'|^2)^{\ell-1/2}\,|\hat{\varphi}(\xi')|^2 \underbrace{\left(\int_{-\infty}^{\infty} m^2(\xi',\xi_d)\,d\xi_d\right)}_{= C_\ell} d\xi'$$

$$= C_\ell \,\|\varphi\|^2_{H^{\ell-1/2}(\mathbb{R}^{d-1})}.$$

Hence, there exists a unique $u \in W^{\ell,2}(\mathbb{R}^d)$ s.t.

$$\mathcal{F}u = \frac{1}{C_\ell} m^2(\xi',\xi_d)\,\hat{\varphi}(\xi')$$

(since $\mathcal{F}: W^{\ell,2}(\mathbb{R}^d) \to L^2_\ell(\mathbb{R}^d)$ is an isomorphism)

and $\|u\|_{W^{\ell,2}(\mathbb{R}^d)} \le C_\ell \|\varphi\|_{H^{\ell-1/2}(\mathbb{R}^{d-1})}$.

Since $\varphi \in \mathcal{D}(\mathbb{R}^{d-1})$, it follows $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^{d-1})$,

hence $\mathcal{F}u = m^2(\xi',\xi_d)\,\hat{\varphi}(\xi') \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

By Fourier inversion theorem,

$$u(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, m^2(\xi',\xi_d)\,\hat{\varphi}(\xi')\,d\xi$$

for a.e. $x \in \mathbb{R}^d$. By Riemann-Lebesgue, $u \in C_0(\mathbb{R}^d)$ and the formula holds for all $x \in \mathbb{R}^d$ [we may change u on a set of measure zero]. However

$$(T_1 u)(x') = u(x',0) = \int_{\mathbb{R}^d} e^{ix'\cdot\xi'}\,\hat{\varphi}(\xi')\,d\xi' \overset{\text{Fourier inversion for } \varphi \in \mathcal{D}(\mathbb{R}^{d-1})}{=} \varphi(x')$$

So that we may define

$$Z_1 \varphi := u, \quad \varphi \in \mathcal{D}(\mathbb{R}^{d-1}).$$

The claim now follows by density. ∎

Thm: (Trace theorem for $\mathbb{H}^d_+$)

Let $l > 1/2$, let $L \in \mathbb{N}$ s.t. $L \geq l$, and denote by $E_{\mathbb{H}^d_+ \to \mathbb{R}^d}$ the associated extension operator.

Then: **(a)** $T_1 \circ E_{\mathbb{H}^d_+ \to \mathbb{R}^d} : W^{l,2}(\mathbb{H}^d_+) \to W^{l-1/2,2}(\partial\mathbb{H}^d_+)$ is a continuous and linear operator such that

$$(T_1 \varphi) = \varphi(\cdot, 0) \text{ in } W^{l-\frac{1}{2},2}(\partial\mathbb{H}^d_+)$$

for all $\varphi \in C_0(\overline{\mathbb{H}^d_+}) \cap W^{l,2}(\mathbb{H}^d_+)$.

(b) $R_{\mathbb{R}^d \to \mathbb{H}^d_+} \circ Z_1 : W^{l-\frac{1}{2},2}(\partial\mathbb{H}^d_+) \to W^{l,2}(\mathbb{H}^d_+)$ is a continuous linear operator, which is a right-inverse to $T_1 \circ E_{\mathbb{H}^d_+ \to \mathbb{R}^d}$.

In what follows, we will abuse notation by simply writing again T_1, Z_1.

(c) $T_1 : W^{1,2}(\mathbb{H}^d_+) \to W^{\frac{1}{2},2}(\partial\mathbb{H}^d_+)$ satisfies $\ker T_1 = W^{1,2}_0(\mathbb{H}^d_+)$.

Proof: (b) is immediate, whereas for

(a) one only has to read off from our recipe for $E_{\mathbb{H}^d_+ \to \mathbb{R}^d}$ that $E_{\mathbb{H}^d_+ \to \mathbb{R}^d}\varphi \in C_0(\mathbb{R}^d)$ for all $\varphi \in C_0(\overline{\mathbb{H}^d_+})$.

(c) Since trivially $W^{1,2}_0(\mathbb{H}^d_+) \subseteq \ker T_1$, we only have to show the converse. For that, recall $W^{1,2}_0(\mathbb{H}^d_+) = \{\varphi \in W^{1,2}(\mathbb{H}^d) : \exists\, \tilde{\varphi} \in W^{1,2}(\mathbb{R}^d)$ s.t. $\operatorname{supp}\tilde{\varphi} \subseteq \mathbb{H}^d_+$ and $\tilde{\varphi}|_{\mathbb{H}^d_+} = \varphi\}$.

Let $\varphi \in \ker T_1$, and we make the educated guess $\bar{\varphi} := E_0\varphi := \begin{cases} \varphi, & \text{in } \mathbb{H}^d_+, \\ 0, & \text{else.} \end{cases}$

Take $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ s.t. $\varphi_n|_{\mathbb{H}^d_+} \to \varphi$ in $W^{1,2}(\mathbb{H}^d_+)$.

Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} D_j\psi\, E_0\varphi_n = \int_{\mathbb{H}^d_+} (D_j\psi)\,\varphi_n = -\int_{\partial\mathbb{H}^d_+} \vec{e}_j\cdot\vec{e}_n\,\psi\,\varphi_n - \int_{\mathbb{H}^d_+} \psi D_j\varphi_n$$

$$\xrightarrow{n\to\infty} \int_{\mathbb{H}^d_+} D_j\psi\,\varphi \qquad \xrightarrow{n\to\infty} -\int_{\partial\mathbb{H}^d_+} \vec{e}_j\cdot\vec{e}_n\,\psi\,\underbrace{T_1\varphi}_{=0} - \int_{\mathbb{H}^d_+} \psi\, D_j\varphi,$$

hence, as wanted, $\int_{\mathbb{R}^d} D_j\psi\, E_0\varphi = -\int_{\mathbb{R}^d} \psi\, E_0 D_j\varphi$. ∎