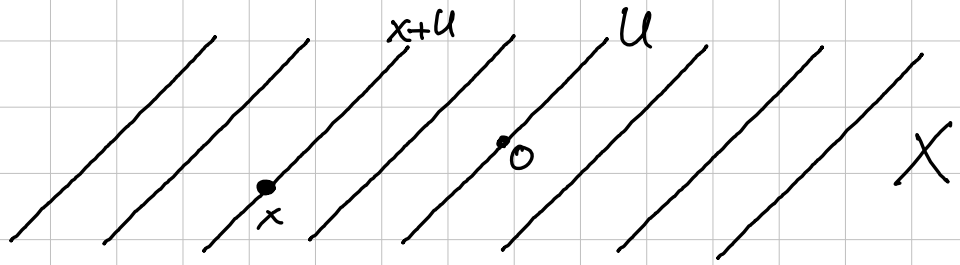
Talk 0 - Preliminaries from functional analysis

X is always a normed space, $U \leq X$ and $V \leq X'$ linear subspaces.

(1) The quotient space X/U

$$X/U = \{ [x] := \hat{x} := x + U : x \in X \}.$$



Thm: (a) $\|[x]\|_{X/U} := \operatorname{dist}(x, U)$ is a seminorm.

(b) U closed $\Rightarrow \|\cdot\|_{X/U}$ is a norm.

(c) X Banach, U closed $\Rightarrow (X/U, \|\cdot\|_{X/U})$ Banach

(2) Annihilators

Def: We define closed subspaces

$$X' \geq U^\perp := \{x' \in X' : x'(x) = 0, \forall x \in U\}$$
$$= \{x' \in X' : U \subseteq \ker x'\},$$

$$X \geq V_\perp := \{x \in X : x'(x) = 0, \forall x' \in V\}.$$

Lemma: (i) U closed, then $(X/U)' \simeq U^\perp$.

(iii) $U' \simeq X'/U^\perp$.

Proof: (i) Let $\pi : X \to X/U$, $x \mapsto [x]$, then $(X/U)' \ni \ell \mapsto \ell \circ \pi \in U^\perp$.

(iii) $x_1' \sim x_2'$ iff $U \subseteq \ker(x_1' - x_2')$
iff $x_1'|_U = x_2'|_U$,
so consider $x' + U^\perp \mapsto x'|_U \in U'$.

Lemma: Let $T \in L(X,Y)$, thus $T' \in L(Y',X')$ via $T'y'(x) := y'(Tx)$.

(i) $\ker T = (\operatorname{im} T')_\perp$,

(ii) $\ker T' = (\operatorname{im} T)^+$,

(iii) $\overline{\operatorname{im} T} = (\ker T')_\perp$.

Proof: (i) Let $x \in \ker T$, $x' \in \operatorname{im} T'$, i.e., $x' = T'y'$ for some $y' \in Y'$. Then

$$x'(x) = T'y'(x) = y'(Tx) = y'(0) = 0,$$

hence $\ker T \subseteq (\operatorname{im} T')_\perp$. If $x \notin \ker T$, then $y := Tx \neq 0$ and by Hahn-Banach there exists $y' \in Y'$ s.t. $y'(y) \neq 0$, i.e.

$$T'y'(x) = y'(Tx) \neq 0$$

so that $x \notin (\operatorname{im} T')_\perp$. ∎

(ii) Let $y' \in \ker T'$, hence for all x:

$$y'(Tx) = T'y'(x) = 0,$$

hence $\ker T' \subseteq (\operatorname{im} T)^+$.

If $y' \notin \ker T'$ then $x' := T'y' \neq 0$, i.e.,
there exists $x \in X$ s.t. $x'(x) \neq 0$, i.e.,

$$y'(Tx) = T'y'(x) = x'(x) \neq 0.$$

Hence $y' \notin (\operatorname{im} T)^\perp$. ∎

(iii) Let $y \in \operatorname{im} T$, i.e. $y = Tx$ for some $x \in X$.
Let $y' \in \ker T'$: $y'(y) = T'y'(x) = 0$,
hence $\operatorname{im} T \subseteq (\ker T')_\perp$, and since
$(\ker T')_\perp$ is closed, $\overline{\operatorname{im} T} \subseteq (\ker T')_\perp$.
If $y \notin \overline{\operatorname{im} T} =: U$ closed, choose by
Hahn-Banach $y' \in Y'$ s.t.

$$\begin{cases} y'(y) \neq 0, \\ y'|_U = 0. \end{cases}$$

$\Rightarrow y' \in \ker T'$ and $y \notin (\ker T')_\perp$. ∎

3 Complementary subspaces

Lemma: Let X be a Banach space.

(i) Let $W \leq X$ be a finite dimensional subspace. Then, $\exists\, U \leq X$, U closed s.t.

$$X = U \oplus W.$$

(ii) Let $U \leq X$ s.t. $\dim(X/U) < \infty$.

Then, there exists $W \leq X$, $\dim W < \infty$, s.t. $X = U \oplus W$.

Proof: (i) Let $n := \dim W$, $\ell : W \xrightarrow{\sim} \mathbb{K}^n$, and $\ell_i : W \to \mathbb{K} \in W'$. By Hahn-Banach,

$$\exists\, \tilde{\ell}_i \in X' \text{ s.t. } \tilde{\ell}_i|_W = \ell_i.$$

Define $\tilde{\ell} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n) : X \to \mathbb{K}^n$

and $P := \ell^{-1} \circ \tilde{\ell}$, i.e.

$$P : X \xrightarrow{\tilde{\ell}} \mathbb{K}^n \xrightarrow{\ell^{-1}} W.$$

Then: • $W = \operatorname{im} P$,

• $P(Px) = Px$, $\forall x \in X$.

We call P a projection onto W, so it holds

$$X = \underbrace{\ker P}_{=:U} \oplus \operatorname{im} P = U \oplus W.$$

[$x = \underbrace{(x - Px)}_{\in \ker P} + \underbrace{Px}_{\in \operatorname{im} P}$ and $w \in \ker P \cap W$ $\Rightarrow w = 0$ since $w = Px = P(Px) = Pw = 0$.]

(ii) Pick $x_{1}, \dots, x_n$ s.t. $\hat{x}_1, \dots, \hat{x}_n$ is a basis of X/U. Define a dual basis $\hat{l}_1, \dots, \hat{l}_n$ in $(X/U)'$, i.e., $\hat{l}_i(\hat{x}_j) = \delta_{ij}$. Define $l_i := \hat{l}_i \circ \pi \in X'$, $\pi : X \to X/U$ the canonical projection, and define

$$P : X \to X, \quad x \mapsto \sum_{i=1}^{n} l_i(x) x_i.$$

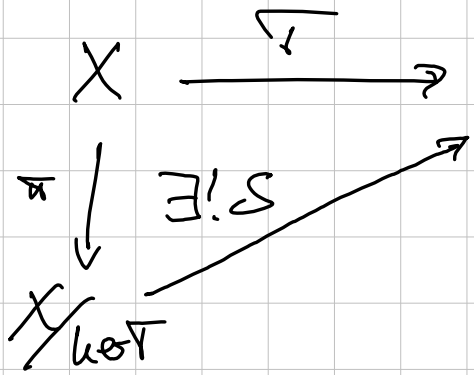
$\Rightarrow$ P is a projection onto $W := \operatorname{im} P$ with kernel $U = \ker P$. ∎

Lemma: Let X, Y be Banach spaces, $T \in L(X,Y)$. If

$$\dim \operatorname{coker} T := \dim Y/\operatorname{im} T < \infty,$$

then $\operatorname{im} T$ is closed.

Proof: Consider the commutative diagram



and choose $W \leq Y$, $\dim W < \infty$, s.t. $Y = \operatorname{im} T \oplus W$. (in particular, W is Banach)

Then, $\hat{S} : X/\ker T \times W \to Y$

$$(\hat{x}, w) \mapsto S\hat{x} + w$$

is a cont. linear & bijective operator

between Banach spaces, hence, by the open mapping theorem, $\hat{S}^{-1}$ is also continuous. Since $X/\ker T \times \{0\}$ is closed in $X/\ker T \times W$,

$$\operatorname{im} T = \hat{S}\left(X/\ker T \times \{0\}\right)$$

is closed in Y. ∎