WEAK SOLUTIONS OF MULLINS–SEKERKA FLOW AS A HILBERT SPACE GRADIENT FLOW

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Abstract. We propose a novel weak solution theory for the Mullins–Sekerka equation in dimensions $d = 2$ and 3 primarily motivated from a gradient flow perspective. Previous existence results on weak solutions due to Luckhaus and Sturzenhecker (Calc. Var. PDE 3, 1995) or Röger (SIAM J. Math. Anal. 37, 2005) left open the inclusion of both a sharp energy dissipation principle and a weak formulation of the contact angle at the intersection of the interface and the domain boundary. To incorporate these, we introduce a functional framework encoding a weak solution concept for Mullins–Sekerka flow essentially relying only on *i)* a single sharp energy dissipation inequality in the spirit of De Giorgi, and *ii)* a weak formulation for an arbitrary fixed contact angle through a distributional representation of the first variation of the underlying capillary energy. Both ingredients are intrinsic to the interface of the evolving phase indicator and an explicit distributional PDE formulation with potentials can be derived from them. Existence of weak solutions is established via subsequential limit points of the naturally associated minimizing movements scheme. Smooth solutions are consistent with the classical Mullins–Sekerka flow, and even further, we expect our solution concept to be amenable, at least in principle, to the recently developed relative entropy approach for curvature driven interface evolution.

Keywords: Mullins–Sekerka flow, gradient flows, weak solutions, energy dissipation inequality, De Giorgi metric slope, contact angle, Young's law

Mathematical Subject Classification: 35D30, 49J27, 49Q20, 53E10

Contents

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1. Introduction

1.1. Context and motivation.

The purpose of this paper is to develop the gradient flow perspective for the Mullins–Sekerka equation at the level of a weak solution theory in dimensions $d = 2$ and 3. The Mullins–Sekerka equation is a curvature driven evolution equation for a mass preserved quantity, see (1a)–(1e) below, typically used to model a coarsening behavior. Stability properties of such solidification processes were first studied by Mullins and Sekerka in the early 60s [49], but general existence results for such equations didn't come until almost 30 years later. The ground breaking results of the 90s showed that when strong solutions exist, this equation is in fact the sharp interface limit of the Cahn–Hilliard equation, a fourth order diffuse interface model for phase separation in materials [4] (see also, e.g., [9], [50]). However, as for mean curvature flows, one of the critical challenges in studying such sharp interface models is the existence of solutions after topological change. As a result, many different weak solution concepts have been developed, and in analogy with the development of weak solution theories for PDEs and the introduction of weak function spaces, a variety of weak notions of smooth surfaces have been applied for solution concepts. In the case that a surface arises as the common boundary of two sets (i.e., an interface), a powerful solution concept has been the BV solution, first developed for the Mullins–Sekerka flow in the seminal work of Luckhaus and Sturzenhecker [40].

For BV solutions in the sense of [40], the evolving phase is represented by a time-dependent family of characteristic functions which are of bounded variation. Furthermore, both the evolution equation for the phase and the Gibbs–Thomson law are satisfied in a distributional form. The corresponding existence result for such solutions crucially leverages the well-known fact that the Mullins–Sekerka flow can be formally obtained as an H^{-1}-type gradient flow of the perimeter functional (see, e.g., [26]). Indeed, BV solutions for the Mullins–Sekerka flow are constructed in [40] as subsequential limit points of the associated minimizing movements scheme. However, due to the discontinuity of the first variation of the perimeter functional with respect to weak-$*$ convergence in BV, Luckhaus and Sturzenhecker [40] relied on the additional assumption of convergence of the perimeters in order to obtain a BV solution in their sense. Based on geometric measure theoretic results of Schätzle [55] and a pointwise interpretation of the Gibbs–Thomson law in terms of a generalized curvature intrinsic to the interface [51], Röger [52] was later able to remove the energy convergence assumption (see also [2]).

However, the existence results of Luckhaus and Sturzenhecker [40] and Röger [52] still leave two fundamental questions unanswered. First, both weak formulations of the Gibbs–Thomson law do not encompass a weak formulation for the boundary condition of the interface where it intersects the domain boundary. For instance, if the energy is proportional to the surface area of the interface, one expects a constant ninety degree contact angle condition at the intersection points, which quantitatively accounts for the fact that minimizing energy in the bulk, the surface will travel the shortest path to the boundary. Second, neither of the two works establishes a sharp energy dissipation principle, which, because of the formal gradient flow structure of the Mullins–Sekerka equation, is a natural ingredient for a weak solution concept as we will further discuss below. A second motivation to prove a sharp energy dissipation inequality stems from its crucial role in the recent progress concerning weak-strong uniqueness principles for curvature driven interface evolution problems (see, e.g., [23], [25] or [29]).

Turning to approximations of the Mullins–Sekerka flow via the Cahn–Hilliard equation Chen [16] introduced an alternative weak solution concept, which does include an energy dissipation inequality. To prove existence, Chen developed powerful estimates (that have been used in numerous applications, e.g., [1], [2], [43]) to control the sign of the discrepency measure, an object which captures the distance of a solution from an equipartition of energy. Critically these estimates do not rely on the maximum principle and are applicable to the fourth-order Cahn–Hilliard equation. However, in contrast to Ilmanen's proof for the convergence of the Allen–Cahn equation to mean curvature flow [32], where the discrepancy vanishes in the limit, Chen is restricted to proving non-positivity in the limit. As a result, the proposed solution concept requires a varifold lifting of the energy for the dissipation inequality and a modified varifold for the Gibbs–Thomson relation. In the interior of the domain, the modified Gibbs–Thomson relation no longer implies the pointwise interpretation of the evolving surface's curvature in terms of the trace of the chemical potential and, on the boundary, cannot account for the contact angle. Further, Chen's solution concept does not use the optimal dissipation inequality to capture the full dynamics of the gradient flow.

Looking to apply the framework of evolutionary Gamma-convergence developed by Sandier and Serfaty [54] to the convergence of the Cahn–Hilliard equation, Le [36] introduces a gradient flow solution concept for the Mullins–Sekerka equation, which principally relies on an optimal dissipation inequality. However, interpretation of the limiting interface as a solution in this sense requires that the surface is regular and does not intersect the domain boundary, i.e., there is no contact angle. As noted by Serfaty [56], though the result of Le [36] sheds light on the gradient flow structure of the Mullins–Sekerka flow in a smooth setting, it is of interest to develop a general framework for viewing solutions of the Mullins–Sekerka flow as curves of maximal slope even on the level of a weak solution theory. This is one of the primary contributions of the present work.

Though still in the spirit of the earlier works by Le [36], Luckhaus and Sturzenhecker [40], and Röger [52], the solution concept we introduce includes both a weak formulation for the constant contact angle and a sharp energy dissipation principle.

The boundary condition for the interface is in fact not only implemented for a constant contact angle $\alpha = \frac{\pi}{2}$ but even for general constant contact angles $\alpha \in (0, \pi)$. For the formulation of the energy dissipation inequality, we exploit a gradient flow perspective encoded in terms of a De Giorgi type inequality. Recall to this end that for smooth gradient flows, the gradient flow equation $\dot{u} = -\nabla E[u]$ can equivalently be represented by the inequality

$$E[u(T)] + \int_0^T \frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}|\nabla E[u(t)]|^2\,dt \leq E[u(0)]$$

(for a discussion of gradient flows and their solution concepts in further detail see Subsection 1.3). Representation of gradient flow dynamics through the above dissipation inequality allows one to generalize to the weak setting and is often amenable to typical variational machinery such as weak compactness and lower semi-continuity.

The main conceptual contribution of this work consists of the introduction of a functional framework for which a weak solution of the Mullins–Sekerka flow is essentially characterized through only *i)* a single sharp energy dissipation inequality, and *ii)* a weak formulation for the contact angle condition in the form of a suitable distributional representation of the first variation of the energy (cf. Kagaya and Tonegawa [33, 34] and Modica [47]). We emphasize that both these ingredients are intrinsic to the trajectory of the evolving phase indicator. Beyond proving existence of solutions via a minimizing movements scheme (Theorem 1), we show that our solution concept extends Le's [36] to the weak setting (Subsection 2.4), a more classical distributional PDE formulation with potentials can be derived from it (Lemma 3), smooth solutions are consistent with the classical Mullins-Sekerka equation (Lemma 4), and that the underlying varifold for the energy is of bounded variation.

A natural question arising from the present work is whether solutions of the Cahn–Hilliard equation converge subsequentially to weak solutions of the Mullins–Sekerka flow in our sense, which would improve the seminal result of Chen [16] that relies on a weaker formulation of the Mullins–Sekerka flow. An investigation of this question will be the subject of a future work.

1.2. **Mullins–Sekerka motion law: Strong PDE formulation.** Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and C^2-boundary $\partial\Omega$. Consider also a finite time horizon $T_* \in (0,\infty)$ and let $\mathscr{A} = (\mathscr{A}(t))_{t\in[0,T_*)}$ be a time-dependent family of smoothly evolving open subsets $\mathscr{A}(t) \subset \Omega$ with $\partial\mathscr{A}(t) = \overline{\partial^*\mathscr{A}(t)}$ and $\mathcal{H}^{d-1}(\partial\mathscr{A}(t) \setminus \partial^*\mathscr{A}(t)) = 0$, $t \in [0, T_*)$, where $\partial^*\mathscr{A}$ refers to the reduced boundary [6]. Denoting for every $t \in [0, T_*)$ by $V_{\partial\mathscr{A}(t)}$ and $H_{\partial\mathscr{A}(t)}$ the associated normal velocity and mean curvature vector, respectively, the family $\mathscr{A}$ is said to evolve by *Mullins–Sekerka flow* if for each $t \in (0, T_*)$ there exists a chemical potential $\overline{u}(\cdot, t)$ so that

$$\Delta\overline{u}(\cdot,t) = 0 \qquad \text{in } \Omega \setminus \partial\mathscr{A}(t), \tag{1a}$$

$$V_{\partial\mathscr{A}(t)} = -\big(n_{\partial\mathscr{A}(t)} \cdot [\![\nabla\overline{u}(\cdot,t)]\!]\big)n_{\partial\mathscr{A}(t)} \qquad \text{on } \partial\mathscr{A}(t)\cap\Omega, \tag{1b}$$

$$c_0 H_{\partial\mathscr{A}(t)} = \overline{u}(\cdot,t)n_{\partial\mathscr{A}(t)} \qquad \text{on } \partial\mathscr{A}(t)\cap\Omega, \tag{1c}$$

$$(n_{\partial\Omega}\cdot\nabla)\overline{u}(\cdot,t) = 0 \qquad \text{on } \partial\Omega \setminus \overline{\partial\mathscr{A}(t)\cap\Omega}. \tag{1d}$$

Here, we denote by $c_0 \in (0,\infty)$ a fixed surface tension constant, by $n_{\partial\mathscr{A}(t)}$ the unit normal vector field along $\partial\mathscr{A}(t)$ pointing inside the phase $\mathscr{A}(t)$, and similarly $n_{\partial\Omega}$ is the inner normal on the domain boundary $\partial\Omega$. Furthermore, the jump $[\![\cdot]\!]$ across the interface $\partial\mathscr{A}(t)\cap\Omega$ is taken for a function v as

$$[\![v]\!] := \mathrm{trace}_{\mathscr{A}}(v) - \mathrm{trace}_{\Omega\setminus\mathscr{A}}(v),$$

which is in particular the correct orientation for the following integration by parts formula to hold:

$$\begin{aligned} -\int_{\Omega\setminus\partial\mathscr{A}(t)} \eta\nabla\cdot v\,dx = \int_{\Omega}\nabla\eta\cdot v\,dx + \int_{\partial\mathscr{A}(t)\cap\Omega}\eta\big(n_{\partial\mathscr{A}(t)}\cdot[\![v]\!]\big)\,d\mathcal{H}^{d-1} \\ + \int_{\partial\Omega}\eta\big(n_{\partial\Omega}\cdot v\big)\,d\mathcal{H}^{d-1} \end{aligned} \tag{2}$$

for all sufficiently regular functions $v\colon\overline{\Omega}\to\mathbb{R}^d$ and $\eta\colon\overline{\Omega}\to\mathbb{R}$.

For sufficiently smooth evolutions, it is a straightforward exercise to verify that the Mullins–Sekerka flow conserves the mass of the evolving phase as

$$\frac{d}{dt}\int_{\mathscr{A}(t)} 1\,dx = -\int_{\partial\mathscr{A}(t)\cap\Omega} V_{\partial\mathscr{A}(t)}\cdot n_{\partial\mathscr{A}(t)}\,d\mathcal{H}^{d-1} = 0. \tag{3}$$

To compute the change of interfacial surface area, we first need to fix a boundary condition for the interface. In the present work, we consider the setting of a fixed contact angle $\alpha\in(0,\pi)$ in the sense that for all $t\in[0,T_*)$ it is required that

$$n_{\partial\Omega}\cdot n_{\partial\mathscr{A}(t)} = \cos\alpha \qquad \text{on } \partial\Omega\cap\overline{\partial\mathscr{A}(t)\cap\Omega}. \tag{1e}$$

Then, it is again straightforward to compute that

$$\begin{aligned} &\frac{d}{dt}\bigg(\int_{\partial\mathscr{A}(t)\cap\Omega} c_0\,d\mathcal{H}^{d-1} + \int_{\partial\mathscr{A}(t)\cap\partial\Omega} c_0\cos\alpha\,d\mathcal{H}^{d-1}\bigg) \\ &= -\int_{\partial\mathscr{A}(t)\cap\Omega} V_{\partial\mathscr{A}(t)}\cdot c_0 H_{\partial\mathscr{A}(t)}\,d\mathcal{H}^{d-1} = -\int_{\Omega}|\nabla\overline{u}(\cdot,t)|^2\,dx \le 0. \end{aligned} \tag{4}$$

In view of the latter inequality, one may wonder whether the Mullins–Sekerka flow can be equivalently represented as a gradient flow with respect to interfacial surface energy. That this is indeed possible is of course a classical observation (see [26] and references therein) and, at least for smooth evolutions, may be realized in terms of a suitable H^{-1}-type metric on a manifold of smooth surfaces.

1.3. **Gradient flow perspective assuming smoothly evolving geometry.** To take advantage of the insight provided by (4), we recall two methods for gradient flows. In parallel, our approach is inspired by De Giorgi's methods for curves of maximal slope in metric spaces and the approach for Gamma-convergence of evolutionary equations developed by Sandier and Serfaty in [54], which has been applied to the Cahn–Hilliard approximation of the Mullins–Sekerka flow by Le [36].

Looking to the school of thought inspired by De Giorgi (see [7] and references therein), in a generic metric space (X,d) equipped with energy $E : X\to\mathbb{R}\cup\{\infty\}$, a curve $t\mapsto u(t)\in X$ is said to be a solution of the differential inclusion $-\frac{d}{dt}u\in\partial E[u]$ if it is a curve of maximal slope, that is, it satisfies the optimal dissipation relation

$$E[u(T)] + \frac{1}{2}\int_0^T \Big|\frac{d}{dt}u\Big|^2 + |\partial E[u]|^2\,dt \le E[u(0)] \tag{5}$$

for almost all $T \in (0, T_*)$, where $|\frac{d}{dt}u|$ is interpreted in the metric sense and

$$|\partial E[u]| := \limsup_{v \to u} \frac{(E[u] - E[v])_+}{d(u,v)}.$$

One motivation for this solution concept is in the Banach setting where, for sufficiently nice energies E, the optimal dissipation (5) is equivalent to solving the differential inclusion [7].

The energy behind the gradient flow structure of the Mullins–Sekerka flow is the perimeter functional, for which we have the classical result of Modica [46] (see also [48]) at the static level:

$$E_\epsilon[u] := \int_\Omega \frac{1}{\epsilon} W(u) + \epsilon \|\nabla u\|^2 \, dx \rightharpoonup_\Gamma c_0 \mathrm{Per}_\Omega(\chi) =: E[\chi].$$

The energy E_ϵ is often referred to as the Cahn-Hilliard energy, which was introduced to model phase separation in multiphase materials [12]. However, as we are interested in the dynamics driven by the energy, the perspective of Sandier and Serfaty [54] for evolutionary Γ-convergence is relevant.

Abstractly, given Γ-converging (see, e.g., [11], [18]) energies $E_\epsilon \rightharpoonup_\Gamma E$, this approach gives conditions for when a curve $t \mapsto u(t) \in Y$, which is the limit of $t \mapsto u_\epsilon(t) \in X_\epsilon$ solving $-\frac{d}{dt}u_\epsilon \in \nabla_{X_\epsilon} E_\epsilon[u_\epsilon]$, is a solution the gradient flow $-\frac{d}{dt}u \in \nabla_Y E[u]$ associated with the limiting energy. Specifically, this requires the lower semi-continuity of the time derivative and the variations given by

$$\int_0^T \left\| \frac{d}{dt}u \right\|_Y^2 dt \le \liminf_{\epsilon \downarrow 0} \int_0^T \left\| \frac{d}{dt}u_\epsilon \right\|_{X_\epsilon}^2 dt,$$
$$\int_0^T \left\| \nabla_Y E[u] \right\|_Y^2 dt \le \liminf_{\epsilon \downarrow 0} \int_0^T \left\| \nabla_{X_\epsilon} E_\epsilon[u_\epsilon] \right\|_{X_\epsilon}^2 dt,$$

which are precisely the relations needed to maintain an optimal dissipation inequality (5) in the limit. We note that this idea was precisely developed in finite dimensions with C^1-functionals, and extending this approach to geometric evolution equations seems to require re-interpretation in general.

This process of formally applying the Sandier–Serfaty approach to the Cahn–Hilliard equation was carried out by Le in [36] (see also [37] and [42]). As the Cahn-Hilliard equation and Mullins-Sekerka flow are mass preserving, it is necessary to introduce the Sobolev space $H^1_{(0)} := H^1(\Omega) \cap \{u : \fint u \, dx = 0\}$ with dual $H^{-1}_{(0)}$. Then, for a set $A \subset \Omega$ with $\Gamma := \partial A \cap \Omega$ a piecewise Lipschitz surface, Le recalls the space $H^{1/2}_{(0)}(\Gamma)$, the trace space of $H^1(\Omega \setminus \Gamma)$ with constants quotiented out, and introduces a norm with Hilbert structure given by

$$\|f\|_{H^{1/2}_{(0)}(\Gamma)} = \sqrt{(f,f)_{H^{1/2}_{(0)}(\Gamma)}} := \|\nabla \tilde{f}\|_{L^2(\Omega)},$$

where $\tilde{f}$ satisfies the Dirichlet problem

$$-\Delta \tilde{f} = 0 \text{ in } \Omega \setminus \Gamma, \quad \tilde{f} = f \text{ on } \Gamma. \tag{6}$$

Additionally, $H^{-1/2}_{(0)}(\Gamma)$ is the naturally associated dual space with a Hilbert space structure induced by the corresponding Riesz isomorphism.

With these concepts, Le shows that in the smooth setting the Mullins–Sekerka flow is the gradient flow of the perimeter functional on a formal Hilbert manifold with tangent space given by $H^{-1/2}_{(0)}(\Gamma)$, which for a characteristic function $u(t)$ with interface $\Gamma_t := \partial\{u(t){=}1\}$ can summarily be written as

$$-\frac{d}{dt}u(t) \in \nabla_{H^{-1/2}_{(0)}(\Gamma_t)} E[u(t)]. \tag{7}$$

Further, solutions u_ϵ of the Cahn–Hilliard equation

$$\begin{aligned} \partial_t u_\epsilon = \Delta v_\epsilon \quad \text{where} \quad v_\epsilon = \delta E_\epsilon[u_\epsilon] = \frac{1}{\epsilon} f'(u_\epsilon) - \epsilon\Delta u_\epsilon \quad \text{in } \Omega,\\ (n_{\partial\Omega}\cdot\nabla)u_\epsilon = 0 \quad \text{and} \quad (n_{\partial\Omega}\cdot\nabla)v_\epsilon = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

are shown to converge to a trajectory $t \mapsto u(t) \in BV(\Omega;\{0,1\})$, such that if the evolving surface Γ_t is C^3 in space-time, then u is a solution of the Mullins-Sekerka flow (7) in the sense that

$$\begin{aligned} \int_0^T \Big\|\frac{d}{dt}u(t)\Big\|^2_{H^{-1/2}_{(0)}(\Gamma_t)}\,dt &\leq \liminf_{\epsilon\downarrow 0}\int_0^T \Big\|\frac{d}{dt}u_\epsilon(t)\Big\|^2_{H^{-1}_{(0)}(\Omega)}\,dt,\\ \int_0^T \big\|H_{\Gamma_t}\big\|^2_{H^{1/2}_{(0)}(\Gamma_t)}\,dt &= \int_0^T \big\|\nabla_{H^{-1/2}_{(0)}(\Gamma_t)}E[u(t)]\big\|^2_{H^{-1/2}_{(0)}(\Gamma_t)}\,dt\\ &\leq \liminf_{\epsilon\downarrow 0}\int_0^T \big\|\nabla_{H^{-1}_{(0)}(\Omega)}E_\epsilon[u_\epsilon(t)]\big\|^2_{H^{-1}_{(0)}(\Omega)}\,dt, \end{aligned}$$

where H_Γ is the scalar mean curvature of a sufficiently regular surface Γ. As developed by Le, interpretation of the left-hand side of the above inequalities is only possible for regular Γ. However, we note that lower semi-continuity of the gradient term is intimately connected to the Gibbs–Thomson law which expresses the curvature of the underlying interface in terms of an ambient Sobolev function (i.e., the chemical potential). Understanding when the diffuse interface approximation of the Gibbs-Thomson law is stable in the limit is a challenging problem and a full understanding of this may provide critical tools for studying the sharp interface limit of the Cahn–Hilliard equation in the context of a weak solution theory. We refer the reader to Röger and Tonegawa [53] for a conditional result.

In the next section, we will introduce function spaces and a solution concept that allow us to extend the quantities introduced by Le to the weak setting.

2. Main results and relation to previous works

So as not to waylay the reader, we first introduce in Subsection 2.1 a variety of function spaces necessary for our weak solution concept and then state our main existence theorem. Further properties of the associated solution space, an interpretation of our solution concept from the viewpoint of classical PDE theory (i.e., in terms of associated chemical potentials), as well as further properties of the time-evolving oriented varifolds associated with solutions which are obtained as limit points of the natural minimizing movements scheme are presented in Subsection 2.2. In Subsection 2.3, we return to a discussion of the function spaces introduced in Subsection 2.1 to further illuminate the intuition behind their choice. We then proceed in Subsection 2.4 with a discussion relating our functional framework to the one introduced by Le [36] for the smooth setting. In Subsection 2.5, we finally take the opportunity to highlight the potential of our framework in terms of the recent developments concerning weak-strong uniqueness for curvature driven interface evolution.

2.1. **Weak formulation: Gradient flow structure and existence result.** At the level of a weak formulation, we will describe the evolving interface, arising as the boundary of a phase region, in terms of a time-evolving family of characteristic functions of bounded variation. This strongly motivates us to formulate the gradient flow structure over a manifold of $\{0,1\}$-valued BV functions in Ω. To this end, let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable C^2 boundary $\partial\Omega$. Fixing the mass to be $m_0 \in (0, \mathcal{L}^d(\Omega))$, we define the "manifold"

$$\mathcal{M}_{m_0} := \Big\{\chi \in BV(\Omega;\{0,1\})\colon \int_\Omega \chi\,dx = m_0\Big\}. \tag{8}$$

For the definition of the associated energy functional E on $\mathcal{M}_{m_0}$, recall that we aim to include contact point dynamics with fixed contact angle in this work. Hence, in addition to an isotropic interfacial energy contribution in the bulk, we also incorporate a capillary contribution. Precisely, for a fixed set of three positive surface tension constants $(c_0, \gamma_+, \gamma_-)$ we consider an interfacial energy $E[\chi]$, $\chi \in \mathcal{M}_{m_0}$, of the form

$$\int_\Omega c_0\,d|\nabla\chi| + \int_{\partial\Omega} \gamma_+\chi\,d\mathcal{H}^{d-1} + \int_{\partial\Omega} \gamma_-(1-\chi)\,d\mathcal{H}^{d-1},$$

where by an abuse of notation we do not distinguish between χ and its trace along $\partial\Omega$. Furthermore, the surface tension constants are assumed to satisfy Young's relation $|\gamma_+-\gamma_-| < c_0$ so that there exists an angle $\alpha \in (0,\pi)$ such that

$$(\cos\alpha)c_0 = \gamma_+ - \gamma_-. \tag{9}$$

For convenience, we will employ the following convention: switching if needed the roles of the sets indicated by χ and $1-\chi$, we may assume that $\gamma_- < \gamma_+$ and hence $\alpha \in (0, \frac{\pi}{2}]$. In particular, by subtracting a constant, we may work with the following equivalent formulation of the energy functional on $\mathcal{M}_{m_0}$:

$$E[\chi] := \int_\Omega c_0\,d|\nabla\chi| + \int_{\partial\Omega} (\cos\alpha)c_0\chi\,d\mathcal{H}^{d-1}, \quad \chi \in \mathcal{M}_{m_0}. \tag{10}$$

As usual in the context of weak formulations for curvature driven interface evolution problems, it will actually be necessary to work with a suitable (oriented) varifold relaxation of E. We refer to Definition 3 below for details in this direction.

In order to encode a weak solution of the Mullins–Sekerka equation as a Hilbert space gradient flow with respect to the interfacial energy E, it still remains to introduce the associated Hilbert space structure. To this end, we first introduce a class of regular test functions, which give rise to infinitesimally volume preserving inner variations, denoted by

$$\mathcal{S}_\chi := \Big\{B \in C^1(\overline{\Omega};\mathbb{R}^d)\colon \int_\Omega \chi\nabla\cdot B\,dx = 0,\ B\cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\Big\}. \tag{11}$$

As in Subsection 1.3, we recall the Sobolev space of functions with mass-average zero given by $H^1_{(0)} := \{u \in H^1(\Omega)\colon \fint_\Omega u\,dx = 0\}$ with norm $\|u\|_{H^1_{(0)}} := \|\nabla u\|_{L^2(\Omega)}$ and dual $H^{-1}_{(0)} := (H^1_{(0)})^*$. Based on the test function space $\mathcal{S}_\chi$, we can introduce the space $\mathcal{V}_\chi \subset H^{-1}_{(0)}$ as the closure of regular mass preserving normal velocities generated on the interface associated with $\chi \in \mathcal{M}_{m_0}$:

$$\mathcal{V}_\chi := \overline{\big\{B\cdot\nabla\chi\colon B \in \mathcal{S}_\chi\big\}}^{H^{-1}_{(0)}} \subset H^{-1}_{(0)}, \tag{12}$$

where $B \cdot \nabla\chi$ acts on elements $u \in H^1_{(0)}$ in the distributional sense, i.e., recalling that $B \cdot n_{\partial\Omega} = 0$ along $\partial\Omega$ for $B \in \mathcal{S}_\chi$ we have

$$\langle B \cdot \nabla\chi, u\rangle_{H^{-1}_{(0)},H^1_{(0)}} := -\int_\Omega \chi \nabla \cdot (uB)\,dx. \tag{13}$$

The space $\mathcal{V}_\chi$ carries a Hilbert space structure directly induced by the natural Hilbert space structure of $H^{-1}_{(0)}$. The latter in turn is induced by the inverse Δ_N^{-1} of the weak Neumann Laplacian $\Delta_N \colon H^1_{(0)} \to H^{-1}_{(0)}$ (which for the Hilbert space $H^1_{(0)}$ is in fact nothing else but the associated Riesz isomorphism) in the form of

$$(F, \widetilde{F})_{H^{-1}_{(0)}} := \int_\Omega \nabla\Delta_N^{-1}(F) \cdot \nabla\Delta_N^{-1}(\widetilde{F})\,dx \quad \text{for all } F, \widetilde{F} \in H^{-1}_{(0)}, \tag{14}$$

so that we may in particular define

$$\|F\|^2_{\mathcal{V}_\chi} := \|F\|^2_{H^{-1}_{(0)}} = (F, F)_{H^{-1}_{(0)}}, \quad F \in \mathcal{V}_\chi. \tag{15}$$

We remark that the operator norm on $H^{-1}_{(0)}$ is recovered from the inner product in (14). For the Mullins–Sekerka flow, the space $\mathcal{V}_\chi$ is the natural space associated with the action of the first variation (i.e., the gradient) of the interfacial energy on $\mathcal{S}_\chi$, see (17m) in Definition 5 below.

In view of the Sandier–Serfaty perspective on Hilbert space gradient flows, cf. Subsection 1.3, it would be desirable to capture the time derivative of a trajectory $t \mapsto \chi(\cdot, t) \in \mathcal{M}_{m_0}$ within the same bundle of Hilbert spaces. However, given the a priori lack of regularity of weak solutions, it will be necessary to introduce a second space of velocities $\mathcal{T}_\chi$ (containing the space $\mathcal{V}_\chi$) which can be thought of as a maximal tangent space of the formal manifold; this is given by

$$\mathcal{T}_\chi := \overline{\big\{\mu \in H^{-1}_{(0)} \cap \mathrm{M}(\Omega) \colon \operatorname{supp}\mu \subset \operatorname{supp}|\nabla\chi|\big\}}^{H^{-1}_{(0)}} \subset H^{-1}_{(0)}, \tag{16}$$

where $\mathrm{M}(\Omega)$ denotes the space of Radon measures on Ω. Both spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$ are spaces of velocities, and from the PDE perspective, associated with these will be spaces for the (chemical) potential. We will discuss this and quantify the separation between $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$ in Subsection 2.3. However, despite the necessity to work with two spaces, we emphasize that our gradient flow solution concept still only requires use of the above formal metric/manifold structure and the above energy functional.

As already said, it is in general necessary to work with a varifold relaxation of the energy functional (10). We recall the notion of varifold below.

Definition 1. For a locally compact and separable metric space X, we denote by $\mathrm{M}(X)$ the space of finite Radon measures on X. An (oriented) varifold μ on $\overline{\Omega}$ is a positive Radon measure $\mu \in \mathrm{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$. The mass measure associated with the varifold is denoted by $|\mu| \in \mathrm{M}(\overline{\Omega})$ with $|\mu|_{\mathbb{S}^{d-1}}(O) := \mu(O \times \mathbb{S}^{d-1})$. Here, μ is said to be $(d-1)$-rectifiable if $|\mu|_{\mathbb{S}^{d-1}}$ is $(d-1)$-rectifiable. We note that the prototypical example of a varifold is the measure $\theta\mathcal{H}^{d-1}\llcorner M \otimes \delta_{\Pi_x}$, where M is a rectifiable subset of $\overline{\Omega}$, Π_x is the approximate tangent space of M at x, and θ is a positive function prescribing the density. We say that μ is $(d-1)$-integer-rectifiable if in addition the $(d-1)$-density of $|\mu|_{\mathbb{S}^{d-1}}$ is integer valued.

We remark that often one uses the more general notion of varifold, where instead μ belongs to $M(\overline{\Omega} \times G_{d-1})$, where G_{d-1} is the Grassmanian manifold or the collection of all $(d-1)$-dimensional planes. In this setting, the varifold keeps track of the tangent plane at each point. By considering measures on $\mathbb{S}^{d-1}$ one keeps track of the effective normal vector. As our varifolds will be defined via the boundary of a set of finite perimeter, we are able to keep this information and draw a closer connection between the varifold and the underlying set of finite perimeter (see, e.g., (17e)). Though this is not the case in our paper, in a similar setting, keeping track of the orientation can be essential [10].

We will further be interested the first variation of a varifold, which is found by taking the inner variation of the mass measure. As we will often take tangential variations of the domain, we take this moment to highlight this as a definition.

Definition 2. *A tangential variation* will refer to a $B \in C^1(\overline{\Omega};\mathbb{R}^d)$, $n_{\partial\Omega}\cdot B \equiv 0$ along $\partial\Omega$. The *tangential first variation* of a varifold μ is the first inner variation of the varifold computed with respect to a tangential variation and is denoted by $\delta\mu(B)$. Applying Allard's classical result to this case gives

$$\delta\mu(B) = \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s) : \nabla B(x)\, d\mu(x,s).$$

We refer the reader to [5] for further information on inner variations.

Below we introduce a notion of *admissible varifold*. This definition imposes structural restrictions on a varifold μ so that is is tied the underlying set (or phase indicator χ) in a natural way. In particular the varifold μ lives on top of the boundary of the set, in the sense that $c_0|\nabla\chi| \le \mu$, and the varifold is a regular surface measure, in the sense that it is a rectifiable measure with globally bounded first variation. This definition underlies our solution concept, but is not related to the dynamics of the evolution equation. While it is admittedly rather lengthy and at points technical, with this structural relation between the varifold and phase indicator fixed, we may afterwards introduce streamlined notion of solution concept that is principally focused on capturing the dynamics of the interface.

Definition 3 (Admissible couples of evolving phase indicators and varifolds)**.** Let $d \in \{2,3\}$, consider a finite time horizon $T_* \in (0,\infty)$, and let $\chi \in L^\infty(0,T_*;\mathcal{M}_{m_0}) \cap C([0,T_*);H^{-1}_{(0)}(\Omega))$. For a locally compact and separable metric space X, we denote by $\mathrm{M}(X)$ the space of finite Radon measures on X, and consider a family $\mu = (\mu_t)_{t\in(0,T_*)}$ of oriented varifolds $\mu_t \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$, $t \in (0,T_*)$. The couple (χ,μ) is called *admissible* if the following properties are satisfied.

i) *(Structure of oriented varifolds)* For almost every $t \in (0,T_*)$, the oriented varifold $\mu_t \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ decomposes as $\mu_t = c_0\mu_t^{\Omega} + (\cos\alpha)c_0\mu_t^{\partial\Omega}$ for two separate oriented varifolds given in their disintegrated form by

$$\mu_t^{\Omega} =: |\mu_t^{\Omega}|_{\mathbb{S}^{d-1}} \otimes (\lambda_{x,t})_{x\in\overline{\Omega}} \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1}) \tag{17a}$$

and

$$\mu_t^{\partial\Omega} =: |\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}} \otimes (\delta_{n_{\partial\Omega}(x)})_{x\in\partial\Omega} \in \mathrm{M}(\partial\Omega\times\mathbb{S}^{d-1}). \tag{17b}$$

For almost every $t \in (0,T_*)$, the measure $|\mu_t^{\Omega}|_{\mathbb{S}^{d-1}}$ is $(d-1)$-integer-rectifiable, i.e., the $(d-1)$-dimensional density takes values in $\mathbb{N}$, and further, the measure $\mu_t^{\partial\Omega}$ is given by $g\, n_{\partial\Omega}\mathcal{H}^{d-1}\llcorner\partial\Omega$, where $g \in BV(\partial\Omega;\{0,1\})$.

ii) *(Compatibility with phase indicator)* We also require that these oriented varifolds contain the interfaces associated with the phase modeled by χ in the sense of

$$
\begin{aligned}
&c_0|\nabla\chi(\cdot,t)|\llcorner\Omega \\
&\quad = c_0\sum_{k=1}^{\infty}\frac{1}{2k-1}|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\big(\Omega\cap\{\theta^{d-1}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega,\cdot)=2k-1\}\big)
\end{aligned}
\tag{17c}
$$

and

$$
(\cos\alpha)\chi(\cdot,t)\,\mathcal{H}^{d-1}\llcorner\partial\Omega\leq(\cos\alpha)\big(|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}+|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega\big)
\tag{17d}
$$

for almost every $t\in(0,T_*)$. Furthermore, for almost every $t\in(0,T_*)$ it holds that for every $\eta\in C^1(\overline{\Omega};\mathbb{R}^d)$ such that $n_{\partial\Omega}\cdot\eta=0$ on $\partial\Omega$, and every $\xi\in C^1(\overline{\Omega};\mathbb{R}^d)$ with $n_{\partial\Omega}\cdot\xi=(\cos\alpha)$ on $\partial\Omega$, it holds

$$
\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} s\cdot\eta(x)\,d\mu_t^\Omega(x,s)=\int_\Omega\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot\eta(\cdot)\,d|\nabla\chi(\cdot,t)|,
\tag{17e}
$$

$$
\begin{aligned}
-\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} s\cdot\xi(x)\,d\mu_t^\Omega(x,s)&=-\int_\Omega\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot\xi(\cdot)\,d|\nabla\chi(\cdot,t)|, \\
&\quad+(\cos\alpha)\Big(|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}(\partial\Omega)-\int_{\partial\Omega}\chi(\cdot,t)\,d\mathcal{H}^{d-1}\Big).
\end{aligned}
\tag{17f}
$$

iii) *(Existence of generalized mean curvature vector)* For almost every $t\in(0,T_*)$, there exists a map $\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}\colon\operatorname{supp}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega)\to\mathbb{R}^d$ such that

$$
\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}\in L^s\big(\Omega;d|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega\big)
\tag{17g}
$$

where $s\in[1,4]$ if $d=3$ and $s\in[1,\infty)$ if $d=2$, and such that the first variation $\delta\mu_t$ of μ_t in the direction of a tangential variation $B\in C^1(\overline{\Omega};\mathbb{R}^d)$, $n_{\partial\Omega}\cdot B\equiv0$ along $\partial\Omega$, is given by

$$
\delta\mu_t(B)=-\int_\Omega c_0\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}\cdot B\,d|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega.
\tag{17h}
$$

Furthermore, μ_t is of bounded first variation on $\overline{\Omega}$ such that

$$
\begin{aligned}
&\sup_{B\in C^1(\overline{\Omega};\mathbb{R}^d),\,\|B\|_{L^\infty}\leq1}|\delta\mu_t(B)| \\
&\qquad\leq C(\Omega)|\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega})+\widetilde{C}\|\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}\|_{L^1(\Omega,d|\mu|_{\mathbb{S}^{d-1}})}
\end{aligned}
\tag{17i}
$$

for some universal constant $\widetilde{C}>0$ and some constant $C(\Omega)>0$, the latter only depending on the second fundamental form of $\partial\Omega$.

iv) *(Structure of generalized mean curvature)* The generalized mean curvature vector $\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}\colon\operatorname{supp}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega)\to\mathbb{R}^d$ satisfies

$$
\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}=0\qquad\text{in }\Omega\cap\{\theta^{d-1}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega,\cdot)>1\},
\tag{17j}
$$

$$
\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega}=H_{\chi(\cdot,t)}\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\qquad\text{in }\operatorname{supp}(|\nabla\chi(\cdot,t)|\llcorner\Omega),
\tag{17k}
$$

where $H_{\chi(\cdot,t)}\colon\operatorname{supp}(|\nabla\chi(\cdot,t)|\llcorner\Omega)\to\mathbb{R}$ denotes the generalized mean curvature of $\operatorname{supp}(|\nabla\chi(\cdot,t)|\llcorner\Omega)$ in the sense of Röger [51, Definition 1.1].

v) *(Measurability in time of energy and slope)* The total mass measure associated with the oriented varifold μ_t, i.e.,

$$
E[\mu_t]:=|\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega})=c_0|\mu_t^\Omega|_{\mathbb{S}^{d-1}}(\overline{\Omega})+(\cos\alpha)c_0|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}(\partial\Omega),
\tag{17l}
$$

is a measurable map $(0,T_*) \ni t \mapsto E[\mu_t] \in [0,\infty)$. Second, defining by a slight abuse of notation but still in the spirit of the usual metric slope à la De Giorgi (cf. (20) and (59) below)

$$\frac{1}{2}\big|\partial E[\mu_t]\big|^2_{\mathcal{V}_{\chi(\cdot,t)}} := \sup_{B\in\mathcal{S}_{\chi(\cdot,t)}} \Big\{\delta\mu_t(B) - \frac{1}{2}\big\|B\cdot\nabla\chi(\cdot,t)\big\|^2_{\mathcal{V}_{\chi(\cdot,t)}}\Big\}, \tag{17m}$$

the slope is measurable as a map $(0,T_*) \ni t \mapsto \big|\partial E[\mu_t]\big|_{\mathcal{V}_{\chi(\cdot,t)}} \in [0,\infty]$.

We note that the above definition implicitly enforces uniqueness of the decomposition of the bulk and boundary varifolds used to define the surface energy.

Remark 4. Note that as $(\cos\alpha) \in [0,1)$, the integer-rectifiability of the measure μ_t^{Ω} along with the density of $\mu_t^{\partial\Omega}$ being 0 or 1 guarantees that the decomposition is unique, so long as when $\alpha = \pi/2$ one takes $\mu_t^{\partial\Omega} \equiv 0$. Further, we remark that in (17d), one cannot a priori expect the contribution of $|\mu_t^{\Omega}|$ to be zero on $\partial\Omega$. Though these measures will be constructed from a local minimization problem, in the limit, this will allow for surfaces which lay tangentially on the boundary $\partial\Omega$, as these do not charge the first variation in (17h) with a lower dimensional measure on $\partial\Omega$.

We have everything in place to state our solution concept for Mullins–Sekerka flow (1a)–(1e).

Definition 5 (Varifold solutions of Mullins–Sekerka flow as curves of maximal slope)**.** Let $d \in \{2,3\}$, consider a finite time horizon $T_* \in (0,\infty)$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable C^2 boundary $\partial\Omega$. Fix $\chi_0 \in \mathcal{M}_{m_0}$.

A measurable map $\chi\colon \Omega\times(0,T_*) \to \{0,1\}$ together with a family $\mu = (\mu_t)_{t\in(0,T_*)}$ of oriented varifolds $\mu_t \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$, $t \in (0,T_*)$, is called a *varifold solution for Mullins–Sekerka flow* (1a)–(1e) *with time horizon T_* and initial data χ_0* if (χ,μ) is an admissible couple in the sense of Definition 3, $\chi(0) = \chi_0$ in $H^{-1}_{(0)}$, and for almost all $0 < s < T < T_*$ it holds

$$E[\mu_T] + \int_s^T \frac{1}{2}\big\|(\partial_t\chi)(\cdot,t)\big\|^2_{\mathcal{T}_{\chi(\cdot,t)}} + \frac{1}{2}\big|\partial E[\mu_t]\big|^2_{\mathcal{V}_{\chi(\cdot,t)}}\,dt \leq E[\mu_s] \leq E[\chi_0]. \tag{18}$$

We call χ a *BV solution for evolution by Mullins–Sekerka flow* (1a)–(1e) *with initial data χ_0* if there exists $\mu = (\mu_t)_{t\in(0,T_*)}$ such that (χ,μ) is a varifold solution in the above sense and the family of varifolds μ is given by the canonical lift of χ, i.e., for almost every $t \in (0,T_*)$ it holds (recall (17a))

$$|\mu_t|_{\mathbb{S}^{d-1}} = c_0\,|\nabla\chi(\cdot,t)|\llcorner\Omega + (\cos\alpha)c_0\chi(\cdot,t)\,\mathcal{H}^{d-1}\llcorner\partial\Omega, \tag{19a}$$

$$\lambda_{x,t} = \delta_{\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}(x)} \quad \text{for } (|\nabla\chi(\cdot,t)|\llcorner\Omega)\text{-almost every } x\in\Omega. \tag{19b}$$

Remark 6. We note the role the varifold plays in the above definition. First and foremost, the varifold ensures that there is a regular object living on top of the boundary $|\nabla\chi|$, which is differentiable in the sense of a first variation. However, the first variation is uniquely defined by χ, that is, $|\partial E[\mu]|^2_{\mathcal{V}_\chi}$ is determined by χ, and consequently, the dissipation in inequality (18) is independent of μ.

Remark 7. We note that implicit in the above solution definition is that $\partial_t\chi \in L^2(0,T_*;H^{-1}_{(0)})$ and, in particular, $(\partial_t\chi)(\cdot,t) \in \mathcal{T}_{\chi(\cdot,t)}$ for almost every $t \in (0,T_*)$.

Before we state the main existence result of this work, let us provide two brief comments on the above definition. First, we note that in Lemma 3 we show that if (χ, μ) is a varifold solution to Mullins–Sekerka flow in the sense of Definition 5, then it is also a solution from a more typical PDE perspective. Second, to justify the metric slope notation within (17m), we refer the reader to Lemma 2 where it is shown that for a varifold solution (χ, μ) it holds that

$$\big|\partial E[\mu]\big|_{\mathcal{V}_\chi} = \sup_{\Psi} \limsup_{s\to 0} \frac{(E[\mu] - E[\mu\circ\Psi_s^{-1}])_+}{\|\chi - \chi\circ\Psi_s^{-1}\|_{H^{-1}_{(0)}}},$$

where the supremum runs over all one-parameter families of diffeomorphisms $s \mapsto \Psi_s \in C^1\text{-Diffeo}(\overline{\Omega}, \overline{\Omega})$ which are differentiable in an open neighborhood of $s = 0$ and further satisfy $\Psi_0 = \mathrm{Id}$, $\int_\Omega \chi\circ\Psi_s^{-1}\,dx = m_0$ and $\partial_s\Psi_s|_{s=0} = B \in \mathcal{S}_\chi$. Note that the relation $\partial_s(\chi\circ\Psi_s^{-1})|_{s=0} + (B\cdot\nabla)\chi = 0$ enforced by the chain rule, $(\chi\circ\Psi_s^{-1})|_{s=0} = \chi$ as well as $\partial_s\Psi_s^{-1}|_{s=0} = -\partial_s\Psi_s|_{s=0} = -B$ motivates us to consider $\mathcal{V}_\chi$ as the tangent space for the formal manifold at $\chi \in \mathcal{M}_{m_0}$.

Theorem 1 (Existence of varifold solutions of Mullins–Sekerka flow)**.** *Let $d \in \{2,3\}$, $T_* \in (0,\infty)$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable C^2 boundary $\partial\Omega$. Let $m_0 \in (0, \mathcal{L}^d(\Omega))$, $\chi_0 \in \mathcal{M}_{m_0}$, $c_0 \in (0,\infty)$ and $\alpha \in (0, \frac{\pi}{2}]$.*

Then, there exists a varifold solution for Mullins–Sekerka flow (1a)–(1e) *with initial data χ_0 in the sense of Definition 5.*

In fact, each limit point of the minimizing movements scheme associated with the Mullins–Sekerka flow (1a)–(1e)*, cf. Subsection 3.1, is a solution in the sense of Definition 5. In case of convergence of the time-integrated energies of the approximations (cf.* (60)*), the corresponding limit point of the minimizing movements scheme is even a BV solution in the sense of Definition 5.*

The proof of Theorem 1 is the content of Subsections 3.1–3.3.

2.2. **Further properties of varifold solutions.** The purpose of this subsection is to collect a variety of further results complementing our main existence result, Theorem 1. Proofs of these are postponed until Subsection 3.4.

Lemma 2 (Interpretation as a De Giorgi metric slope)**.** *Let $\chi \in BV(\Omega;\{0,1\})$ and $\mu \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$. Suppose in addition that the tangential first variation of μ is given by a curvature $H_\chi \in L^1(\Omega;|\nabla\chi|)$ in the sense of equations* (17h)–(17k)*. Then, it holds that*

$$\big|\partial E[\mu]\big|_{\mathcal{V}_\chi} = \sup_{\Psi} \limsup_{s\to 0} \frac{(E[\mu] - E[\mu\circ\Psi_s^{-1}])_+}{\|\chi - \chi\circ\Psi_s^{-1}\|_{H^{-1}_{(0)}}}, \tag{20}$$

where the supremum runs over all one-parameter families of diffeomorphisms $s \mapsto \Psi_s \in C^1\text{-Diffeo}(\overline{\Omega}, \overline{\Omega})$ which are differentiable in a neighborhood of the origin and further satisfy $\Psi_0 = \mathrm{Id}$, $\int_\Omega \chi\circ\Psi_s^{-1}\,dx = m_0$ and $\partial_s\Psi_s|_{s=0} = B \in \mathcal{S}_\chi$. Without assuming (17h)–(17k)*, the right hand side of* (20) *provides at least an upper bound.*

Next, we aim to interpret the information provided by the sharp energy inequality (18) from a viewpoint which is more in the tradition of classical PDE theory. More precisely, we show that (18) together with the representation (17h)–(17k) already encodes the evolution equation for the evolving phase as well as the Gibbs-Thomson law–both in terms of a suitable distributional formulation featuring an associated potential. We emphasize, however, that without further regularity assumptions on the evolving geometry these two potentials may *a priori* not agree.

This flexibility is in turn a key strength of the gradient flow perspective to allow for less regular evolutions (i.e., a weak solution theory).

Lemma 3 (Interpretation from a PDE perspective)**.** *Let (χ,μ) be a varifold solution for Mullins–Sekerka flow with initial data χ_0 in the sense of Definition 5. For a given $\chi\in\mathcal{M}_{m_0}$, define for each of the two velocity spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$ an associated space of potentials via $\mathcal{G}_\chi := \Delta_N^{-1}(\mathcal{V}_\chi)\subset H^1_{(0)}$ and $\mathcal{H}_\chi := \Delta_N^{-1}(\mathcal{T}_\chi)\subset H^1_{(0)}$, respectively.*

i) There exists a potential $u\in L^2(0,T_*;\mathcal{H}_{\chi(\cdot,t)})\subset L^2(0,T_*;H^1_{(0)})$ such that $\partial_t\chi = \Delta u$ in $\Omega\times(0,T_*)$, $\chi(\cdot,0)=\chi_0$ in Ω, and $(n_{\partial\Omega}\cdot\nabla)u = 0$ on $\partial\Omega\times(0,T_*)$, in the precise sense of

$$\begin{aligned}&\int_\Omega \chi(\cdot,T)\zeta(\cdot,T)\,dx - \int_\Omega \chi_0\zeta(\cdot,0)\,dx\\&= \int_0^T\int_\Omega \chi\partial_t\zeta\,dxdt - \int_0^T\int_\Omega \nabla u\cdot\nabla\zeta\,dxdt\end{aligned}\tag{21}$$

for almost every $T\in(0,T_*)$ and all $\zeta\in C^1(\overline{\Omega}\times[0,T_*))$.

ii) There exists a potential $w\in L^2(0,T_*;H^1(\Omega))$ such that for $w_0 := w - \fint_\Omega w\,dx$ one has $w_0\in L^2(0,T_*;\mathcal{G}_{\chi(\cdot,t)})\subset L^2(0,T_*;H^1_{(0)})$, and further satisfies the following three properties: first, the Gibbs–Thomson law

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s):\nabla B(x)\,d\mu_t(x,s) = \int_\Omega \chi(\cdot,t)\nabla\cdot(w(\cdot,t)B)\,dx \tag{22}$$

holds true for almost every $t\in(0,T_*)$ and all $B\in C^1(\overline{\Omega};\mathbb{R}^d)$ such that $(B\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv 0$; second, it holds that

$$\int_\Omega \frac{1}{2}|\nabla w(\cdot,t)|^2\,dx = \frac{1}{2}\|w_0(\cdot,t)\|^2_{\mathcal{G}_{\chi(\cdot,t)}} = \frac{1}{2}\big|\partial E[\chi(\cdot,t),\mu_t]\big|^2_{\mathcal{V}_{\chi(\cdot,t)}} \tag{23}$$

for almost every $t\in(0,T_*)$; and third, there is $C=C(\Omega,d,c_0,m_0,\chi_0)>0$ such that

$$\|w(\cdot,t)\|_{H^1(\Omega)} \le C(1+\|\nabla w(\cdot,t)\|_{L^2(\Omega)}) \tag{24}$$

for almost every $t\in(0,T_*)$.

iii) The sharp energy dissipation inequality holds true in the sense that for almost all choices $0<s<T<T_*$

$$E[\mu_T] + \int_s^T\int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2\,dxdt \le E[\mu_s] \le E[\chi_0]. \tag{25}$$

Note that in view of (17c), (17h)–(17k), (22) and the trace estimate (32) from below, if (χ,μ) is a varifold solution we may in particular deduce that

$$c_0 H_{\chi(\cdot,t)} = w(\cdot,t) \qquad \text{on } \operatorname{supp}(|\nabla\chi(\cdot,t)|\llcorner\Omega), \tag{26}$$

$$w(\cdot,t) = 0 \qquad \text{on } \Omega\cap\big\{\theta^{d-1}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega,\cdot)\in 2\mathbb{N}{+}1\big\} \tag{27}$$

for almost every $t\in(0,T_*)$ up to sets of $(|\nabla\chi(\cdot,t)|\llcorner\Omega)$-measure zero.

Next, we show subsequential compactness of our solution concept and consistency with classical solutions. To formulate the latter, we make use of the notion of a time-dependent family $\mathscr{A} = (\mathscr{A}(t))_{t\in[0,T_*)}$ of smoothly evolving subsets $\mathscr{A}(t)\subset\Omega$, $t\in[0,T_*)$. More precisely, each set $\mathscr{A}(t)$ is open and consists of finitely many connected components (the number of which is constant in time). Furthermore, the reduced boundary of $\mathscr{A}(t)$ in $\mathbb{R}^d$ differs from its topological boundary only by a finite number of contact sets on $\partial\Omega$ (the number of which is again constant in time) represented by $\partial(\partial^*\mathscr{A}(t)\cap\Omega) = \partial(\partial^*\mathscr{A}(t)\cap\partial\Omega) \subset \partial\Omega$. The remaining parts of $\partial\mathscr{A}(t)$, i.e., $\partial^*\mathscr{A}(t)\cap\Omega$ and $\partial^*\mathscr{A}(t)\cap\partial\Omega$, are smooth manifolds with boundary (which for both is given by the contact points manifold).

Lemma 4 (Properties of the space of varifold solutions)**.** *Let the assumptions and notation of Theorem 1 be in place.*

i) (Consistency) *Let (χ,μ) be a varifold solution for Mullins–Sekerka flow in the sense of Definition 5 which is smooth, i.e., $\chi(x,t) = \chi_{\mathscr{A}}(x,t) := \chi_{\mathscr{A}(t)}(x)$ for a smoothly evolving family $\mathscr{A} = (\mathscr{A}(t))_{t\in[0,T_*)}$. Furthermore, assume that* (17h) *also holds with $\delta\mu_t$ replaced on the left hand side by $\delta E[\chi(\cdot,t)]$ (which for a BV solution does not represent an additional constraint). Then, $\mathscr{A}$ is a classical solution for Mullins–Sekerka flow in the sense of* (1a)–(1e)*. Further, it also holds that*

$$|\nabla\chi(\cdot,t)|\llcorner\Omega = |\mu_t^{\Omega}|_{\mathbb{S}^{d-1}}\llcorner\Omega, \qquad |\mu_t^{\Omega}|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega = 0, \tag{28}$$

$$\chi(\cdot,t)\,\mathcal{H}^{d-1}\llcorner\partial\Omega = |\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}} \tag{29}$$

for a.e. $t\in(0,T_*)$.

Vice versa, any classical solution $\mathscr{A}$ of Mullins–Sekerka flow (1a)–(1e) *gives rise to a (smooth) BV solution $\chi = \chi_{\mathscr{A}}$ in the sense of Definition 5.*

ii) (Subsequential compactness of the solution space) *Let $(\chi_k,\mu_k)_{k\in\mathbb{N}}$ be a sequence of varifold solutions with initial data $\chi_{k,0}$ and time horizon $0<T_*<\infty$ in the sense of Definition 5. Assume that $\sup_{k\in\mathbb{N}} E[\mu_{k,0}] < \infty$, and that the sequence $(|\nabla\chi_{k,0}|\llcorner\Omega)_{k\in\mathbb{N}}$ is tight. Then, one may find a subsequence $\{k_n\}_{n\in\mathbb{N}}$, data χ_0, and a varifold solution (χ,μ) with initial data χ_0 and time horizon T_* in the sense of Definition 5 such that $\chi_{k_n}\to\chi$ in $L^1(\Omega\times(0,T_*))$ as well as for almost all $t\in(0,T_*)$, $(\mu_{k_n})_t \overset{*}{\rightharpoonup} \mu_t$ subsequentially in $\mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ as $n\to\infty$.*

Remark 8. We note that the assumption in *i)* of the above lemma that (17h) holds with $\delta\mu_t$ replaced on the left hand side by $\delta E[\chi(\cdot,t)]$ enforces the correct contact angle. As χ prescribes a smooth geometry, the curvature of the surface at a given time already coincides with $H_{\chi(\cdot,t)}$, and hence (17h) already holds for compactly supported $B\in C^1_{cpt}(\Omega;\mathbb{R}^2)$ for free. Of course, if (χ,μ) is a BV-solution, this assumption is immediately satisfied.

One can alternatively formulate compactness over the space $GMM(\chi_{k,0})$ of generalized minimizing movements, introduced by Ambrosio et al. [7]. The space $GMM(\chi_{k,0})$ is given by all limit points as $h\to 0$ of the minimizing movements scheme introduced in Subsection 3.1. By Theorem 1, every element of $GMM(\chi_{k,0})$ is a varifold solution of Mullins–Sekerka flow with initial value $\chi_{k,0}$. Though we do not prove this, a diagonalization argument shows that for a sequence of initial data as in Part ii) of Lemma 4, (χ_k,μ_k) belonging to $GMM(\chi_{k,0})$ are precompact and up to a subsequence (in the above sense) converge to (χ,μ) in $GMM(\chi_0)$.

Finally, the proof of our main result, Theorem 1, is based upon the following two auxiliary results, which we believe are worth mentioning on their own.

Proposition 5 (First variation estimate up to the boundary for tangential variations)**.** *Let $d\in\{2,3\}$, let $\Omega\subset\mathbb{R}^d$ be a bounded domain with orientable C^2 boundary $\partial\Omega$, let $w\in H^1(\Omega)$, let $\chi\in BV(\Omega;\{0,1\})$, and let $\mu = |\mu|_{\mathbb{S}^{d-1}}\otimes(\lambda_x)_{x\in\overline{\Omega}} \in$*

$\mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ *be an oriented varifold such that* $c_0|\nabla\chi|\llcorner\Omega \leq |\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega$ *in the sense of measures for some constant* $c_0 > 0$. *Assume moreover that the Gibbs–Thomson law holds true in form of*

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B \, d\mu = \int_\Omega \chi\nabla\cdot(wB)\, dx \tag{30}$$

for all tangential variations $B\in C^1(\overline{\Omega};\mathbb{R}^d)$, $(B\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv 0$.

There exists $r = r(\partial\Omega)\in(0,1)$ *such that for all* $x_0\in\overline{\Omega}$ *with* $\mathrm{dist}(x_0,\partial\Omega) < r$ *and all exponents* $s\in[2,4]$ *if* $d=3$ *or otherwise* $s\in[2,\infty)$ *there exists a constant* $C = C(r,s,d) > 0$ *such that*

$$\bigg(\int_{B_r(x_0)\cap\overline{\Omega}} |w|^s \, d|\mu|_{\mathbb{S}^{d-1}}\bigg)^{\frac{1}{s}} \leq C\big(1 + |\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \|w\|^d_{H^1(\Omega)}\big)^{1+\frac{1}{s}}. \tag{31}$$

In particular, the varifold μ *is of bounded variation with respect to tangential variations (with generalized mean curvature vector* H^Ω *trivially given by* $\rho^\Omega\frac{w}{c_0}\frac{\nabla\chi}{|\nabla\chi|}$ *where* $\rho^\Omega := \frac{c_0|\nabla\chi|\llcorner\Omega}{|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega}\in[0,1]$, *cf.* (30)*) and the potential satisfies*

$$\bigg(\int_{\overline{\Omega}} |w|^s \, d|\mu|_{\mathbb{S}^{d-1}}\bigg)^{\frac{1}{s}} \leq C\big(1 + |\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \max\big\{1, \|w\|^d_{H^1(\Omega)}\big\}\big)^{1+\frac{1}{s}}. \tag{32}$$

By a recent work of De Masi [19], one may post-process the previous result to the following statement.

Corollary 6 (First variation estimate up to the boundary)**.** *In the setting of Proposition 5, the varifold* μ *is in fact of bounded variation on* $\overline{\Omega}$. *More precisely, there exist* H^Ω, $H^{\partial\Omega}$ *and* σ_μ *with the properties*

$$H^\Omega = \rho^\Omega\frac{w}{c_0}\frac{\nabla\chi}{|\nabla\chi|}, \quad \rho^\Omega := \frac{c_0|\nabla\chi|\llcorner\Omega}{|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega}\in[0,1], \tag{33}$$

$$H^{\partial\Omega}\in L^\infty(\partial\Omega, d|\mu|_{\mathbb{S}^{d-1}}), \quad H^{\partial\Omega}(x)\perp \mathrm{Tan}_x\partial\Omega \;\; \textit{for } |\mu|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega\textit{-a.e. } x\in\overline{\Omega}, \tag{34}$$

$$\sigma_\mu\in\mathrm{M}(\partial\Omega), \tag{35}$$

such that the first variation $\delta\mu$ *of* μ *is represented by*

$$\delta\mu(B) = -\int_{\overline{\Omega}} (H^\Omega + H^{\partial\Omega})\cdot B \, d|\mu|_{\mathbb{S}^{d-1}} + \int_{\partial\Omega} B\cdot n_{\partial\Omega}\, d\sigma_\mu \tag{36}$$

for all $B\in C^1(\overline{\Omega};\mathbb{R}^d)$. *Furthermore, there exists* $C = C(\Omega) > 0$ *(depending only on the second fundamental form of the domain boundary* $\partial\Omega$*) such that*

$$\sup_{B\in C^1(\overline{\Omega}),\, \|B\|_{L^\infty}\leq 1} |\delta\mu(B)| \leq C|\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + 2\|H^\Omega\|_{L^1(\Omega, d|\mu|_{\mathbb{S}^{d-1}})}, \tag{37}$$

$$\|H^{\partial\Omega}\|_{L^\infty(\partial\Omega, d|\mu|_{\mathbb{S}^{d-1}})} \leq C, \tag{38}$$

$$\sigma_\mu(\partial\Omega) \leq C|\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \|H^\Omega\|_{L^1(\Omega, d|\mu|_{\mathbb{S}^{d-1}})}. \tag{39}$$

2.3. **A closer look at the functional framework.** In this subsection, we characterize the difference between the velocity spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$, defined in (12) and (16) respectively, by expressing the quotient space $\mathcal{T}_\chi/\mathcal{V}_\chi$ in terms of a distributional trace space and quasi-everywhere trace space (see (50)). As an application, this result will show that if $|\nabla\chi|$ is given by the surface measure of a Lipschitz graph, then the quotient space collapses to a point and $\mathcal{V}_\chi = \mathcal{T}_\chi$. A modification of this idea will allow us to show that in dimension $d=2$, under an energy convergence assumption, the formal tangent spaces coincide for solutions constructed via the minimizing movements scheme (see Corollary 7).

Both spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$ are spaces of velocities, and associated with these will be spaces of potentials where one expects to find the chemical potential. For this, we recall that the inverse of the weak Neumann Laplacian $\Delta_N^{-1}\colon H^{-1}_{(0)} \to H^1_{(0)}$ is defined by $u_F := \Delta_N^{-1}(F)$, $F \in H^{-1}_{(0)}$, where $u_F \in H^1_{(0)}$ is the unique weak solution of the Neumann problem

$$\begin{aligned} \Delta u_F &= F \quad \text{in } \Omega, \\ (n_{\partial\Omega}\cdot\nabla)u_F &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{40}$$

Recall also that $\Delta_N^{-1}\colon H^{-1}_{(0)} \to H^1_{(0)}$ defines an isometric isomorphism (with respect to the Hilbert space structures on $H^1_{(0)}$ and $H^{-1}_{(0)}$ defined in Subsection 2.1), and since Δ_N is nothing else but the Riesz isomorphism for the Hilbert space $H^1_{(0)}$, the relation

$$(u_F, v)_{H^1_{(0)}} = \langle F, v\rangle_{H^{-1}_{(0)}, H^1_{(0)}} \tag{41}$$

holds for all $v \in H^1_{(0)}$. We then introduce a space of potentials associated with $\mathcal{V}_\chi$ given by

$$\mathcal{G}_\chi := \Delta_N^{-1}(\mathcal{V}_\chi) \subset H^1_{(0)}. \tag{42}$$

Likewise we can introduce the space of potentials associated to the "maximal tangent space" $\mathcal{T}_\chi$ given by

$$\mathcal{H}_\chi := \Delta_N^{-1}(\mathcal{T}_\chi) \subset H^1_{(0)}. \tag{43}$$

To understand the relation between the spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$, we will develop annihilator relations for $\mathcal{G}_\chi$ and $\mathcal{H}_\chi$ in $H^1_{(0)}$. Throughout the remainder of this subsection, we identify $H^1_{(0)}$ with $H^1(\Omega)/\mathbb{R}$, the Sobolev space quotiented by constants, which allows us to consider any $v \in H^1(\Omega)$ as an element of $H^1_{(0)}$.

By [3, Corollary 9.1.7] of Adams and Hedberg,

$$\mathcal{T}_\chi = \{F \in H^{-1}_{(0)}\colon \langle F, v\rangle_{H^{-1}_{(0)}, H^1_{(0)}} = 0 \text{ for all } v \in C^1_c(\Omega \setminus \operatorname{supp}|\nabla\chi|)\}.$$

Using (41) and (43), this implies that the space

$$H^1_{0,\operatorname{supp}|\nabla\chi|} := \overline{\{v \in C^1_c(\Omega\setminus\operatorname{supp}|\nabla\chi|)\}}^{H^1_{(0)}}, \tag{44}$$

satisfies the annihilator relation with $(\cdot,\cdot)_{H^1_{(0)}}$

$$H^1_{0,\operatorname{supp}|\nabla\chi|} = \mathcal{H}_\chi^\perp. \tag{45}$$

Further by [3, Corollary 9.1.4], for the trace operator $\mathrm{Tr}_{\operatorname{supp}|\nabla\chi|}$ defined Cap_2 quasi-everywhere for $H^1(\Omega)$-functions, (45) is the same as

$$\ker\left(\mathrm{Tr}_{\operatorname{supp}|\nabla\chi|}\right) = \mathcal{H}_\chi^\perp, \tag{46}$$

where $u \in H^1_{(0)} \cap \ker\left(\mathrm{Tr}_{\operatorname{supp}|\nabla\chi|}\right)$ if and only if $\mathrm{Tr}_{\operatorname{supp}|\nabla\chi|} u \equiv c$ for some $c \in \mathbb{R}$.

Similarly, one may use the definition (13) and the relation (41) to show that

$$\mathcal{G}_\chi^\perp = \left\{u \in H^1_{(0)}\colon \int_\Omega \chi\nabla u\cdot B\,dx = -\int_\Omega \chi u\nabla\cdot B\,dx \text{ for all } B \in \mathcal{S}_\chi\right\}. \tag{47}$$

For $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ such that $\int_\Omega \chi\nabla\cdot B\,dx \neq 0$, one can consider fixed $\xi \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $\xi\cdot n_{\partial\Omega} = 0$ on $\partial\Omega$ such that $\int_\Omega \chi\nabla\cdot\xi\,dx \neq 0$ and use the corrected function $\tilde{B} := B - \frac{\int_\Omega \chi\nabla\cdot B\,dx}{\int_\Omega \chi\nabla\cdot\xi\,dx}\xi$ in (47) to see that the above relation is equivalent to

$$\mathcal{G}_\chi^\perp = \Big\{ u \in H^1_{(0)} \colon \int_\Omega \chi\nabla(u+c)\cdot B\,dx = -\int_\Omega \chi(u+c)\nabla\cdot B\,dx \qquad (48)$$

$$\text{for some } c\in\mathbb{R} \text{ and all } B \in C^1(\bar{\Omega};\mathbb{R}^d) \text{ with } B\cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega \Big\}.$$

Thus, $\mathcal{G}_\chi^\perp$ is the space functions in $H^1_{(0)}$ which have vanishing trace on $\operatorname{supp}|\nabla\chi|$ in a distributional sense.

We now show that $\mathcal{G}_\chi \subset \mathcal{H}_\chi$, which is equivalent to $\mathcal{V}_\chi \subset \mathcal{T}_\chi$. First note that (45) implies

$$\mathcal{H}_\chi = \big\{ u \in H^1_{(0)} \colon \Delta u = 0 \text{ in } \Omega\setminus\operatorname{supp}|\nabla\chi| \big\}. \qquad (49)$$

As a technical tool, we remark that for fixed $v \in C^1_c(\Omega\setminus\operatorname{supp}|\nabla\chi|)$, up to a representative, $v\chi \in C^1_c(\Omega)$. To see this, let $\chi = \chi_A$ for $A \subset \Omega$ and note for any $x \in \operatorname{supp} v$ we can find $r > 0$ such that $|B(x,r)\cap A| = |B(x,r)|$ or $|B(x,r)\cap(\Omega\setminus A)| = |B(x,r)|$. We construct a finite cover of $\operatorname{supp} v$ given by $\mathcal{C} := \cup_i B(x_i,r_i)$, and define the set

$$\mathcal{C}' := \bigcup_{x_i \colon |B(x_i,r_i)\cap A| = |B(x_i,r_i)|} B(x_i,r_i).$$

We have that

$$v\chi = \begin{cases} v & \text{in } \mathcal{C}', \\ 0 & \text{otherwise,} \end{cases}$$

which is smooth as the balls used in $\mathcal{C}'$ are disjoint from those balls such that $|B(x,r)\cap(\Omega\setminus A)| = |B(x,r)|$, completing the claim. Now, for $u \in \mathcal{G}_\chi$ given by $u = u_{B\cdot\nabla\chi}$ with $B \in \mathcal{S}_\chi$, by (41) we compute

$$(u,v)_{H^1_{(0)}} = -\int_\Omega \nabla\cdot(vB)\chi\,dx = -\int_\Omega \nabla\cdot((v\chi)B)\,dx = 0.$$

It follows that $\mathcal{G}_\chi \subset \mathcal{H}_\chi$ by (44) and (45).

Using the quotient space isomorphism $Y/X \simeq X^\perp$ for a closed subspace X of Y [17, Theorem III.10.2] and the subset relation $\mathcal{G}_\chi \subset \mathcal{H}_\chi$, we have $\mathcal{H}_\chi/\mathcal{G}_\chi \simeq \mathcal{G}_\chi^\perp/\mathcal{H}_\chi^\perp$. Consequently, unifying the results of this subsection, the following characterization of the difference between the velocity spaces follows:

$$\mathcal{T}_\chi/\mathcal{V}_\chi \simeq \mathcal{H}_\chi/\mathcal{G}_\chi \simeq \qquad (50)$$

$$\Big\{ u \in H^1_{(0)} \colon \int_\Omega \chi\nabla u\cdot B\,dx = -\int_\Omega \chi u\nabla\cdot B\,dx \text{ for all } B \in \mathcal{S}_\chi \Big\} \Big/ \ker \mathrm{Tr}_{\operatorname{supp}|\nabla\chi|}.$$

In summary, the gap in the velocity spaces $\mathcal{V}_\chi$ and $\mathcal{T}_\chi$ is exclusively due to a loss in regularity of the interface and amounts to the gap between having the trace in a distributional sense (see (48)) versus a quasi-everywhere sense.

2.4. **On the relation to Le's functional framework.** We now have sufficient machinery to discuss our solution concept in relation to the framework developed by Le in [36]. Within Le's work, the critical dissipation inequality for $\Gamma_t :=$

supp $|\nabla\chi(\cdot,t)|$, a C^3 space-time interface, to be a solution of the Mullins–Sekerka flow is given by

$$E[\chi(\cdot,T)] + \int_0^T \frac{1}{2}\|\partial_t\chi\|^2_{H^{-1/2}(\Gamma_t)} + \frac{1}{2}\|H_{\Gamma_t}\|^2_{H^{1/2}(\Gamma_t)}\,dt \le E[\chi_0],$$

where $\|H_\Gamma\|_{H^{1/2}(\Gamma)} = \|\nabla\tilde{f}\|_{L^2(\Omega)}$ for $\tilde{f}$ satisfying (6) with $f = H_\Gamma$ (the curvature) and $H^{-1/2}(\Gamma)$ again defined by duality and normed by means of the Riesz representation theorem (see also Lemma 2.1 of Le [36]),

$$H^{-1/2}(\Gamma) \simeq \Delta_N(\{u \in H^1_{(0)} \colon \Delta u = 0 \text{ in } \Omega\setminus\Gamma\}). \tag{51}$$

As this is simply the image under the weak Neumann Laplacian of functions u associated with the problem (6), we can rewrite this as

$$H^{-1/2}(\Gamma) = \Delta_N(H^{1/2}(\Gamma)). \tag{52}$$

Considering our solution concept now, let (χ,μ) be a solution in the sense of Definition 5 such that $\Gamma_t := \mathrm{supp}|\nabla\chi(\cdot,t)|$ is a Lipschitz surface for a.e. t. By (49) and (51), $\mathcal{T}_{\chi(\cdot,t)} = H^{-1/2}(\Gamma_t)$. Then as the classical trace space is well-defined, the isomorphism (50) collapses to the identity showing that

$$\mathcal{T}_{\chi(\cdot,t)} = \Delta_N(\mathcal{G}_{\chi(\cdot,t)}), \tag{53}$$

verifying the analogue of (52) and implying that $\mathcal{G}_{\chi(\cdot,t)} = H^{1/2}(\Gamma_t)$. Further, this discussion and (26) (letting $c_0 = 1$ for convenience) show that

$$\|\partial_t\chi\|_{\mathcal{T}_{\chi(\cdot,t)}} = \|\partial_t\chi\|_{H^{-1/2}(\Gamma_t)} \quad\text{and}\quad \|w(\cdot,t)\|_{\mathcal{G}_{\chi(\cdot,t)}} = \|H_{\Gamma_t}\|_{H^{1/2}(\Gamma_t)}.$$

Looking to (18) and (23), we see that our solution concept naturally subsumes Le's, preserves structural relations on the function spaces, and works without any regularity assumptions placed on Γ.

A natural question following from the discussion of this subsection and the prior is when does the relation (53) or the inclusion $\partial_t\chi \in \mathcal{V}_\chi \subset \mathcal{T}_\chi$ hold. By (50), both will follow if *zero* distributional trace is equivalent to having *zero* trace in the quasi-everywhere sense. Looking towards results on traces (see, e.g., [14], [44], and [45]), characterization of this condition will be a nontrivial result, and applying similar ideas to the Mullins–Sekerka flow may require a fine characterization of the singular set from Allard's regularity theory [5].

Despite the general question being beyond the scope of this paper, we can show that in dimension $d = 2$ under an energy convergence hypothesis, solutions constructed via the minimizing movement scheme satisfy $\mathcal{T}_\chi = \mathcal{V}_\chi$. We note that the energy convergence assumption forces the limiting varifold to lie entirely on the surface associated with χ. In general, one must extract the even multiplicities from μ to obtain $|\nabla\chi|$ (see (17c)), and it is not clear this extracted set will interact well with the notion of capacity used to define "quasi-everywhere."

Corollary 7. *Let $\Omega \subset \mathbb{R}^2$ be an open set with C^2 boundary and suppose that the time integrated energies of the approximating sequence converge, in the sense of (60). Then it follows that the solutions of Mullins–Sekerka flow constructed in Theorem 1 satisfy*

$$\mathcal{V}_\chi = \mathcal{T}_\chi.$$

We defer the proof of this result till after that of Theorem 1, as we make use of the tools therein.

2.5. **Motivation from the viewpoint of weak-strong uniqueness.** Another major motivation for our weak solution concept, especially for the inclusion of a sharp energy dissipation principle, is drawn from the recent progress on uniqueness properties of weak solutions for various curvature driven interface evolution problems. More precisely, it was established that for incompressible Navier–Stokes two-phase flow with surface tension [23] and for multiphase mean curvature flow [25] (cf. also [29] or [30]), weak solutions with sharp energy dissipation rate are unique within a class of sufficiently regular strong solutions (as long as the latter exist, i.e., until they undergo a topology change). Such weak-strong uniqueness principles are optimal in the sense that weak solutions in geometric evolution may in general be non-unique after the first topology change. Extensions to constant contact angle problems as considered in the present work are possible as well, see [31] for Navier–Stokes two-phase flow with surface tension or [28] for mean curvature flow.

The weak-strong uniqueness results of the previous works rely on a Gronwall stability estimate for a novel notion of distance measure between a weak and a sufficiently regular strong solution. The main point is that this distance measure is in particular able to penalize the difference in the location of the two associated interfaces in a sufficiently strong sense. Let us briefly outline how to construct such a distance measure in the context of the present work (i.e., interface evolution in a bounded container with constant contact angle (1e)). We claim that the following functional represents a natural candidate for the desired error functional:

$$\begin{aligned} E_{\mathrm{rel}}[\chi,\mu|\mathscr{A}](t) := |\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) &- \int_{\partial^* A(t)\cap\Omega} c_0 \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot\xi(\cdot,t)\,d|\nabla\chi(\cdot,t)| \\ &- \int_{\partial\Omega} (\cos\alpha)c_0\chi(\cdot,t)\,d\mathcal{H}^{d-1}, \end{aligned}$$

where $\xi(\cdot,t)\colon\overline{\Omega}\to\{|x|\leq 1\}$ denotes a suitable extension of the unit normal vector field $n_{\partial\mathscr{A}(t)}$ of $\partial\mathscr{A}(t)\cap\Omega$. Due to the compatibility conditions (17c) and (17d) as well as the length constraint $|\xi|\leq 1$, it is immediate that $E_{\mathrm{rel}}\geq 0$. The natural boundary condition for $\xi(\cdot,t)$ turns out to be $(\xi(\cdot,t)\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv\cos\alpha$. Indeed, this shows by means of an integration by parts that

$$E_{\mathrm{rel}}[\chi,\mu|\mathscr{A}](t) = |\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \int_{\Omega} c_0\chi(\cdot,t)\nabla\cdot\xi\,dx.$$

The merit of the previous representation of E_{rel} is that it allows one to compute the time evolution of E_{rel} relying in a first step only on the De Giorgi inequality (18) and using $\nabla\cdot\xi$ as a test function in the evolution equation (21). Furthermore, the compatibility condition (17f) yields that

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \frac{1}{2}|s-\xi|^2\,d\mu_t^{\Omega} \leq \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} 1 - s\cdot\xi\,d\mu_t^{\Omega} = E_{\mathrm{rel}}[\chi,\mu|\mathscr{A}](t),$$

which in turn implies a tilt-excess type control provided by E_{rel} at the level of the varifold interface. Further coercivity properties may be derived based on the compatibility conditions (17c) and (17d) in form of the associated Radon–Nikodỳm derivatives $\rho_t^{\Omega} := \frac{c_0|\nabla\chi(\cdot,t)|\llcorner\Omega}{|\mu_t^{\Omega}|_{\mathbb{S}^{d-1}}\llcorner\Omega} \in [0,1]$ and $\rho_t^{\partial\Omega} := \frac{(\cos\alpha)c_0\chi(\cdot,t)\mathcal{H}^{d-1}\llcorner\partial\Omega}{|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}+|\mu_t^{\Omega}|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega} \in [0,1]$, respectively. More precisely, one obtains the representation

$$E_{\mathrm{rel}}[\chi,\mu|\mathscr{A}](t)$$

$$
\begin{aligned}
&= \int_{\Omega} 1 - \rho_t^{\Omega}\, d|\mu_t^{\Omega}|_{\mathbb{S}^{d-1}} + \int_{\partial\Omega} 1 - \rho_t^{\partial\Omega}\, d\big(|\mu_t^{\Omega}|_{\mathbb{S}^{d-1}} + |\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}\big) \\
&\quad + \int_{\Omega} c_0 \Big(1 - \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|} \cdot \xi(\cdot,t)\Big)\, d|\nabla\chi(\cdot,t)|.
\end{aligned}
$$

The last of these right hand side terms ensures tilt-excess type control at the level of the BV interface

$$
c_0 \int_{\Omega} \frac{1}{2} \Big| \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|} - \xi(\cdot,t) \Big|^2 d|\nabla\chi(\cdot,t)| \leq E_{\mathrm{rel}}[\chi,\mu|\mathscr{A}](t).
$$

The other three simply penalize the well-known mass defects (i.e., mass moving out from the bulk to the domain boundary, or the creation of hidden boundaries within the bulk) originating from the lack of continuity of the perimeter functional under weak-$*$ convergence in BV.

In summary, the requirements of Definition 5 allow one to define a functional which on one side penalizes, in various ways, the "interface error" between a varifold and a classical solution, and which on the other side has a structure supporting at least in principle the idea of proving a Gronwall-type stability estimate for it. One therefore may hope that varifold solutions for Mullins–Sekerka flow in the sense of Definition 5 satisfy a weak-strong uniqueness principle together with a weak-strong stability estimate based on the above error functional. In the simplest setting of $\alpha = \frac{\pi}{2}$, a BV solution χ, and assuming no boundary contact for the interface of the classical solution $\mathscr{A}$, this is at the time of this writing work in progress [24].

For the present contribution, however, we content ourselves with the above existence result (i.e., Theorem 1) for varifold solutions to Mullins–Sekerka flow in the sense of Definition 5 together with establishing further properties of these.

3. Existence of varifold solutions to Mullins–Sekerka flow

3.1. **Key players in minimizing movements.** To construct weak solutions for the Mullins–Sekerka flow (1a)–(1e) in the precise sense of Definition 5, it comes at no surprise that we will employ the gradient flow perspective in the form of a minimizing movements scheme, which we pass to the limit. Given an initial condition $\chi_0 \in \mathcal{M}_{m_0}$ (see (8)), a fixed time step size $h \in (0,1)$, and E as in (10), we let $\chi_0^h := \chi_0$ and choose inductively for each $n \in \mathbb{N}$

$$
\chi_n^h \in \operatorname*{arg\,min}_{\widetilde{\chi} \in \mathcal{M}_{m_0}} \Big\{ E[\widetilde{\chi}] + \frac{1}{2h} \big\| \chi_{n-1}^h - \widetilde{\chi} \big\|_{H^{-1}_{(0)}}^2 \Big\}, \tag{54}
$$

an approximation via the backward-Euler scheme. Note that this minimization problem is indeed solvable by the direct method in the calculus of variations; see, for instance, the result of Modica [47, Proposition 1.2] for the lower-semicontinuity of the capillary energy.

By a telescoping argument, it is immediate that the associated piecewise constant interpolation

$$
\chi^h(t) := \chi_{n-1}^h \quad \text{for all } t \in [(n-1)h, nh),\ n \in \mathbb{N}, \tag{55}
$$

satisfies the energy dissipation estimate

$$
E[\chi^h(T)] + \int_0^T \frac{1}{2h^2} \big\| \chi^h(t+h) - \chi^h(t) \big\|_{H^{-1}_{(0)}}^2 dt \leq E[\chi_0] \quad \text{for all } T \in \mathbb{N}h. \tag{56}
$$

Although the previous inequality is already enough for usual compactness arguments, it is obviously not sufficient, however, to establish the expected sharp energy dissipation inequality (cf. (4)) in the limit as $h \to 0$. It goes back to ideas of De Giorgi how to capture the remaining half of the dissipation energy at the level of the minimizing movements scheme, versus, for example, recovering the dissipation from the regularity of a solution to the limit equation. The key ingredient for this is a finer interpolation than the piecewise constant one, which in the literature usually goes under the name of De Giorgi (or variational) interpolation and is defined as follows:

$$\begin{aligned}
&\bar{\chi}^h((n-1)h) := \chi^h((n-1)h) = \chi^h_{n-1}, \quad n \in \mathbb{N},\\
&\bar{\chi}^h(t) \in \operatorname*{arg\,min}_{\widetilde{\chi} \in \mathcal{M}_{m_0}} \Big\{ E[\widetilde{\chi}] + \frac{1}{2(t-(n-1)h)} \big\| \widetilde{\chi} - \bar{\chi}^h((n-1)h) \big\|^2_{H^{-1}_{(0)}} \Big\},\ t \in ((n-1)h, nh).
\end{aligned} \tag{57}$$

The merit of this second interpolation consists of the following improved (and now sharp) energy dissipation inequality

$$E[\chi^h(T)] + \int_0^T \frac{1}{2h^2} \big\| \chi^h(t+h) - \chi^h(t) \big\|^2_{H^{-1}_{(0)}}\,dt + \int_0^T \frac{1}{2} \big|\partial E[\bar{\chi}^h(t)]\big|^2_{\mathrm{d}}\,dt \le E[\chi_0], \tag{58}$$

with $T \in \mathbb{N}h$ [7]. The quantity $|\partial E[\chi]|_{\mathrm{d}}$ is usually referred to as the metric slope of the energy E at a given point $\chi \in \mathcal{M}_{m_0}$, and in our context may more precisely be defined by

$$\big|\partial E[\chi]\big|_{\mathrm{d}} := \limsup_{\widetilde{\chi} \in \mathcal{M}_{m_0} :\, \|\chi - \widetilde{\chi}\|_{H^{-1}_{(0)}} \to 0} \frac{\big(E[\chi] - E[\widetilde{\chi}]\big)_+}{\big\|\chi - \widetilde{\chi}\big\|_{H^{-1}_{(0)}}}. \tag{59}$$

We remind the reader that (58) is a general result for abstract minimizing movement schemes requiring only a metric space to work. However, as it turns out, we will be able to preserve a formal manifold structure even in the limit. This in turn is precisely the reason why the "De Giorgi metric slope" appearing in our energy dissipation inequality (18) is computed only in terms of inner variations, see (20).

With these main ingredients and properties of the minimizing movements scheme in place, our main task now consists of passing to the limit $h \to 0$ and identifying the resulting (subsequential but unconditional) limit object as a varifold solution to Mullins–Sekerka flow (1a)–(1e) in our sense. Furthermore, to obtain a BV solution, we will *additionally assume*, following the tradition of Luckhaus and Sturzenhecker [40], that for a subsequential limit point χ obtained from (102) below, it holds that

$$\int_0^{T_*} E[\chi^h(t)]\,dt \to \int_0^{T_*} E[\chi(t)]\,dt. \tag{60}$$

3.2. **Four technical auxiliary results.** For the Mullins–Sekerka equation, a mass preserving flow, it will be helpful to construct "smooth" mass-preserving flows corresponding to infinitesimally mass-preserving velocities, i.e., velocities in the test function class $\mathcal{S}_\chi$ (see (11)). Using these flows as competitors in (57) and considering the associated Euler–Lagrange equation, it becomes apparent that an approximate Gibbs–Thomson relation holds for infinitesimally mass-preserving velocities. To extend this relation to arbitrary variations (tangential at the boundary) we must control the Lagrange multiplier arising from the mass constraint. Though the first lemma and the essence of the subsequent lemma is contained in the work of Abels and Röger [2] or Chen [15], we include the proofs for both completeness and to show that the result is unperturbed if the energy exists at the varifold level.

Lemma 8. *Let $\chi \in \mathcal{M}_0$ and $B \in \mathcal{S}_\chi$. Then there exists $\eta > 0$ and a family of C^1 diffeomorphisms $\Psi_s \colon \overline{\Omega} \to \overline{\Omega}$ depending differentiably on $s \in (-\eta, \eta)$ such that for all $x \in \overline{\Omega}$ one has $\Psi_0(x) = x$, $\partial_s \Psi_s(x)|_{s=0} = B(x)$, and*

$$\int_\Omega \chi \circ \Psi_s^{-1}\,dx = m_0 \quad \text{for all } s \in (-\eta, \eta).$$

Proof. Fix $\xi \in C^\infty(\overline{\Omega})$ such that $(\xi \cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$ and $\int_\Omega \chi \nabla \cdot \xi\,dx \neq 0$. Naturally associated to B and ξ are flow-maps β_s and γ_r solving $\partial_s \beta_s(x) = B(\beta_s(x))$ and $\partial_r \gamma_r(x) = \xi(\gamma_r(x))$, each with initial condition given by the identity map, i.e., $\beta_0(x) = x$. Define the function f, which is locally differentiable near the origin, by

$$f(s,r) := \int_\Omega \chi \circ (\beta_s \circ \gamma_r)^{-1}\,dx - m_0 = \int_\Omega \chi\left(\det(\nabla(\beta_s \circ \gamma_r)) - 1\right)dx.$$

As $f(0,0) = 0$ and $\partial_r f(0,0) = \int_\Omega \chi \nabla \cdot \xi\,dx \neq 0$ by assumption, we may apply the implicit function theorem to find a differentiable function $r = r(s)$ with $r(0) = 0$ such that $f(s, r(s)) = 0$ for s near 0. We can further compute that (see (72))

$$\partial_s f(0,0) = \int_\Omega \chi\big(\nabla \cdot B + r'(0)\nabla \cdot \xi\big)\,dx.$$

Rearranging, we find

$$r'(0) = -\frac{\int_\Omega \chi \nabla \cdot B\,dx}{\int_\Omega \chi \nabla \cdot \xi\,dx} = 0,$$

and thus the flow given by $\beta_s \circ \gamma_{r(s)}$ satisfies $\partial_s(\beta_s \circ \gamma_{r(s)})|_{s=0} = B + r'(0)\xi = B$, thereby providing the desired family of diffeomorphisms. ☐

Lemma 9. *Let $\chi \in \mathcal{M}_0$, $w \in H^1_{(0)}$, and $\mu \in \mathrm{M}(\overline{\Omega} \times \mathbb{S}^{d-1})$ be an oriented varifold such that*

$$\delta\mu(B) = \int_\Omega \chi \nabla \cdot (wB)\,dx \quad \text{for all } B \in \mathcal{S}_\chi. \tag{61}$$

Then there is $\lambda \in \mathbb{R}$ such that

$$\delta\mu(B) = \int_\Omega \chi \nabla \cdot \big((w+\lambda)B\big)\,dx \quad \text{for all } B \in C^1(\overline{\Omega}; \mathbb{R}^d) \text{ with } (B \cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$$

and there exists $C = C(\Omega, d, c_0, m_0)$

$$\|w+\lambda\|_{H^1(\Omega)} \leq C(1 + |\nabla\chi|(\Omega))\left(|\mu|(\overline{\Omega}) + \|\nabla w\|_{L^2(\Omega)}\right). \tag{62}$$

Proof. For $\xi \in C^\infty(\overline{\Omega})$ such that $(\xi \cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$ and $\int_\Omega \chi \nabla \cdot \xi\,dx \neq 0$, we have that $\tilde{B} := B - \frac{\int_\Omega \chi \nabla \cdot B\,dx}{\int_\Omega \chi \nabla \cdot \xi\,dx}\xi$ belongs to $\mathcal{S}_\chi$, and plugging this into (61) and rearranging, one finds

$$\delta\mu(B) - \int_\Omega \chi \nabla \cdot (wB)\,dx = \lambda \int_\Omega \chi \nabla \cdot B\,dx,$$

where

$$\lambda = \frac{\delta\mu(\xi) - \int_\Omega \chi \nabla \cdot (w\xi)\,dx}{\int_\Omega \chi \nabla \cdot \xi\,dx}. \tag{63}$$

To conclude the lemma, it suffices to make a careful selection of ξ such that

$$|\lambda| \leq C(1+|\nabla\chi|(\Omega))\left(|\mu|(\overline{\Omega}) + \|\nabla w\|_{L^2(\Omega)}\right). \tag{64}$$

Let ρ_ϵ be a standard mollifier for $\epsilon > 0$, and let $\chi_\epsilon := \chi * \rho_\epsilon$ with $m_\epsilon := \fint_\Omega \chi_\epsilon \, dx$. We solve the Poisson problem

$$\begin{cases} \Delta\phi_\epsilon = \chi_\epsilon - m_\epsilon & \text{in } \Omega, \\ (n_{\partial\Omega}\cdot\nabla)\phi_\epsilon = 0 & \text{on } \partial\Omega, \\ \displaystyle\fint_\Omega \phi_\epsilon \, dx = 0. \end{cases} \tag{65}$$

As $\|\chi_\epsilon - m_\epsilon\|_{C^1} \leq C(\Omega)/\epsilon$, we can apply Schauder estimates to find

$$\|\phi_\epsilon\|_{C^2(\Omega)} \leq C(\Omega)/\epsilon. \tag{66}$$

Noting the L^1 estimate

$$\|\chi_\epsilon - \chi\|_{L^1(\Omega)} \leq C(\Omega)\epsilon\,(1+|\nabla\chi|(\Omega))$$

and $m_\epsilon \leq m_0$, we have

$$\begin{aligned} \int_\Omega \chi\nabla\cdot\nabla\phi_\epsilon \, dx &= \int_\Omega \chi(\chi_\epsilon - m_\epsilon)\, dx = (1-m_\epsilon)m_0|\Omega| + \int_\Omega \chi(\chi_\epsilon - \chi)\, dx \\ &\geq (1-m_0)m_0|\Omega| - C(\Omega)\epsilon\,(1+|\nabla\chi|(\Omega)) \geq C(m_0,\Omega) \end{aligned} \tag{67}$$

where we have now fixed $\epsilon = \frac{(1-m_0)m_0|\Omega|}{4C(\Omega)(1+|\nabla\chi|(\Omega))}$. Choosing $\xi = \nabla\phi_\epsilon$, by (66), (67), the Poincaré inequality for w, and the first variation formula

$$\delta\mu(\xi) = \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla\xi(x)\, d\mu(x,s),$$

we conclude (64) from (63). □

We finally state and prove a result which is helpful for the derivation of approximate Gibbs–Thomson laws from the optimality condition (57) of De Giorgi interpolants and is also needed in the proof of Lemma 2.

Lemma 10. *Let $\chi \in \mathcal{M}_0$ and $B \in \mathcal{S}_\chi$, and let $(\Psi_s)_{s\in(-\eta,\eta)}$ be an associated family of diffeomorphisms from Lemma 8. Then, for any $\phi \in H^1_{(0)}$ it holds*

$$\left| \int_\Omega \phi \frac{\chi\circ\Psi_s^{-1} - \chi}{s}\, dx + \langle B\cdot\nabla\chi, \phi\rangle_{H^{-1}_{(0)},H^1_{(0)}} \right| \leq \|\phi\|_{H^1_{(0)}} o_{s\to 0}(1). \tag{68}$$

In particular, taking the supremum over $\phi \in H^1_{(0)}$ with $\|\phi\|_{H^1_{(0)}} \leq 1$ in (68) implies $\frac{\chi\circ\Psi_s^{-1}-\chi}{s} \to -B\cdot\nabla\chi$ strongly in $H^{-1}_{(0)}$ as $s \to 0$.

Proof. To simplify the notation, we denote $\chi_s := \chi\circ\Psi_s^{-1}$. Heuristically, one expects (68) by virtue of the formal relation $\partial_s\chi_s|_{s=0} = -(B\cdot\nabla)\chi$ mentioned after (20). A rigorous argument is given as follows.

Using the product rule, we first expand (13) as

$$\begin{aligned} &\int_\Omega \phi\frac{\chi_s - \chi}{s}\, dx + \langle B\cdot\nabla\chi, \phi\rangle_{H^{-1}_{(0)},H^1_{(0)}} \\ &= \int_\Omega \phi\frac{\chi_s - \chi}{s}\, dx - \int_\Omega \chi\big((B\cdot\nabla)\phi + \phi\nabla\cdot B\big)\, dx. \end{aligned} \tag{69}$$

Recalling $\chi_s = \chi\circ\Psi_s^{-1}$, using the change of variables formula for the map $x \mapsto \Psi_s(x)$, and adding zero entails

$$\int_\Omega \phi\frac{\chi_s - \chi}{s}\,dx = \int_\Omega \chi\frac{\phi\circ\Psi_s - \phi}{s}\,dx + \int_\Omega \chi(\phi\circ\Psi_s)\frac{|\det\nabla\Psi_s| - 1}{s}\,dx. \tag{70}$$

Inserting (70) into (69), we have that

$$\int_\Omega \phi\frac{\chi_s - \chi}{s}\,dx + \langle B\cdot\nabla\chi, \phi\rangle_{H^{-1}_{(0)},H^1_{(0)}} = I + II, \tag{71}$$

where

$$\begin{aligned} I &:= \int_\Omega \chi\frac{\phi\circ\Psi_s - \phi}{s}\,dx - \int_\Omega \chi(B\cdot\nabla)\phi\,dx,\\ II &:= \int_\Omega \chi(\phi\circ\Psi_s)\frac{|\det\nabla\Psi_s| - 1}{s} - \int_\Omega \chi\phi\nabla\cdot B\,dx. \end{aligned}$$

To estimate II, we first Taylor expand $\nabla\Psi_s(x) = \mathrm{Id} + s\nabla B(x) + F_s(x)$ where, by virtue of the regularity of $s\mapsto\Psi_s$ and B, the remainder satisfies the upper bound $\sup_{x\in\Omega}|F_s(x)| \leq s o_{s\to0}(1)$. In particular, from the Leibniz formula we deduce

$$\det\nabla\Psi_s(x) - 1 = s(\nabla\cdot B)(x) + f_s(x), \tag{72}$$

where the remainder satisfies the same qualitative upper bound as $F_s(x)$. Note that by restricting to sufficiently small s, we may ensure that $\det\nabla\Psi_s = |\det\nabla\Psi_s|$. Hence, using (72), then adding zero to reintroduce the determinant for a change of variables, and applying the continuity of translation (by a diffeomorphism) in $L^2(\Omega)$, we have

$$\begin{aligned} II &= \|\phi\|_{L^2(\Omega)}o_{s\to0}(1) + \int_\Omega \chi\nabla\cdot B(\phi\circ\Psi_s - \phi)\,dx\\ &= \|\phi\|_{L^2(\Omega)}o_{s\to0}(1) + \int_\Omega \phi\Big((\chi\nabla\cdot B)\circ\Psi_s^{-1} - \chi\nabla\cdot B\Big)\,dx\\ &\quad + \int_\Omega \chi\nabla\cdot B\,(\phi\circ\Psi_s)\,(1-|\det\nabla\Psi_s(x)|)\,dx\\ &\leq \|\phi\|_{L^2(\Omega)}o_{s\to0}(1). \end{aligned} \tag{73}$$

To estimate I, we first make use of the fundamental theorem of calculus along the trajectories determined by Ψ_s, reintroduce the determinant by adding zero as in (73), and apply $\frac{d}{d\lambda}\Psi_\lambda(x) = B(x) + o_{\lambda\to0}(1)$ to see

$$\begin{aligned} \int_\Omega \chi\frac{\phi\circ\Psi_s - \phi}{s}\,dx &= \fint_0^s \int_\Omega \chi\nabla\phi\big(\Psi_\lambda(x)\big)\cdot\frac{d}{d\lambda}\Psi_\lambda(x)\,dx d\lambda\\ &= \fint_0^s \int_\Omega \chi\nabla\phi\big(\Psi_\lambda(x)\big)\cdot B(x)|\det\nabla\Psi_\lambda(x)|\,dx d\lambda\\ &\quad + \|\nabla\phi\|_{L^2(\Omega)}o_{s\to0}(1). \end{aligned}$$

Hence, by undoing the change of variables in the first term on the right hand side of the previous identity, an argument analogous to the one for II guarantees

$$I \leq \|\nabla\phi\|_{L^2(\Omega)}o_{s\to0}(1). \tag{74}$$

Looking to (71), we use the two respective estimates for I and II given in (74) and (73). By an application of Poincaré's inequality, we arrive at (68). $\square$

We now turn to proving a result which gives us further structural information on the minimizers $\bar{\chi}^h$ coming from the minimizing movements scheme (57). In particular, we will show that the reduced boundary associated with the minimizer is a $C^{1,\beta}$-surface intersecting the boundary of the container Ω with angle given by Young's law. For this we will apply a regularity result of Taylor [59] (see also De Phillipis and Maggi [20]).

Proposition 11. *Let $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3 be an open set with C^2 boundary and suppose $\bar{\chi}$ is a minimizer of the energy*

$$G[\chi] := \begin{cases} E[\chi] + C\|\chi - \chi_0\|^2_{H^{-1}_{(0)}} & \fint \chi \, dx = m_0, \\ +\infty & \text{else,} \end{cases}$$

where χ_0 is a fixed characteristic function with $\fint \chi_0 \, dx = m_0$. Letting A be such that $\bar{\chi} = \chi_A$, we have that (up to a representative) $\partial A \cap \Omega = \partial^* A \cap \Omega$ is locally given by a $C^{1,\beta}$-graph that meets the boundary of Ω at angle α, where $\beta := \beta(d) > 0$.

Proof. To prove the statement, it suffices to show that χ is an almost-minimizer of the energy E with gauge $\epsilon(r) := Cr^{2\beta}$ with $\beta > 0$: that is, there exists $r_0 > 0$ such that if $r < r_0$, then

$$E[\bar{\chi}] \leq E[\chi_{\mathrm{comp}}] + \epsilon(r) r^{d-1} \tag{75}$$

for all competitors χ_{comp} with $\operatorname{supp}(\bar{\chi} - \chi_{\mathrm{comp}}) \subset\subset B(x_0, r)$ for $x_0 \in \bar{\Omega}$ and $r < r_0$. With this we may apply a theorem (see Section 5 and Section 6) of Taylor [59]. Note that Taylor proves the claim in dimension $N = 3$, but the regularity result may be directly obtained for the simpler $d = 2$ case by extending an almost-minimizer $\chi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ to $d = 3$ with $\chi_{\mathrm{ext}}(x_1, x_2, x_3) = \chi(x_1, x_2)$. As the extension is flat in the third direction, it is still a perimeter almost-minimizer in $\Omega \times \mathbb{R} \subset \mathbb{R}^3$.

We remark a related regularity result is contained in Theorem 1.2 (see also Theorem 1.10) of De Phillipis and Maggi [20], proven in arbitrary dimension with gauge Cr (hence there $\beta = 1/2$), a technically convenient choice.

Turning to the proof of (75), we must show that the nonlocal term $\|\chi - \chi_0\|^2_{H^{-1}_{(0)}}$ gives rise to an appropriate gauge function, and we must remove the mass constraint on competitors.

First, suppose that $\tilde{\chi} \in BV(\Omega; \{0, 1\})$ with $\fint \tilde{\chi} \, dx = m_0$. Necessarily $G[\tilde{\chi}] < \infty$, and we may compare $G[\bar{\chi}] \leq G[\tilde{\chi}]$ to find that

$$\begin{aligned} E[\bar{\chi}] \leq& E[\tilde{\chi}] + C(\|\tilde{\chi} - \chi_0\|^2_{H^{-1}_{(0)}} - \|\bar{\chi} - \chi_0\|^2_{H^{-1}_{(0)}}) \\ \leq& E[\tilde{\chi}] + C(\|\tilde{\chi} - \chi_0\|_{H^{-1}_{(0)}} + \|\bar{\chi} - \chi_0\|_{H^{-1}_{(0)}})\|\bar{\chi} - \tilde{\chi}\|_{H^{-1}_{(0)}} \\ \leq& E[\tilde{\chi}] + C_\Omega \|\bar{\chi} - \tilde{\chi}\|_{H^{-1}_{(0)}}. \end{aligned} \tag{76}$$

To estimate the last term, we use that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\bar{\chi} - \tilde{\chi}$ is identified as an element of the dual space via the embedding $L^2(\Omega) \hookrightarrow H^{-1}_{(0)}$. Consequently, by Hölder's inequality we have

$$\|\bar{\chi} - \tilde{\chi}\|_{H^{-1}_{(0)}} \leq C_\Omega \|\bar{\chi} - \tilde{\chi}\|_{L^{6/5}(\Omega)}. \tag{77}$$

To use this for a generic competitor χ_{comp} as in (75), we note that by Lemma 17.21 (see also Example 21.3 therein) of [41] there is $\tilde{\chi}$ with $\fint \tilde{\chi} \, dx = m_0$ such that $\|\bar{\chi} - \tilde{\chi}\|_{L^1(\Omega)} \leq Cr^d$ and $E[\tilde{\chi}] \leq E[\chi_{\mathrm{comp}}] + Cr^d$ for $r < r_0$, a fixed radius that depends on $\bar\chi$. Putting this together with (76) and (77), we recover (75) with $\epsilon(r) := Cr^{2\beta} := Cr^{5d/6-d+1}$ and $\beta > 0$. □

3.3. **Proof of Theorem 1.** We proceed in several steps.

Step 1: Approximate solution and approximate energy inequality. For time discretization parameter $h \in (0,1)$ and initial condition $\chi_0 \in BV(\overline{\Omega};\{0,1\})$, we define the sequence $\{\chi_n^h\}_{n\in\mathbb{N}_0}$ as in (54), and recall the piecewise constant function χ^h in (55) and the De Giorgi interpolant $\bar\chi^h$ in (57). We further define the linear interpolant $\hat\chi^h$ by

$$\hat\chi^h(t) = \frac{nh-t}{h}\chi_{n-1}^h + \frac{t-(n-1)h}{h}\chi_h^n \quad \text{for all } t \in [(n-1)h, nh),\ n \in \mathbb{N}. \tag{78}$$

To capture fine scale behavior of the energy in the limit, we will introduce measures $\mu^h = (\mu_t^h)_{t\in(0,T_*)}$, so that for each $t \in (0,T_*)$ the total mass of the mass measure $|\mu_t^h|_{\mathbb{S}^{d-1}} \in \mathrm{M}(\overline{\Omega})$ associated with the oriented varifold $\mu_t^h \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ is naturally associated to the energy of the De Giorgi interpolant at time t. More precisely, we define varifolds associated to the varifold lift of $\bar\chi^h$ in the interior and on the boundary by

$$\mu_t^{h,\Omega} := |\nabla\bar\chi^h(\cdot,t)|\llcorner\Omega \otimes (\delta_{\frac{\nabla\bar\chi^h(\cdot,t)}{|\nabla\bar\chi^h(\cdot,t)|}(x)})_{x\in\Omega}, \tag{79}$$

respectively

$$\mu_t^{h,\partial\Omega} := \bar\chi^h(\cdot,t)\llcorner\partial\Omega \otimes (\delta_{n_{\partial\Omega}(x)})_{x\in\partial\Omega}, \tag{80}$$

where $n_{\partial\Omega}$ denotes the inner normal on $\partial\Omega$ and where we again perform an abuse of notation and do not distinguish between $\bar\chi^h(\cdot,t)$ and its trace along $\partial\Omega$. We finally define the total approximate varifold by

$$\mu_t^h := c_0\mu_t^{h,\Omega} + (\cos\alpha)c_0\mu_t^{h,\partial\Omega}, \quad t \in (0,T_*). \tag{81}$$

The remainder of the first step is concerned with the proof of the following approximate version of the energy dissipation inequality (18): for τ, κ, s and T such that $0 < s < \kappa < \tau < T < T_*$, and $0 < h < \min\{T-\tau, \kappa-s\}$, we claim that

$$E[\mu_T^h] + \int_\kappa^\tau \frac{1}{2}\|\partial_t\hat\chi^h\|^2_{H^{-1}_{(0)}} + \frac{1}{2}\left\|\frac{\bar\chi^h(t)-\bar\chi^h(\lfloor t/h\rfloor h)}{t-\lfloor t/h\rfloor h}\right\|^2_{H^{-1}_{(0)}} dt \le E[\mu_s^h] \le E[\chi_0]. \tag{82}$$

As a first step towards (82), we claim for all $n \in \mathbb{N}$ (cf. [7] and [13])

$$\begin{aligned} &E[\bar\chi^h(nh)] + \frac{h}{2}\left\|\frac{\chi_n-\chi_{n-1}}{h}\right\|^2_{H^{-1}_{(0)}} + \frac{1}{2}\int_{(n-1)h}^{nh}\left\|\frac{\bar\chi^h(t)-\bar\chi^h((n-1)h)}{t-h(n-1)}\right\|^2_{H^{-1}_{(0)}} dt \\ &\le E[\bar\chi^h((n-1)h)] \end{aligned} \tag{83}$$

and

$$E[\bar\chi^h(t)] \le E[\bar\chi^h((n-1)h)] \quad \text{for all } n \in \mathbb{N} \text{ and all } t \in ((n-1)h, nh). \tag{84}$$

In particular, using the definition (78) and then telescoping over n in (83) provides for all $n, m \in \mathbb{N}$ with $m < n$ the discretized dissipation inequality

$$E[\bar\chi^h(nh)] + \frac{1}{2}\int_{mh}^{nh}\frac{1}{2}\|\partial_t\hat\chi^h\|^2_{H^{-1}_{(0)}} + \frac{1}{2}\left\|\frac{\bar\chi^h(t)-\bar\chi^h(\lfloor t/h\rfloor h)}{t-\lfloor t/h\rfloor h}\right\|^2_{H^{-1}_{(0)}} dt \le E[\bar\chi^h(mh)]. \tag{85}$$

The bound (84) is a direct consequence of the minimality of the interpolant (57) at t. To prove (83), and thus also (85), we restrict our attention to the interval $(0,h)$ and temporarily drop the superscript h. We define the function

$$f(t) := E[\bar{\chi}(t)] + \frac{1}{2t}\,\|\bar{\chi}(t)-\chi_0\|^2_{H^{-1}_{(0)}}, \quad t\in(0,h), \tag{86}$$

and prove f is locally Lipschitz in $(0,h)$ with

$$\frac{d}{dt}f(t) = -\frac{1}{2t^2}\|\bar{\chi}(t)-\chi_0\|^2_{H^{-1}_{(0)}} \quad \text{for a.e. } t\in(0,h). \tag{87}$$

To deduce (87), we first show

$$(0,h]\ni t\mapsto \|\bar{\chi}(t)-\chi_0\|_{H^{-1}_{(0)}} \qquad \text{is non-decreasing,} \tag{88}$$

$$(0,h]\ni t\mapsto f(t) \qquad \text{is non-increasing.} \tag{89}$$

Indeed, for $0<s<t\leq h$ we obtain from minimality of the interpolant (57) at s, then adding zero, and then from minimality of the interpolant (57) at t that

$$\begin{aligned} f(s) &\leq E[\bar{\chi}(t)] + \frac{1}{2s}\|\bar{\chi}(t)-\chi_0\|^2_{H^{-1}_{(0)}} \\ &\leq E[\bar{\chi}(s)] + \frac{1}{2t}\|\bar{\chi}(s)-\chi_0\|^2_{H^{-1}_{(0)}} + \Big(\frac{1}{2s}-\frac{1}{2t}\Big)\|\bar{\chi}(t)-\chi_0\|^2_{H^{-1}_{(0)}}. \end{aligned}$$

Recalling definition (86), this is turn immediately implies (88). For a proof of (89), we observe for $s,t\in(0,h]$ by minimality of the interpolant (57) at t that

$$f(t)-f(s) \leq \frac{1}{2t}\|\bar{\chi}(s)-\chi_0\|^2_{H^{-1}_{(0)}} - \frac{1}{2s}\|\bar{\chi}(s)-\chi_0\|^2_{H^{-1}_{(0)}}.$$

Rearranging one finds for $0<s<t\leq h$

$$\frac{f(t)-f(s)}{t-s} \leq -\frac{1}{2ts}\|\bar{\chi}(s)-\chi_0\|^2_{H^{-1}_{(0)}} \leq 0, \tag{90}$$

proving (89).

Likewise using minimality of the interpolant (57) at s, one also concludes for $0<s<t<h$ the lower bound

$$\frac{f(t)-f(s)}{t-s} \geq -\frac{1}{2ts}\|\bar{\chi}(t)-\chi_0\|^2_{H^{-1}_{(0)}}. \tag{91}$$

As the discontinuity set of a monotone function is at most countable, we infer (87) from (90), (91) and (88). Integrating (87) on (s,t), using optimality of the interpolant (57) at s in the form of $f(s)\leq E[\bar{\chi}(0)] = E[\chi_0]$, and using monotonicity of f from (89), we have

$$E[\bar{\chi}(h)] + \frac{1}{2h}\,\|\bar{\chi}(h)-\chi_0\|^2_{H^{-1}_{(0)}} + \int_s^t \frac{1}{2\tau^2}\,\|\bar{\chi}(\tau)-\chi_0\|^2_{H^{-1}_{(0)}}d\tau \leq E[\chi_0].$$

Sending $s\downarrow 0$ and $t\uparrow h$, we recover (83) and thus also (85).

It remains to post-process (85) to (82). Note first that by definitions (79)–(81) it holds $|\mu_t^h|_{\mathbb{S}^{d-1}}(\overline{\Omega}) = E[\bar{\chi}^h(t)]$ for all $t\in(0,T_*)$. We claim that for all $h\in(0,1)$

$$(0,T_*)\ni t\mapsto |\mu_t^h|_{\mathbb{S}^{d-1}}(\overline{\Omega}) = E[\mu_t^h] \text{ is non-increasing.} \tag{92}$$

Indeed, for $(n-1)h<s<t\leq nh$ we simply get from the minimality of the De Giorgi interpolant (57) at time t

$$E[\bar{\chi}^h(t)] + \frac{1}{2(t-(n-1)h)}\|\bar{\chi}^h(t)-\chi^h((n-1)h)\|^2_{H^{-1}_{(0)}}$$

$$\leq E[\bar{\chi}^h(s)] + \frac{1}{2(t-(n-1)h)}\|\bar{\chi}^h(s) - \chi^h((n-1)h)\|^2_{H^{-1}_{(0)}}$$

so that (92) follows from (88) and (84). Restricting to $0 < h < \min\{T-\tau, s-\kappa\}$, there is $n_0 \in \mathbb{N}$ such that $\tau < n_0 h < T$ as well as $m_0 \in \mathbb{N}$ such that $s < m_0 h < \kappa$. Hence, by (92) and positivity of the integrand, we may bound the left-hand side of (82) from above by the left-hand side of (85) with $n = n_0$, as well as the right-hand side of (82) from below by the right-hand side of (85) with $m = m_0$, completing the proof of (82).

Step 2: Approximate Gibbs–Thomson law. Naturally associated to the De Giorgi interpolant (57) is the potential $w^h \in L^2(0,T_*;H^1_{(0)})$ satisfying

$$\begin{aligned} \Delta w^h(\cdot,t) &= \frac{\bar{\chi}^h(t) - \bar{\chi}^h(\lfloor t/h\rfloor h)}{t - \lfloor t/h\rfloor h} && \text{in } \Omega, \\ (n_{\partial\Omega}\cdot\nabla)w^h(\cdot,t) &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{93}$$

for $t \in (0,T_*)$. Note that this equivalently expresses (82) in the form of

$$E[\mu^h_T] + \int_\kappa^\tau \frac{1}{2}\|\partial_t\hat{\chi}^h\|^2_{H^{-1}_{(0)}} + \frac{1}{2}\|\nabla w^h\|^2_{L^2(\Omega)}\,dt \leq E[\mu^h_s] \leq E[\chi_0], \tag{94}$$

valid for all $0 < s < \kappa < \tau < T < T_*$ and $0 < h < \min\{T-\tau, \kappa-s\}$.

By the minimizing property (57) of the De Giorgi interpolant, Allard's first variation formula [5], and Lemma 10, it follows that the De Giorgi interpolant furthermore satisfies the approximate Gibbs–Thomson relation

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B(x)\,d\mu^h_t(x,s) = \int_\Omega \bar{\chi}^h(t)\nabla\cdot(w^h(\cdot,t)B)\,dx \tag{95}$$

for all $t \in (0,T_*)$ and all $B \in \mathcal{S}_{\bar{\chi}^h(t)}$.

Applying the result of Lemma 9 to control the Lagrange multiplier arising from the mass constraint, and using the uniform bound on the energy from the dissipation relation (94), we find there exist functions $\lambda^h \in L^2(0,T)$ and a constant $C = C(\Omega,d,c_0,m_0,\chi_0) > 0$ such that for all t in $(0,T_*)$

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B(x)\,d\mu^h_t(x,s) = \int_\Omega \bar{\chi}^h(t)\nabla\cdot\big((w^h(\cdot,t)+\lambda^h(t))B\big)\,dx \tag{96}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$, and

$$\|w^h(\cdot,t) + \lambda^h(t)\|_{H^1(\Omega)} \leq C(1 + \|\nabla w^h(\cdot,t)\|_{L^2(\Omega)}). \tag{97}$$

Step 3: Compactness, part I: Auxiliary limit potential. Compactness for w^h and λ^h follows immediately from bounds (94) and (97) as well as the Poincaré inequality, showing there is $\widetilde{w} \in L^2(0,T;H^1_{(0)})$ and $\widetilde{\lambda} \in L^2(0,T)$ such that, up to a subsequence $h \downarrow 0$,

$$w^h \rightharpoonup \widetilde{w} \quad \text{in } L^2(0,T;H^1_{(0)}) \tag{98}$$

and

$$\lambda^h \rightharpoonup \widetilde{\lambda} \quad \text{in } L^2(0,T). \tag{99}$$

In particular, $w^h + \lambda^h \rightharpoonup \widetilde{w} + \widetilde{\lambda}$ in $L^2(0,T;H^1(\Omega))$ for such $h \downarrow 0$.

Step 4: Compactness, part II: Limit phase indicator. To obtain compactness of $\hat{\chi}^h$ and $\bar{\chi}^h$, we will use the classical Aubin–Lions–Simon compactness theorem (see [8], [39], and [57]). By (82), the fundamental theorem of calculus, and Jensen's inequality, it follows that

$$\int_0^{T_*-\delta} \|\chi^h(t+\delta) - \chi^h(t)\|^2_{H^{-1}_{(0)}}\,dt \to 0 \quad \text{uniformly in } h \text{ as } \delta \to 0 \tag{100}$$

(see, e.g., [58, Lemma 4.2.7]). Looking to the dissipation (82) and recalling the definition of w^h in (93), one sees that

$$\int_0^{T_*} \|\bar\chi^h(t) - \chi^h(t)\|^2_{H^{-1}_{(0)}}\,dt \le C_1 h^2. \tag{101}$$

We *claim* (100) is also satisfied for $\bar\chi^h$ for a given sequence $h \to 0$. Fix $\epsilon > 0$, and choose $h_1 > 0$ such that $C_1 h_1^2 < \epsilon$. Choose $\delta_1 > 0$ such that the left hand side of (100) is bounded by ϵ uniformly in h for $0 < \delta < \delta_1$. By continuity of translation, which follows from density of smooth functions, we can suppose

$$\int_0^{T_*-\delta} \|\bar\chi^h(t+\delta) - \bar\chi^h(t)\|^2_{H^{-1}_{(0)}}\,dt < \epsilon \quad \text{for } h > h_1 \text{ and } 0 < \delta < \delta_1.$$

Then by the triangle inequality one can directly estimate that for all h (in the sequence) and $0 < \delta < \delta_1$, we have

$$\int_0^{T_*-\delta} \|\bar\chi^h(t+\delta) - \bar\chi^h(t)\|^2_{H^{-1}_{(0)}}\,dt < 3\epsilon,$$

proving the claim. With (100), we may apply the Aubin–Lions–Simon compactness theorem to $\bar\chi^h$ and $\hat\chi^h$ in the embedded spaces $BV(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega) \hookrightarrow H^{-1}_{(0)}$ for some $6/5 < p < 1^*$ to obtain $\chi \in L^2(0,T_*;BV(\Omega;\{0,1\})) \cap H^1(0,T_*;H^{-1}_{(0)})$ such that

$$\chi^h, \bar\chi^h, \hat\chi^h \to \chi \quad \text{in } L^2(0,T_*;L^2(\Omega)) \tag{102}$$

(where we have used the Lebesgue dominated convergence theorem to move up to L^2 convergence). To see that the target of each approximation is in fact correctly written as a single function χ, both χ^h and $\bar\chi^h$ must converge to the same limit by (101). Further, by the fundamental theorem of calculus, we have

$$\begin{aligned}\|\chi^h(t) - \hat\chi^h(t)\|_{H^{-1}_{(0)}} &= \|\chi^h(ih) - \hat\chi^h(t)\|_{H^{-1}_{(0)}} \\ &\le \int_{ih}^t \|\partial_t\hat\chi^h\|\,dt \le h^{1/2}\|\partial_t\hat\chi^h\|_{L^2(0,T;H^{-1}_{(0)})},\end{aligned} \tag{103}$$

for some $i \in \mathbb{N}_0$, which shows that χ^h and $\hat\chi^h$ also converge to the same limit, thereby justifying (102). Finally, note the dimension dependent embedding was introduced for technical convenience to ensure that $L^p(\Omega) \hookrightarrow H^{-1}_{(0)}$ is well defined, but can be circumnavigated (see, e.g., [35]).

We finally note that the distributional formulation of the initial condition survives passing to the limit for $\hat\chi^h$ as

$$\int_0^{T_*}\left(\langle\partial_t\chi,\zeta\rangle_{H^{-1}_{(0)},H^1_{(0)}} + \int_\Omega \chi\partial_t\zeta\,dx\right)dt = -\int_\Omega \chi_0\zeta(x,0)\,dx \tag{104}$$

for all $\zeta \in C^1_c(\overline\Omega\times[0,T_*)) \cap H^1(0,T_*;H^1_{(0)})$. As the trace of a function in $H^1(0,T;H^{-1}_{(0)})$ exists in $H^{-1}_{(0)}$ (see [38]), (104) implies

$$\mathrm{Tr}|_{t=0}\chi = \chi_0 \quad \text{in } H^{-1}_{(0)}. \tag{105}$$

Step 5: Compactness, part III: Auxiliary limit varifold. Based on the uniform bound on the energy from the dissipation relation (94), we have by weak-$*$ compactness of finite Radon measures, up to selecting a subsequence $h \downarrow 0$,

$$\mathcal{L}^1 \llcorner (0,T_*) \otimes (\mu_t^{h,\Omega})_{t\in(0,T_*)} \overset{*}{\rightharpoonup} \widetilde{\mu} \qquad \text{in } \mathrm{M}((0,T_*)\times\overline{\Omega}\times\mathbb{S}^{d-1}). \tag{106}$$

Thanks to the monotonicity (92), one may apply [29, Lemma 2] to obtain that the limit measure $\widetilde{\mu}$ can be sliced in time as

$$\widetilde{\mu} = \mathcal{L}^1 \llcorner (0,T_*) \otimes (\widetilde{\mu}_t)_{t\in(0,T_*)},\ \widetilde{\mu}_t \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1}) \text{ for all } t\in(0,T_*). \tag{107}$$

Step 6: Compactness, part IV: Limit varifolds. The monotonicity (92) and the uniform bound on the energy from the dissipation relation (94) ensure in view of Helly's selection theorem that there exists a non-increasing $e\colon (0,T_*) \to [0,\infty)$ such that

$$E[\mu_t^h] \to e(t) \quad \text{for all } t\in(0,T_*) \text{ for a subsequence } h\downarrow 0. \tag{108}$$

Select a subsequence $h \downarrow 0$ (not relabeled) such that the convergences (98), (99), (102), (106) and (108) hold true as $h \downarrow 0$.

For each $t \in (0,T^*)$, we apply Proposition 11 to see that $\mu_t^{h,\Omega}$ is given by an oriented surface measure of a $C^{1,\beta}$-manifold with boundary. Further, this tells us that $\mu_t^{h,\partial\Omega}$ is the oriented surface measure of manifold contained in $\partial\Omega$ with $C^{1,\beta}$ boundary. Using (96) to apply Corollary 6, we have that μ_t^h is of globally bounded first variation. But given the explicit contact angle coming from Proposition 11, we have that $\sigma_{\mu_t^h}$ (within Corollary 6) is given by finite number of Dirac measures for $d=2$ or the $\mathcal{H}^1$ measure on a finite collection of curves in $d=3$, up to a constant multiple depending on α. We denote these points or curves by $\{\gamma_k^h\}$. It follows by bound (39), Proposition 5, (96), and (97), we have that

$$\mathcal{H}^{d-2}(\cup_k\gamma_k) \le C(1+|\mu_t^h|(\overline{\Omega})^{3/2} + \|\nabla w^h\|_{L^2(\Omega)}^{3d/2}). \tag{109}$$

It follows that there is $g^h \in BV(\partial\Omega;\{0,1\})$ such that $\mu_t^h = g^h n_{\partial\Omega}\mathcal{H}^{d-1}\llcorner\partial\Omega$ and $|\nabla_{\partial\Omega} g^h|(\partial\Omega) = \mathcal{H}^{d-2}(\cup_k\gamma_k^h)$. Further, decomposing the measure μ_t^h, using the contact angle, and (109) (or (39)), both measures $\mu_t^{h,\Omega}$ and $\mu_t^{h,\partial\Omega}$ have bounded first variation with

$$|\delta\mu_t^{h,\Omega}|(\mathbb{R}^d) + |\delta\mu_t^{h,\partial\Omega}|(\mathbb{R}^d) \le C(1+|\mu_t^h|(\overline{\Omega})^{3/2} + \|\nabla w^h\|_{L^2(\Omega)}^{3d/2}). \tag{110}$$

Noting by Fatou's lemma and the energy dissipation inequality (94) that

$$t \mapsto \liminf_{h\to 0} \|\nabla w^h(\cdot,t)\|_{L^2(\Omega)}^2$$

is an integrable function, for almost every t, we may choose a subsequence $h_i(t) \downarrow 0$ as $i\to\infty$ with

$$\limsup_{i\to\infty} \|\nabla w^{h_i}(\cdot,t)\|_{L^2(\Omega)}^2 < \infty. \tag{111}$$

By (110), it follows that we may apply Allard's compactness for integer varifolds [5] to conclude that (up to a further sequence not relabeled)

$$\mu_t^{h_i(t),\Omega} \overset{*}{\rightharpoonup} \mu_t^\Omega \text{in } \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1}), \tag{112}$$

where μ_t^Ω is a $(d-1)$-integer-rectifiable varifold. Similarly, we may apply L^1 compactness of BV functions find $g \in BV(\partial\Omega;\{0,1\})$ such that $g_h \to g$ in $L^1(\partial\Omega)$, which gives that

$$\mu_t^{h_i(t),\partial\Omega} \overset{*}{\rightharpoonup} \mu_t^{\partial\Omega} \text{in } \mathrm{M}(\partial\Omega\times\mathbb{S}^{d-1}), \tag{113}$$

where $\mu_t^{\partial\Omega} = g\, n_{\partial\Omega}\mathcal{H}^{d-1}\llcorner\partial\Omega$.

We define the family $\mu = (\mu_t)_{t\in(0,T_*)}$ by

$$\mu_t := c_0\mu_t^{\Omega} + (\cos\alpha)c_0\mu_t^{\partial\Omega}, \quad t \in (0,T_*). \tag{114}$$

Recalling (108), we further find

$$E[\mu_t^{h_i(t)}] \to e(t) = E[\mu_t], \tag{115}$$

$$E[\mu_t] \text{ is non-increasing in } (0,T_*), \tag{116}$$

for almost every $t \in (0,T_*)$ as $i \to \infty$.

***Step 7: Auxiliary Gibbs–Thomson law in the limit and generalized mean curvature in the sense of Röger* [52].** We first note that for $B(x,t)$, which is continuous and compactly supported in $(0,T_*)$ such that for every t one has $B(\cdot,t) \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B(\cdot,t)\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$, by (106), it follows that

$$\int_0^{T_*} \delta\mu_t^h(B)\,dt \to \int_0^{T_*} \delta\widetilde{\mu}_t(B)\,dt. \tag{117}$$

Consequently, we can multiply the Gibbs–Thomson relation (96) by a smooth and compactly supported test function on $(0,T_*)$, integrate in time, pass to the limit as $h \downarrow 0$ using the compactness from (98)–(99) and (102), and then localize in time to conclude that for a.e. $t \in (0,T_*)$, it holds

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B(x)\,d\widetilde{\mu}_t(x,s) = \int_\Omega \chi(\cdot,t)\nabla\cdot\big((\widetilde{w}(\cdot,t) + \widetilde{\lambda}(t))B\big)\,dx, \tag{118}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$ (the null set again does not depend on the choice of B due to the separability of the space $C^1(\overline{\Omega};\mathbb{R}^d)$ normed by $f \mapsto \|f\|_\infty + \|\nabla f\|_\infty$). Note that the left-hand side of (118) is precisely $\delta\widetilde{\mu}_t(B)$ by Allard's first variation formula [5]. Furthermore, by Proposition 5, the Gibbs–Thomson relation (118) can be expressed as

$$\delta\widetilde{\mu}_t(B) = -\int_\Omega c_0 \frac{\widetilde{w}(\cdot,t) + \widetilde{\lambda}(t)}{c_0} \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot B\,d|\nabla\chi(\cdot,t)|. \tag{119}$$

We apply Lemma 4.2 from Röger [52] to find that for $B \in L^2(0,T;C^1_{cpt}(\Omega))$,

$$\int_0^{T_*} \delta\mu_t^h(B)\,dt \to -\int_0^{T_*}\int_\Omega c_0 H_{\chi(\cdot,t)} \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot B\,d|\nabla\chi(\cdot,t)|\,dt, \tag{120}$$

where H_χ is the generalized mean curvature in the sense of Röger [52], intrinsic to the surface $\operatorname{supp}|\nabla\chi(\cdot,t)|\llcorner\Omega$, for almost every t in $(0,T_*)$

Recalling (117) and (119), we localize (120) in time to conclude that the trace of $\frac{\widetilde{w}+\widetilde{\lambda}}{c_0}$ is given by the generalized mean curvature H_χ for $|\nabla\chi|$-almost every $x \in \Omega$ and almost every t in $(0,T_*)$. In particular, for a.e. $t \in (0,T_*)$ it holds

$$\delta\widetilde{\mu}_t(B) = -\int_\Omega c_0 H_{\chi(\cdot,t)} \frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot B\,d|\nabla\chi(\cdot,t)| \tag{121}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$, and due to the integrability guaranteed by Proposition 5,

$$H_{\chi(\cdot,t)} \in L^s\big(\Omega; d|\nabla\chi(\cdot,t)|\llcorner\Omega\big) \tag{122}$$

where $s \in [2,4]$ if $d = 3$ and $s \in [2,\infty)$ if $d = 2$.

Step 8: Preliminary optimal energy dissipation relation. By the compactness from Steps 3, 4 and 6 of this proof, lower semi-continuity of norms, Fatou's inequality, and first taking $h \downarrow 0$ and afterward $\tau \uparrow T$ and $\kappa \downarrow s$ in (94), we obtain for almost all $0 < s < T < T_*$

$$E[\mu_T] + \int_s^T \frac{1}{2}\|\partial_t \chi\|^2_{H^{-1}_{(0)}} + \frac{1}{2}\|\nabla \widetilde{w}\|^2_{L^2(\Omega)}\,dt \le E[\mu_s] \le E[\chi_0]. \tag{123}$$

In order to prove that (χ, μ) is a varifold solution for Mullins–Sekerka flow (1a)–(1e) in the sense of Definition 5, it remains on one side to upgrade (123) to (18) and on the other side to verify admissibility of the couple (χ, μ) in the sense of Definition 3. The upcoming three steps of the proof first take care of the former, whereas the latter will be proven afterward.

Step 9: Metric slope. Fixing $t \in (0, T^*)$ such that (111) holds, by (97), we have $\sup_{h_i(t)\downarrow 0} |\lambda^{h_i(t)}(t)| < \infty$. Up to a further subsequence, there exists $w(\cdot,t) \in H^1(\Omega)$ with $w^{h_i(t)}(\cdot,t) + \lambda^{h_i(t)}(t) \rightharpoonup w(\cdot,t)$ in $H^1(\Omega)$ as $i \to \infty$. Further, we may assume t is such that $\overline{\chi}^h(\cdot,t) \to \chi(\cdot,t)$ in $L^1(\Omega)$. As (96) and the aforementioned convergences hold, we may apply the results of Röger [52, Lemma 4.1] and Schätzle [55, Theorem 1.1 and 1.2] to choose the subsequence $h_i(t) \downarrow 0$ from Step 6 such that in addition to (112)–(115) it holds that:

i) The $(d-1)$-integer-rectifiable measure $|\mu_t^\Omega|_{\mathbb{S}^{d-1}} \llcorner \Omega \in \mathrm{M}(\Omega)$ is such that the compatibility condition (17c) holds true.
ii) The generalized mean curvature vector $\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega} \colon \operatorname{supp}(|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega) \to \mathbb{R}^d$ exists and and coincides with Röger's curvature in the sense of (17j)–(17k) as well as (17h) for all compactly supported variations $B \in C^1_{cpt}(\Omega;\mathbb{R}^d)$. In particular, (17g) holds true due to (122).
iii) The trace of the function $w(\cdot,t)/c_0$ coincides with the curvature μ_t-almost everywhere in Ω, i.e.,

$$\vec{H}_{|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega} = \begin{cases} \frac{w(\cdot,t)}{c_0}\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|} & \text{if } x \in \operatorname{supp}|\nabla\chi(\cdot,t)|, \\ 0 & \text{otherwise.} \end{cases}$$

Passing to the limit in (96) using (112)–(113), we have

$$\begin{aligned} \delta\mu_t(B) &= \int_\Omega \chi(\cdot,t)\nabla\cdot\big(w(\cdot,t)B\big)\,dx \\ &= -\int_\Omega c_0 \frac{w(\cdot,t)}{c_0}\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot B\,d|\nabla\chi(\cdot,t)| \end{aligned} \tag{124}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$.

By the above items, (121), and (124), we may deduce that (17h) holds true since

$$\delta\mu_t(B) = -\int_\Omega c_0 H_{\chi(\cdot,t)}\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}\cdot B\,d|\nabla\chi(\cdot,t)| = \delta\widetilde{\mu}_t(B) \tag{125}$$

for all $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$.

Now, let $t \in (0, T_*)$ be such that the (125) hold true. For $B \in \mathcal{S}_{\chi(\cdot,t)}$, let $w_B \in H^1_{(0)}$ solve the Neumann problem

$$\begin{aligned} \Delta w_B &= B\cdot\nabla\chi(\cdot,t) && \text{in } \Omega, \\ (n_{\partial\Omega}\cdot\nabla)w_B &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{126}$$

Note by definition of the $H^{-1}_{(0)}$ norm, $\|\nabla w_B\|_{L^2(\Omega)} = \|B\cdot\nabla\chi(\cdot,t)\|_{H^{-1}_{(0)}}$. From the Gibbs–Thomson relation (118), we have

$$\delta\widetilde{\mu}_t(B) = \int_\Omega \chi(\cdot,t)\nabla\cdot(\widetilde{w}(\cdot,t)B)\,dx = \int_\Omega \nabla\widetilde{w}(\cdot,t)\cdot\nabla w_B\,dx. \tag{127}$$

Computing the norm of the projection of $\widetilde{w}$ onto $\mathcal{G}_{\chi(\cdot,t)}$ (see (42)) and recalling the identity (125) as well as the inequality $\frac{1}{2}(a/b)^2 \geq a - \frac{1}{2}b^2$, we thus have

$$\begin{aligned}\frac{1}{2}\|\nabla\widetilde{w}(\cdot,t)\|^2_{L^2(\Omega)} &\geq \frac{1}{2}\left(\sup_{B\in\mathcal{S}_{\chi(\cdot,t)}} \frac{\int_\Omega \nabla\widetilde{w}(\cdot,t)\cdot\nabla w_B}{\|\nabla w_B\|_{L^2(\Omega)}}\right)^2 \\ &\geq \sup_{B\in\mathcal{S}_{\chi(\cdot,t)}} \left\{\delta\mu_t(B) - \frac{1}{2}\|B\cdot\nabla\chi(\cdot,t)\|^2_{\mathcal{V}_{\chi(\cdot,t)}}\right\}\end{aligned} \tag{128}$$

for a.e. $t\in(0,T_*)$.

Step 10: Time derivative of phase indicator. In this step, we show

$$\partial_t\chi(\cdot,t)\in\mathcal{T}_{\chi(\cdot,t)}, \tag{129}$$

see (16), for a.e. $t\in(0,T_*)$. As for the metric slope term, cf. (128), we use potentials as a convenient tool.

From the dissipation (123), $\chi\in H^1(0,T_*;H^{-1}_{(0)})$ and there is $u\in H^1(0,T_*;H^1_{(0)})$ such that for almost every $t\in(0,T_*)$ the equation $\partial_t\chi(t)=\Delta_N u(t)$ holds. For any $\zeta\in C^\infty_c(\overline{\Omega}\times[0,T_*))\cap L^2(0,T;H^1_{(0)})$ via a short mollification argument, one can compute the derivative in time of $\langle\chi,\zeta\rangle_{H^{-1}_{(0)},H^1_{(0)}} = (\chi,\zeta)_{L^2(\Omega)}$ to find (recall (105))

$$\begin{aligned}&\int_\Omega \chi(\cdot,T)\zeta(\cdot,T)\,dx - \int_\Omega \chi_0\zeta(\cdot,0)\,dx \\ &= \int_0^T\int_\Omega \chi\partial_t\zeta\,dxdt - \int_0^T\int_\Omega \nabla u\cdot\nabla\zeta\,dxdt\end{aligned} \tag{130}$$

for almost every $T<T^*$. To see that (130) holds for general $\zeta\in C^\infty_c(\overline{\Omega}\times[0,T_*))$ it suffices to check the equation for $c(t)=\fint_\Omega\zeta(\cdot,t)\,dx$. In this case, the left hand side of (130) becomes $m_0(c(T)-c(0))$ and the right hand side becomes $m_0\int_0^T\partial_t c(t) = m_0(c(T)-c(0))$, verifying the assertion. Finally truncating a given test function ζ on the interval (T,T_*), we have that for almost every $T<T^*$, equation (130) holds for all $\zeta\in C^\infty(\overline{\Omega}\times[0,T))$.

Note that for almost every $t\in(0,T_*)$

$$\lim_{\tau\downarrow 0}\fint_{t-\tau}^{t+\tau}\int_\Omega |\nabla u(x,t')-\nabla u(x,t)|^2\,dxdt' = 0. \tag{131}$$

Furthermore, for a.e. $t\in(0,T_*)$ there is a set $A(t)\subset\Omega$ associated to χ in the sense that $\chi(\cdot,t)=\chi_{A(t)}$. As a consequence of (17c), (17g), and (17h) (see equation (2.13) of [55]), for almost every $t\in(0,T_*)$—all of which are already satisfied due to the previous step—, there exists a measurable subset $\widetilde{A}(t)\subset\Omega$ representing a modification of $A(t)$ in the sense

$$\begin{aligned}\mathcal{L}^d\big(A(t)\Delta\widetilde{A}(t)\big)=0,&\quad \widetilde{A}(t)\text{ is open,}\\ \partial\widetilde{A}(t)\cap\Omega = \overline{\partial^*\widetilde{A}(t)}\cap\Omega &\subset \operatorname{supp}|\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\Omega,\\ \operatorname{supp}|\nabla\chi(\cdot,t)| &= \partial\widetilde{A}(t).\end{aligned} \tag{132}$$

We now claim that for almost every $t \in (0, T_*)$ it holds

$$\Delta u(\cdot, t) = 0 \qquad\qquad \text{in } \Omega \setminus \partial\widetilde{A}(t) \tag{133}$$

in a distributional sense. In other words by (43), (49), and (132), for almost every $t \in (0, T_*)$

$$\partial_t \chi(\cdot, t) \in \mathcal{T}_{\chi(\cdot,t)}. \tag{134}$$

For a proof of (133), fix $t \in (0, T_*)$ such that (130), (131), and (132) are satisfied. Fix $\zeta \in C_c^\infty(\Omega \setminus \overline{\widetilde{A}(t)}; [0, \infty))$, consider a sequence $s \downarrow t$ so that one may also apply (130) for the choices $T = s$. Using ζ as a constant-in-time test function in (130) for $T = s$ and $T = t$, respectively, it follows from the nonnegativity of ζ with the first item of (132) that

$$0 \leq \frac{1}{s-t} \int_\Omega \chi_{A(s)} \zeta \, dx = -\fint_t^s \int_\Omega \nabla u \cdot \nabla \zeta \, dx dt'.$$

Hence, for $s \downarrow t$ we deduce from the previous display as well as (131) that

$$\int_\Omega \nabla u \cdot \nabla \zeta \, dx \leq 0 \quad \text{for all } \zeta \in C_c^\infty(\Omega \setminus \overline{\widetilde{A}(t)}; [0, \infty)).$$

Choosing instead a sequence $s \uparrow t$ so that one may apply (130) for the choices $T = s$, one obtains similarly

$$\int_\Omega \nabla u \cdot \nabla \zeta \, dx \geq 0 \quad \text{for all } \zeta \in C_c^\infty(\Omega \setminus \overline{\widetilde{A}(t)}; [0, \infty)).$$

In other words, (133) is satisfied throughout $\Omega \setminus \overline{\widetilde{A}(t)}$ for all nonnegative test functions in $H^1(\Omega \setminus \overline{\widetilde{A}(t)})$, hence also for all nonpositive test functions in $H^1(\Omega \setminus \overline{\widetilde{A}(t)})$, and therefore for all smooth and compactly supported test functions in $\Omega \setminus \overline{\widetilde{A}(t)}$. Adding and subtracting $\int_\Omega \zeta \, dx$ to the left hand side of (130), one may finally show along the lines of the previous argument that (133) is also satisfied throughout $\widetilde{A}(t)$.

Step 11: Optimal energy dissipation relation. Combining the information provided by (123), (128) and (129), we obtain the asserted De Giorgi type energy dissipation inequality (18) for almost all $0 < s < T < T_*$. Since we also already obtained the desired condition on the initial data, see (104), it remains to verify admissibility of the couple (χ, μ).

Step 12: Admissibility of the couple (χ, μ)***.*** We proceed in five substeps.

Proof of item i) of Definition 3. Due to definition (114) of the limits μ_t and the convergences (112)–(113), the oriented varifolds μ_t decompose as required with components as given by (17a) and (17b), respectively. The asserted integer rectifiability of the measures for a.e. $t \in (0, T_*)$ has already been established in Step 6 of this proof.

Proof of item ii) of Definition 3. The validity of (17c) was already part of Step 9 of this proof. Returning to the definition of $\mu_t^{h,\Omega}$, for any $\eta \in C^1(\overline{\Omega}; \mathbb{R}^d)$ with $(\eta \cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$ and all $t \in (0, T_*)$,

$$\int_{\overline{\Omega} \times \mathbb{S}^{d-1}} \eta(x) \cdot s \, d\mu_t^{h,\Omega}(x, s) = \int_\Omega \eta \cdot d\nabla \bar{\chi}^h(\cdot, t) = -\int_\Omega \bar{\chi}^h(x, t) \nabla \cdot \eta \, dx.$$

Using the convergences from (102) and (112), we can pass to the limit on the left- and right-hand side of the above equation and afterward apply the divergence to find that for a.e. $t \in (0,T)$, it holds

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \eta(x)\cdot s\,d\mu_t^{\Omega}(x,s) = \int_{\Omega} \eta\cdot d\nabla\chi(\cdot,t) \tag{135}$$

for all $\eta \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$. This proves (17e).

Let now $\phi \in C(\overline{\Omega})$ and fix a vector field $\xi \in C^1(\overline{\Omega};\mathbb{R}^d)$ such that $\xi\cdot n_{\partial\Omega} = (\cos\alpha)$ along $\partial\Omega$. Recalling the definitions of $\mu_t^{h,\Omega}$ and $\mu_t^{h,\partial\Omega}$, we obtain

$$\begin{aligned}
&\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \phi(x)\xi(x)\cdot s\,d\mu_t^{h,\Omega}(x,s) + (\cos\alpha)\int_{\partial\Omega} \phi\,d|\mu_t^{h,\partial\Omega}|_{\mathbb{S}^{d-1}} \\
&= \int_{\Omega} \phi\xi\cdot d\nabla\bar{\chi}^h(\cdot,t) + \int_{\partial\Omega} \bar{\chi}^h(\cdot,t)\phi\xi\cdot n_{\partial\Omega}\,d\mathcal{H}^{d-1} = -\int_{\Omega} \bar{\chi}^h(x,t)\nabla\cdot(\phi\xi)\,dx,
\end{aligned}$$

which as before implies for a.e. $t \in (0,T_*)$

$$\begin{aligned}
&\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \phi(x)\xi(x)\cdot s\,d\mu_t^{\Omega}(x,s) + (\cos\alpha)\int_{\partial\Omega} \phi\,d|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}} \\
&= \int_{\Omega} \phi\xi\cdot d\nabla\chi(\cdot,t) + \int_{\partial\Omega} \phi(\cos\alpha)\chi(\cdot,t)\,d\mathcal{H}^{d-1}.
\end{aligned}$$

Letting ϕ converge pointwise everywhere to $\chi_{\partial\Omega}$ implies (17f), whereas choosing ξ converging to $(\cos\alpha)n_{\partial\Omega}\chi_{\partial\Omega}$ pointwise everywhere and varying $\phi \in C(\overline{\Omega})$ entails (17d).

Proof of items iii) and iv) of Definition 3. Except for the claim that μ_t is of bounded first variation throughout $\overline{\Omega}$, this was already part of Step 9 of this proof. For a proof of the bound (17i), it suffices to apply Corollary 6, which in turn is admissible thanks to Step 9 of this proof.

Proof of item v) of Definition 3. By (116), it remains to establish measurability of the slope as defined by (17m). To this end, we again employ a construction from the proof of Lemma 9. More precisely, let $\xi \in L^2(0,T_*;H^1(\Omega))$ be defined by $\xi := \nabla\phi_\epsilon$ where, for a suitably chosen $\epsilon \in (0,1)$, we denote by $\phi_\epsilon \in L^2(0,T_*;H^2(\Omega))$ the weak solution of the Poisson problem (65) in the precise sense that

$$-\int_0^{T_*}\int_{\Omega} \nabla\phi_\epsilon\cdot\nabla\eta\,dxdt = \int_0^{T_*}\int_{\Omega} (\chi_\epsilon - m_\epsilon)\eta\,dxdt$$

for all $\eta \in C_c(0,T_*;C^1(\overline{\Omega};\mathbb{R}^d))$. In other words, we simply lifted the spatial PDE (65) to the product space $(0,T_*)\times\Omega$ to obtain a measurable selection of its solution along $(0,T_*)$. As with (67), in this context, thanks to (17c) and the energy dissipation (18), one may indeed choose ϵ independent of $t \in (0,T_*)$ such that

$$\operatorname*{ess\,inf}_{t\in(0,T_*)} \int_{\Omega} \chi(\cdot,t)\nabla\cdot\xi(\cdot,t)\,dx \geq C(m_0,\Omega) > 0.$$

The merit of the previous construction is that for all variations $B \in C^1(\overline{\Omega};\mathbb{R}^d)$ with $B\cdot n_{\partial\Omega} = 0$ along $\partial\Omega$, it holds $\widetilde{B}_\xi(\cdot,t) := B - \frac{\int_\Omega \chi(\cdot,t)\nabla\cdot B\,dx}{\int_\Omega \chi(\cdot,t)\nabla\cdot\xi(\cdot,t)\,dx}\xi(\cdot,t) \in \mathcal{S}_{\chi(\cdot,t)}$ for all $t \in (0,T_*)$, so that

$$\frac{1}{2}\big|\partial E[\mu_t]\big|^2_{\mathcal{V}_{\chi(\cdot,t)}} = \sup_{\substack{B\in C^1(\overline{\Omega};\mathbb{R}^d)\\ (B\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv 0}} \Big\{\delta\mu_t\big(\widetilde{B}_\xi(\cdot,t)\big) - \frac{1}{2}\big\|\widetilde{B}_\xi(\cdot,t)\cdot\nabla\chi(\cdot,t)\big\|^2_{\mathcal{V}_{\chi(\cdot,t)}}\Big\}. \tag{136}$$

Noting that by construction $\widetilde{B}_\xi(\cdot,t) \in L^2(0,T_*;C^1(\overline{\Omega};\mathbb{R}^d))$, it follows from Steps 3 of this proof as well as the identities (118) and (125) that $(0,T_*) \ni t \mapsto \delta\mu_t\big(\widetilde{B}_\xi(\cdot,t)\big)$ is measurable for any fixed tangential variations B from (136). Furthermore, by fixing a solution $w_B \in L^2(0,T_*;H^1_{(0)})$ of the weak formulation

$$-\int_0^{T_*}\int_\Omega \nabla w_B \cdot \nabla \eta \, dxdt = -\int_0^{T_*}\int_\Omega \chi \nabla \cdot \big(\widetilde{B}_\xi \eta\big) \, dxdt,$$

we deduce that $\frac{1}{2}\big\|\widetilde{B}_\xi(\cdot,t)\cdot\nabla\chi(\cdot,t)\big\|^2_{\mathcal{V}_{\chi(\cdot,t)}} = \frac{1}{2}\|\nabla w_B(\cdot,t)\|^2_{L^2(\Omega)}$ is a measurable map $(0,T_*) \to [0,\infty)$ for all tangential variations B from (136).

In summary, since $C^1(\overline{\Omega};\mathbb{R}^d)$ is a separable metric space so that the supremum in (136) may be taken to be countable, we infer measurability of the slope as required.

Step 13: Conclusion. Thanks to the previous two steps, we may conclude the proof that (χ,μ) is a varifold solution for Mullins–Sekerka flow (1a)–(1e) with time horizon T_* and initial data χ_0 in the sense of Definition 5.

Step 14: ***BV solutions.*** Let χ be a subsequential limit point as obtained in (102). We now show that if the time-integrated energy convergence assumption (60) is satisfied then χ is a BV solution. The main difficulty in proving this is showing that there exists a subsequence $h \downarrow 0$ such that the De Giorgi interpolants satisfy

$$\bar{\chi}^h(\cdot,t) \to \chi(\cdot,t) \text{ strictly in } BV(\Omega;\{0,1\}) \text{ for a.e. } t \in (0,T_*) \text{ as } h \downarrow 0. \tag{137}$$

Before proving (137), let us show how this concludes the result. First, since (137) in particular means $|\nabla\bar{\chi}^h(\cdot,t)|(\Omega) \to |\nabla\chi(\cdot,t)|(\Omega)$ for a.e. $t \in (0,T_*)$ as $h \downarrow 0$, it follows from Reshetnyak's continuity theorem, cf. [6, Theorem 2.39], that

$$\mu_t^\Omega := |\nabla\chi(\cdot,t)|\llcorner\Omega \otimes (\delta_{\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|}(x)})_{x\in\Omega}$$

is the weak limit of $\mu_t^{h,\Omega}$, i.e., $\mu_t^{h,\Omega} \to \mu_t^\Omega$ weakly* in $\mathrm{M}(\Omega\times\mathbb{S}^{d-1})$ as $h \to 0$. Second, due to (137) it follows from BV trace theory, cf. [6, Theorem 3.88], that

$$\bar{\chi}^h(\cdot,t) \to \chi(\cdot,t) \text{ strongly in } L^1(\partial\Omega) \text{ for a.e. } t \in (0,T_*) \text{ as } h \downarrow 0. \tag{138}$$

Hence, defining

$$\mu_t^{\partial\Omega} := \chi(\cdot,t)\,\mathcal{H}^{d-1}\llcorner\partial\Omega \otimes (\delta_{n_{\partial\Omega}(x)})_{x\in\partial\Omega},$$

we have $\mu_t^{h,\partial\Omega} \to \mu_t^{\partial\Omega}$ weakly* in $\mathrm{M}(\partial\Omega\times\mathbb{S}^{d-1})$ as $h \to 0$. In summary, the relations (19a) and (19b) hold true as required. Furthermore, defining $\mu_t := c_0\mu_t^\Omega + (\cos\alpha)c_0\mu_t^{\partial\Omega}$, it follows from the arguments of the previous steps that (χ,μ) is a varifold solution in the sense of Definition 5. In other words, χ is a BV solution as claimed.

We now prove (137). To this end, we first show that (102) implies that there exists a subsequence $h \downarrow 0$ such that $E[\bar{\chi}^h(\cdot,t)] \to E[\chi(\cdot,t)]$ for a.e. $t \in (0,T_*)$. Indeed, since $E[\bar{\chi}^h(\cdot,t)] \le E[\chi^h(\cdot,t)]$ for all $t \in (0,T_*)$ by the optimality constraint for a De Giorgi interpolant (57), we may estimate using the elementary relation $|a| = a + 2a_-$ that

$$\begin{aligned}
&\int_0^{T_*} \big|E[\bar{\chi}^h(\cdot,t)] - E[\chi(\cdot,t)]\big| \, dt \\
&= \int_0^{T_*} \big(E[\bar{\chi}^h(\cdot,t)] - E[\chi(\cdot,t)]\big) \, dt + \int_0^{T_*} 2\big(E[\bar{\chi}^h(\cdot,t)] - E[\chi(\cdot,t)]\big)_- \, dt
\end{aligned}$$

$$\leq \int_0^{T_*} \big(E[\chi^h(\cdot,t)] - E[\chi(\cdot,t)]\big)\,dt + \int_0^{T_*} 2\big(E[\bar{\chi}^h(\cdot,t)] - E[\chi(\cdot,t)]\big)_-\,dt.$$

The first right hand side term of the last inequality vanishes in the limit $h \downarrow 0$ by assumption (60). For the second term on the right-hand side, we note that (102) entails that, up to a subsequence, we have $\bar{\chi}^h(\cdot,t) \to \chi(\cdot,t)$ strongly in $L^1(\Omega)$ for a.e. $t \in (0,T_*)$ as $h \downarrow 0$. Hence, the lower-semicontinuity result of Modica [47, Proposition 1.2] tells us that $(E[\bar{\chi}^h(\cdot,t)] - E[\chi(\cdot,t)])_- \to 0$ pointwise a.e. in $(0,T_*)$ as $h \downarrow 0$, which in turn by Lebesgue's dominated convergence theorem guarantees that the second term on the right-hand side of the last inequality vanishes in the limit $h \downarrow 0$. In summary, for a suitable subsequence $h \downarrow 0$

$$E[\bar{\chi}^h(\cdot,t)] \to E[\chi(\cdot,t)] \qquad \text{for a.e. } t \in (0,T_*), \tag{139}$$

$$\bar{\chi}^h(\cdot,t) \to \chi(\cdot,t) \qquad \text{strongly in } L^1(\Omega) \text{ for a.e. } t \in (0,T_*). \tag{140}$$

Now, due to (140) and the definition of strict convergence in $BV(\Omega)$, (137) will follow if the total variations converge, i.e.,

$$|\nabla\bar{\chi}^h(\cdot,t)|(\Omega) \to |\nabla\chi(\cdot,t)|(\Omega) \qquad \text{for a.e. } t \in (0,T_*) \text{ as } h \downarrow 0. \tag{141}$$

However, this is proven in a more general context (i.e., for the diffuse interface analogue of the sharp interface energy (10)) in [28, Lemma 5]. More precisely, thanks to (139) and (140), one may simply apply the argument of [28, Proof of Lemma 5] with respect to the choices $\psi(u) := c_0 u$, $\sigma(u) := (\cos\alpha)c_0 u$ and $\tau(u) := (\sigma\circ\psi^{-1})(u) = (\cos\alpha)u$ to obtain (141), which in turn eventually concludes the proof of Theorem 1. □

We now turn to the proof of Corollary 7, where we show in the special case that $d = 2$ and under an energy convergence hypothesis the tangent spaces coincide.

Proof of Corollary 7. We rely on the structure spelled out in Step 6 of the above proof. In particular, by Proposition 11, we have that $|\nabla\bar{\chi}^h(\cdot,t)| = \mathcal{H}^{d-1}\llcorner(\cup_{k=1}^{N_h}\Gamma_k^h)$, where Γ_k^h are pairwise disjoint $C^{1,\beta}$-curves which intersect the boundary at angle α. We now show that the length of each Γ_k^h is uniformly bounded from below by a constant $c(\Omega,t)$. The essence of this is that if the curve is too short, the squared curvature must blow up. From this, given the bound on the energy, we will have that the number of curves for a good subsequence $h \to 0$ will be controlled by $N_h \leq C(\Omega,t) \approx 1/c(\Omega,t)$.

We now fix one of the curves, and simply denote it by Γ and, by an abuse of notation, let $\Gamma : [0,L] \to \overline{\Omega}$ be an arclength parameterization. By (96), the curve has curvature coinciding with the trace of $w^h + \lambda^h$, which we denote H_Γ (a vector). Then for any $B \in C^1_{cpt}(\Omega;\mathbb{R}^2)$, we have

$$\int_\Gamma \tau\cdot((\tau\cdot\nabla)B)\,d\mathcal{H}^1 = -\int_\Gamma H_\Gamma\cdot B\,d\mathcal{H}^1,$$

where τ is the tangent vector of Γ. Denoting $\tilde{f} := f\circ\Gamma$, we may pull back the relation to the interval $[0,L]$ as

$$\int_0^L \tilde{\tau}\cdot\frac{d}{ds}\tilde{B}\,ds = -\int_0^L \tilde{H}_\Gamma\cdot\tilde{B}\,ds. \tag{142}$$

As Γ is a sufficiently regular surface, for any $\tilde{B} \in C^1_{cpt}(0,L;\mathbb{R}^2)$ there is $B \in C^1_{cpt}(\Omega;\mathbb{R}^2)$ with $\tilde{B} = B\circ\Gamma$. To see this, note that for any $\tilde{B} \in C^1_{cpt}(0,L;\mathbb{R}^2)$ we have

$$\frac{\tilde{B}\circ\Gamma^{-1}(y)-\tilde{B}\circ\Gamma^{-1}(x)-(\frac{d}{ds}\tilde{B})\circ\Gamma^{-1}(x)\tau(x)\cdot(y-x)}{|x-y|}\to 0$$

uniformly on Γ as $|y-x|\to 0$ since

$$\tau(x)\cdot(y-x)=\tau(x)\cdot\tau(x)(\Gamma^{-1}(y)-\Gamma^{-1}(x))+O(|\Gamma^{-1}(y)-\Gamma^{-1}(x)|^{1+\beta}),$$

and $|\Gamma^{-1}(y)-\Gamma^{-1}(x)|$ is proportional to $|y-x|$. With this, we may apply the Whitney extension theorem [22, Theorem 6.10] to find $B\in C^1_{cpt}(\Omega;\mathbb{R}^2)$ with $B|_\Gamma=\tilde{B}\circ\Gamma^{-1}$.

By (142) and the above comment, it follows that $\frac{d}{ds}\tilde{\tau}=\tilde{H}_\Gamma$ in the Sobolev sense of weak derivative. Consequently, by the fundamental theorem of calculus, we may compute the difference in tangent vectors as the integral of the curvature:

$$\int_\Gamma H_\Gamma\,d\mathcal{H}^1=\tau\circ\Gamma(L)-\tau\circ\Gamma(0). \tag{143}$$

Using the above relation and the square integrability of the curvature, we will now find the desired lower bounds on the length of curves.

If Γ_k^h intersects the boundary it must do so twice, and we let $\tilde{\Gamma}_k^h:=\Gamma_k^h$. Let $\tau_k^{h,\pm}$ be the tangent vectors at the contact points of the curve, oriented with the parameterization. Similarly, if Γ_k^h does not intersect the boundary, we let $\tilde{\Gamma}_k^h$ be a subset of the curve such that the (oriented) tangent vectors at the end points, still denoted by $\tau_k^{h,\pm}$, point in opposite directions, i.e., $\tau_k^{h,+}=-\tau_k^{h,-}$.

Consider briefly the case of Γ_k^h which intersects the boundary. Note that one tangent vector points inside the domain and one points outside the domain each having contact angle with $\partial\Omega$ given by α. Letting $\tau_{\mathrm{in}}(x)$ and $\tau_{\mathrm{out}}(x)$ be vectors with contact angle α at $x\in\partial\Omega$ such that τ_{in} enters the domain and τ_{out} exits the domain. It follows that $|\tau_{\mathrm{in}}(x)-\tau_{\mathrm{out}}(x)|\geq 2\sin(\alpha)$. By a continuity argument, there is $\delta>0$ such that for $x,y\in\partial\Omega$ with $|x-y|<\delta$, one always has $|\tau_{\mathrm{in}}(x)-\tau_{\mathrm{out}}(y)|>\sin(\alpha)$. Consequently, if $\mathcal{H}^1(\Gamma_k^h)<\delta$, then $|\tau_k^{h,+}-\tau_k^{h,-}|\geq\sin(\alpha)$.

If $\mathcal{H}^1(\Gamma_k^h)<\delta$, we apply Jensen's inequality and (143) to estimate

$$\int_{\tilde{\Gamma}_k^h}|H_{\Gamma_k^h}|^2\,d\mathcal{H}^1\geq\frac{|\tau_k^{h,+}-\tau_k^{h,-}|^2}{\mathcal{H}^1(\tilde{\Gamma}_k^h)}\geq\frac{C(\alpha)}{\mathcal{H}^1(\tilde{\Gamma}_k^h)}.$$

Recalling Proposition 5, (96), and (111), we have that $\int_{\tilde{\Gamma}_k^h}|H_{\Gamma_k^h}|^2\,d\mathcal{H}^1\leq C(t)<\infty$ for almost every $t\in(0,T^*)$. It follows that

$$c(\Omega,t):=\min\left\{\delta,\frac{C(\alpha)}{C(t)+1}\right\}\leq\mathcal{H}^1(\tilde{\Gamma}_k^h)\leq\mathcal{H}^1(\Gamma_k^h),$$

thereby proving the claim.

This shows that for a subsequence satisfying (111), there is $0\leq C(\Omega,t)<\infty$ such that $|\nabla\overline{\chi}^h|$ is composed of at most $N_h\leq C(\Omega,t)$ regular curves with length bounded strictly away from zero. Up to rescaling the arclength parameterization and recalling the estimates following from (142), we may represent each $\Gamma_k^h=f_k^h([0,1])$ by a function $f\in W^{2,2}(0,1;\overline{\Omega})$ with $|(f_k^h)'|\geq c(\Omega,t)$ and

$$\|f_k^h\|_{W^{2,2}(0,1;\overline{\Omega})}\leq C(\Omega,t).$$

Up to subsequences, we have that each $f_k^h\rightharpoonup f_k$ in $W^{2,2}(0,1;\overline{\Omega})$ as $h\to 0$. By the Sobolev embedding, we have that $|(f_k)'|\geq c(\Omega,t)$, which shows that each curve remains an immersion. Further, as we have assumed the energy convergence hypothesis, we have that $|\nabla\chi| = \mathcal{H}^1 \llcorner (\cup_k \Gamma_k)$, where $\Gamma_k := f_k([0,1])$.

We now use the characterization (50) to show that $\mathcal{T}_\chi = \mathcal{V}_\chi$. Up to accounting for overlap, this effectively will follow from the fact that Γ_k locally (within its parameterization) is prescribed by a Lipschitz graph. We note that to show the identity collapses for a Lipschitz graph one can apply standard trace theorems and the annihilator relations developed in Section 2.3. More generally, one can argue via the capacity. For instance one can show that the $H^{1/2}$-capacity intrinsic to a Lipschitz graph is equivalent to the H^1-capacity defined using the ambient Sobolev space.

Equation (50) will collapse to the identity if we show that a function with distributional trace given by a constant on $|\nabla\chi|$ has quasi-everywhere trace given by constant. Taking this constant to be 0 without loss of generality, suppose that u is a good representative defined (H^1) quasi-everywhere and is such that $\int_\Omega \chi \mathrm{div}(uB)\,dx = 0$ for all tangential variations B.

Using an integration by parts, this directly implies that $u = 0$ almost everywhere on $|\nabla\chi|$. Let $a, b \subset [0,1]$ be such that $f_k((a,b))$ is the graph of a Lipschitz function. Note for $t \in (0,1)$ such a and b can always be found with $a < t < b$ as $W^{2,2}(0,1;\overline{\Omega}) \hookrightarrow C^{1,1/2}(0,1;\overline{\Omega})$. Since u is a good representative of itself in $H^{1/2}(f_k((a,b)))$, the standard trace space, u has trace quasi-everywhere given by 0 on $f_k((a,b))$ (see also Theorem 6.1.4 of [3]). By a covering argument, this implies that $u = 0$ quasi-everywhere on $\cup_k\Gamma_k = \mathrm{supp}|\nabla\chi|$, concluding the proof of the corollary. $\square$

3.4. **Proofs for further properties of varifold solutions.** In this subsection, we present the proofs for the various further results on varifold solutions to Mullins–Sekerka flow as mentioned in Subsection 2.2.

Proof of Lemma 2. The proof is naturally divided into two parts.

Step 1: Proof of "$\leq$" in (20) without assuming (17h)–(17k). To simplify the notation, we denote $\chi_s := \chi \circ \Psi_s^{-1}$ resp. $\mu_s := \mu \circ \Psi_s^{-1}$ and abbreviate the right-hand side of (20) as

$$A := \sup_{\substack{\partial_s\chi_s|_{s=0} = -B\cdot\nabla\chi \\ \chi_s\to\chi,\ B\in\mathcal{S}_\chi}} \limsup_{s\downarrow 0} \frac{(E[\mu]-E[\mu_s])_+}{\|\chi-\chi_s\|_{H^{-1}_{(0)}}}. \tag{144}$$

Fixing a flow χ_s such that $\chi_s \to \chi$ as $s \to 0$ with $\partial_s\chi_s|_{s=0} = -B\cdot\nabla\chi$ for some $B \in \mathcal{S}_\chi$, we claim that the upper bound (20) follows from the assertions

$$\lim_{s\to 0} \frac{1}{s}\big(E[\mu_s]-E[\mu]\big) = \delta\mu(B), \tag{145}$$

$$\lim_{s\to 0} \frac{1}{s}\|\chi_s-\chi\|_{H^{-1}_{(0)}} = \|B\cdot\nabla\chi\|_{H^{-1}_{(0)}}. \tag{146}$$

Indeed, multiplying the definition in (144) by $1 = s/s$, using the inequality $\frac{1}{2}(a/b)^2 \geq a - \frac{1}{2}b^2$, and recalling the notation (15), we find

$$\begin{aligned}\frac{1}{2}A^2 &\geq \lim_{s\to 0}\frac{E[\mu]-E[\mu_s]}{s} - \frac{1}{2}\lim_{s\to 0}\left(\frac{\|\chi-\chi_s\|_{H^{-1}_{(0)}}}{s}\right)^2 \\ &= \delta\mu(-B) - \frac{1}{2}\|B\cdot\nabla\chi\|^2_{\mathcal{V}_\chi}.\end{aligned} \tag{147}$$

Recalling Lemma 8, taking the supremum over $B \in \mathcal{S}_\chi$ thus yields "$\leq$" in (20). It therefore remains to establish (145) and (146). However, the former is a classical and well-known result [5] whereas the latter is established in Lemma 10.

Step 2: Proof of "$\geq$" in (20) *assuming* (17h)–(17k). To show equality under the additional assumption of (17h)–(17k), we may suppose that $|\partial E[\mu]|_{\mathcal{V}_\chi} < \infty$. First, we note that $B\cdot\nabla\chi \mapsto \delta\mu(B)$ is a well defined operator on $\{B\cdot\nabla\chi : B \in \mathcal{S}_\chi\} \subset \mathcal{V}_\chi$ (see definition (12)). To see this, let $B \in \mathcal{S}_\chi$ be any function such that $B\cdot\nabla\chi = 0$ in $\mathcal{V}_\chi \subset H^{-1}_{(0)}$. Recall that $\int_\Omega \chi\nabla\cdot B\,dx = 0$ by the definition of $\mathcal{S}_\chi$ in (11) to find that for all $\phi \in C^1(\overline{\Omega})$, one has

$$\int_\Omega \phi B\cdot\frac{\nabla\chi}{|\nabla\chi|}\,d|\nabla\chi| = \int_\Omega \Big(\phi - \fint_\Omega \phi\,dx\Big) B\cdot\frac{\nabla\chi}{|\nabla\chi|}\,d|\nabla\chi| = 0.$$

The above equation implies that $B\cdot\frac{\nabla\chi}{|\nabla\chi|} = 0$ for $|\nabla\chi|$−almost every x in Ω, which by the representation of the first variation in terms of the curvature in (17h)–(17k) shows $\delta\mu(B) = 0$. Linearity shows the operator is well-defined on $\{B\cdot\nabla\chi : B \in \mathcal{S}_\chi\}$.

With this in hand, the bound $|\partial E[\mu]|_{\mathcal{V}_\chi} < \infty$ implies that the mapping $B\cdot\nabla\chi \mapsto \delta\mu(B)$ can be extended to a bounded linear operator $L\colon \mathcal{V}_\chi \to \mathbb{R}$, which is identified with an element $L \in \mathcal{V}_\chi$ by the Riesz isomorphism theorem. Consequently,

$$\frac{1}{2}|\partial E[\mu]|^2_{\mathcal{V}_\chi} = \sup_{B\in\mathcal{S}_\chi}\Big\{(L, B\cdot\nabla\chi)_{\mathcal{V}_\chi} - \frac{1}{2}\|B\cdot\nabla\chi\|^2_{\mathcal{V}_\chi}\Big\} = \frac{1}{2}\|L\|^2_{\mathcal{V}_\chi}. \tag{148}$$

But recalling (145) and (146), one also has that

$$A = \sup_{\substack{\partial_s\chi_s|_{s=0}=-B\cdot\nabla\chi\\ \chi_s\to\chi,\ B\in\mathcal{S}_\chi}} \frac{\big(\lim_{s\to0}\frac{1}{s}(E[\mu]-E[\mu_s])\big)_+}{\lim_{s\to0}\frac{1}{s}\|\chi-\chi_s\|_{H^{-1}_{(0)}}} = \sup_{B\in\mathcal{S}_\chi}\frac{(\delta\mu(-B))_+}{\|B\cdot\nabla\chi\|_{\mathcal{V}_\chi}} = \|L\|_{\mathcal{V}_\chi}.$$

The previous two displays complete the proof that $A = |\partial E[\mu]|_{\mathcal{V}_\chi}$. □

Proof of Lemma 3. We proceed in three steps.

Proof of item i): We first observe that by (18) as well as taking $s\downarrow 0$ and $T\uparrow T_*$

$$\int_0^{T_*}\frac{1}{2}\big\|\partial_t\chi\big\|^2_{H^{-1}_{(0)}(\Omega)}\,dt = \int_0^{T_*}\frac{1}{2}\big\|\partial_t\chi\big\|^2_{\mathcal{T}_{\chi(\cdot,t)}}\,dt \leq E[\chi_0] < \infty,$$

which in turn simply means that there exists $u \in L^2(0,T_*;\mathcal{H}_{\chi(\cdot,t)}) \subset L^2(0,T_*;H^1_{(0)})$ such that

$$\partial_t\chi(\cdot,t) = \Delta_N u(\cdot,t) \quad \text{for a.e. } t\in(0,T_*). \tag{149}$$

In other words, recalling (41),

$$\int_0^{T_*}\int_\Omega \chi\partial_t\zeta\,dxdt = \int_0^{T_*}\int_\Omega \nabla u\cdot\nabla\zeta\,dxdt \quad \text{for all } \zeta\in C^1_{cpt}(\overline{\Omega}\times(0,T_*)). \tag{150}$$

By standard PDE arguments and the requirement $\chi \in C([0,T_*);H^{-1}_{(0)}(\Omega))$ such that $\mathrm{Tr}|_{t=0}\chi = \chi_0$ in $H^{-1}_{(0)}$, one may post-process the previous display to (21).

Proof of item ii): Thanks to (17h)–(17k), we may apply the argument from the proof of Lemma 2 to infer that the map $B\cdot\nabla\chi(\cdot,t) \mapsto \delta\mu_t(B)$, $B\in\mathcal{S}_{\chi(\cdot,t)}$, is well-defined and extends to a unique bounded and linear functional $L_t\colon \mathcal{V}_{\chi(\cdot,t)} \to \mathbb{R}$. Recalling the definition $\mathcal{G}_{\chi(\cdot,t)} = \Delta_N^{-1}(\mathcal{V}_{\chi(\cdot,t)}) \subset H^1_{(0)}$ and the fact that the weak Neumann Laplacian $\Delta_N\colon H^1_{(0)} \to H^{-1}_{(0)}$ is nothing else but the Riesz isomorphism between $H^1_{(0)}$ and its dual $H^{-1}_{(0)}$, it follows from (17m) and (18) that there exists a potential $w_0 \in L^2(0,T_*;\mathcal{G}_{\chi(\cdot,t)})$ such that

$$\delta\mu_t(B) = L_t\big(B\cdot\nabla\chi(\cdot,t)\big) = -\langle B\cdot\nabla\chi(\cdot,t), w_0(\cdot,t)\rangle_{H^{-1}_{(0)},H^1_{(0)}}$$

for almost every $t\in(0,T_*)$ and all $B\in\mathcal{S}_{\chi(\cdot,t)}$, as well as

$$\frac{1}{2}\|w_0(\cdot,t)\|^2_{\mathcal{G}_{\chi(\cdot,t)}} = \frac{1}{2}\big|\partial E[\mu_t]\big|^2_{\mathcal{V}_{\chi(\cdot,t)}} \tag{151}$$

for almost every $t\in(0,T_*)$. In particular,

$$\delta\mu_t(B) = \int_\Omega \chi(\cdot,t)\nabla\cdot\big(Bw_0(\cdot,t)\big)\,dx \tag{152}$$

for almost every $t\in(0,T_*)$ and all $B\in\mathcal{S}_{\chi(\cdot,t)}$. Due to Lemma 9, there exists a measurable Lagrange multiplier $\lambda\colon(0,T_*)\to\mathbb{R}$ such that

$$\delta\mu_t(B) = \int_\Omega \chi(\cdot,t)\nabla\cdot\big(B\big(w_0(\cdot,t)+\lambda(t)\big)\big)\,dx \tag{153}$$

for almost every $t\in(0,T_*)$ and all $B\in C^1(\overline{\Omega};\mathbb{R}^d)$ with $(B\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv 0$, and that there exists a constant $C=C(\Omega,d,c_0,m_0)>0$ such that

$$|\lambda(t)| \leq C\big(1+|\nabla\chi(\cdot,t)|(\Omega)\big)\big(|\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \|\nabla w_0(\cdot,t)\|_{L^2(\Omega)}\big) \tag{154}$$

for almost every $t\in(0,T_*)$. Due to (151)–(154), (17c) and (18), it follows that the potential $w:=w_0+\lambda$ satisfies the desired properties (22)–(24).

Proof of item iii): The De Giorgi type energy dissipation inequality in the form of (25) now directly follows from (23) and (149). □

Proof of Lemma 4. We start with a proof of the two consistency claims and afterward give the proof of the compactness statement.

Step 1: Classical solutions are BV solutions. Let $\mathscr{A}$ be a classical solution for Mullins–Sekerka flow in the sense of (1a)–(1e). Define $\chi(x,t):=\chi_{\mathscr{A}(t)}(x)$ for all $(x,t)\in\Omega\times[0,T_*)$. We consider the reduced boundary $\partial^*\mathscr{A}$ of $\mathscr{A}$ as a subset of $\mathbb{R}^d$. As one may simply define the associated varifold by means of $|\mu_t^\Omega|_{\mathbb{S}^{d-1}} := |\nabla\chi(\cdot,t)|\llcorner\Omega = \mathcal{H}^{d-1}\llcorner(\partial^*\mathscr{A}(t)\cap\Omega)$, $|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}} := \chi(\cdot,t)\mathcal{H}^{d-1}\llcorner\partial\Omega = \mathcal{H}^{d-1}\llcorner(\partial^*\mathscr{A}(t)\cap\partial\Omega)$, and (19b) with $\frac{\nabla\chi(\cdot,t)}{|\nabla\chi(\cdot,t)|} = n_{\partial\mathscr{A}(t)}$, note that the varifold energy $|\mu_t|_{\mathbb{S}^{d-1}}(\overline{\Omega})$ simply equals the BV energy functional $E[\chi(\cdot,t)]$ from (10).

Now, admissibility of the couple (χ,μ) in the sense of Definition 3 is trivially satisfied by construction and regularity of the classical solution. For instance, due to the smoothness of the geometry and the validity of the contact angle condition (1e) in the pointwise strong sense, an application of the classical first variation formula together with an integration by parts along each of the smooth manifolds $\partial^*\mathscr{A}(t)\cap\Omega$ and $\partial^*\mathscr{A}(t)\cap\partial\Omega$ ensures (recall the notation from Subsection 1.2)

$$\begin{aligned}\delta E[\chi(\cdot,t)](B) &= c_0\int_{\partial^*\mathscr{A}(t)\cap\Omega}\nabla^{\tan}\cdot B\,d\mathcal{H}^{d-1} + c_0(\cos\alpha)\int_{\partial^*\mathscr{A}(t)\cap\partial\Omega}\nabla^{\tan}\cdot B\,d\mathcal{H}^{d-1}\\ &= -c_0\int_{\partial^*\mathscr{A}(t)\cap\Omega} H_{\partial\mathscr{A}(t)}\cdot B\,d\mathcal{H}^{d-1}\end{aligned} \tag{155}$$

for all tangential variations $B\in C^1(\overline{\Omega};\mathbb{R}^d)$. Hence, the requirements (17h)–(17k) hold with $H_{\chi(\cdot,t)} = H_{\partial\mathscr{A}(t)}\cdot n_{\partial\mathscr{A}(t)}$, where the asserted integrability follows from the boundary condition (1c) and a standard trace estimate for the potential $\bar{u}(\cdot,t)$.

It remains to show (18). Starting point for this is again the above first variation formula, now in the form of

$$\frac{d}{dt}E[\chi(\cdot,t)] = -c_0 \int_{\partial^*\mathscr{A}(t)\cap\Omega} H_{\partial\mathscr{A}(t)} \cdot V_{\partial\mathscr{A}(t)}\, d\mathcal{H}^{d-1}. \tag{156}$$

Plugging in (1b) and (1c), integrating by parts in the form of (2), and exploiting afterward (1a) and (1d), we arrive at

$$\frac{d}{dt}E[\chi(\cdot,t)] = -\int_\Omega |\nabla\bar{u}(\cdot,t)|^2\, dx. \tag{157}$$

The desired inequality (18) will follow once we prove

$$\|\partial_t\chi(\cdot,t)\|_{H^{-1}_{(0)}} = \|\nabla\bar{u}(\cdot,t)\|_{L^2(\Omega)}, \tag{158}$$

$$\big|\partial E[\mu_t]\big|_{\mathcal{V}_{\chi(\cdot,t)}} = \|\nabla\bar{u}(\cdot,t)\|_{L^2(\Omega)}. \tag{159}$$

Exploiting that the geometry underlying χ is smoothly evolving, i.e., $(\partial_t\chi)(\cdot,t) = -V_{\partial\mathscr{A}(t)} \cdot (\nabla\chi\llcorner\Omega)(\cdot,t)$, the claim (158) is a consequence of the identity

$$\begin{aligned}\int_\Omega \nabla\Delta_N^{-1}(\partial_t\chi)(\cdot,t)\cdot\nabla\phi\, dx &= -\langle(\partial_t\chi)(\cdot,t),\phi\rangle_{H^{-1}_{(0)},H^1_{(0)}} \\ &= -\int_{\partial^*\mathscr{A}(t)\cap\Omega} n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!]\phi\, d\mathcal{H}^{d-1} \\ &= \int_\Omega \nabla\bar{u}(\cdot,t)\cdot\nabla\phi\, dx\end{aligned}$$

valid for all $\phi \in H^1_{(0)}$, where in the process we again made use of (1b), (1a), (1d), and an integration by parts in the form of (2).

For a proof of (159), we first note that thanks to (155) and (1c) it holds

$$\begin{aligned}\delta E[\chi(\cdot,t)](B) &= -\int_{\partial^*\mathscr{A}(t)\cap\Omega} \bar{u}(\cdot,t) n_{\partial\mathscr{A}(t)}\cdot B\, d\mathcal{H}^{d-1} \\ &= \int_\Omega \nabla\bar{u}(\cdot,t)\cdot\nabla\Delta_N^{-1}(B\cdot\nabla\chi(\cdot,t))\, dx\end{aligned}$$

for all $B \in \mathcal{S}_{\chi(\cdot,t)}$. Hence, in view of (17m) it suffices to prove that for each fixed $t \in (0,T_*)$ there exists $\bar{B}(\cdot,t) \in \mathcal{S}_{\chi(\cdot,t)}$ such that $\bar{u}(\cdot,t) = \Delta_N^{-1}(\bar{B}(\cdot,t)\cdot\nabla\chi(\cdot,t))$.

To construct such a $\bar{B}$, first note that from (1a), (1d) and (2), we have $\Delta_N\bar{u}(\cdot,t) = (n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!])\mathcal{H}^{d-1}\llcorner(\partial^*\mathscr{A}(t)\cap\Omega)$ in the sense of distributions. Consequently, $\bar{B}(\cdot,t) \in C^1(\overline{\Omega};\mathbb{R}^d)$ satisfying

$$\begin{aligned} n_{\partial\mathscr{A}(t)}\cdot\bar{B}(\cdot,t) &= n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!] &&\text{on } \partial^*\mathscr{A}(t)\cap\Omega, \\ n_{\partial\Omega}\cdot\bar{B}(\cdot,t) &= 0 &&\text{on } \partial\Omega,\end{aligned} \tag{160}$$

for which one has (using (1a) and (3))

$$\int_\Omega \chi(\cdot,t)\nabla\cdot\bar{B}(\cdot,t)\, dx = -\int_{\partial\mathscr{A}(t)\cap\Omega} n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!]\, d\mathcal{H}^{d-1} = 0,$$

showing that $\bar{B}(\cdot,t) \in \mathcal{S}_{\chi(\cdot,t)}$, will satisfy the claim.

To this end, one first constructs a C^1 vector field $\mathcal{B}$ defined on $\overline{\partial\mathscr{A}(t)\cap\Omega}$ with the properties that $n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!] = n_{\partial\mathscr{A}(t)}\cdot\mathcal{B}$ and $\mathcal{B}\cdot n_{\partial\Omega} = 0$. Such $\mathcal{B}$ can be constructed by using a partition of unity on the manifold with boundary

$\overline{\partial\mathscr{A}(t)\cap\Omega}$. Away from the boundary, set $\mathcal{B} := (n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!])n_{\partial\mathscr{A}(t)}$, and near the boundary one can define $\mathcal{B}$ as an appropriate lifting of

$$\frac{(n_{\partial\mathscr{A}(t)}\cdot[\![\nabla\bar{u}(\cdot,t)]\!])}{\cos(\pi/2-\alpha)}\tau_{\partial\mathscr{A}(t)\cap\partial\Omega},$$

where $\tau_{\partial\mathscr{A}(t)\cap\partial\Omega}$ is the unit length vector field tangent to $\partial\Omega$ normal to the contact points manifold $\overline{\partial\mathscr{A}(t)\cap\Omega}\cap\partial\Omega$ and with $n_{\partial\mathscr{A}(t)}\cdot\tau_{\partial\mathscr{A}(t)\cap\partial\Omega} = \cos(\pi/2-\alpha)$. With the vector field $\mathcal{B}$ in place, any tangential $\bar{B}(\cdot,t)\in C^1(\overline{\Omega};\mathbb{R}^d)$ extending $\mathcal{B}$ satisfies (160), which in turn concludes the proof of the first step.

Step 2: Smooth varifold solutions are classical solutions. Let (χ,μ) be a varifold solution with smooth geometry, i.e., satisfying Definition 5 and such that the indicator χ can be represented as $\chi(x,t)=\chi_{\mathscr{A}(t)}(x)$, where $\mathscr{A}=(\mathscr{A}(t))_{t\in[0,T_*)}$ is a time-dependent family of smoothly evolving subsets $\mathscr{A}(t)\subset\Omega$, $t\in[0,T_*)$, as in the previous step. It is convenient for what follows to work with the two potentials u and w satisfying the conclusions of Lemma 3, i.e., (21)–(26).

Fix $t\in(0,T_*)$ such that (26) holds. By regularity of $\operatorname{supp}|\nabla\chi(\cdot,t)|\llcorner\Omega = \partial\mathscr{A}(t)\cap\Omega$ and the fact that $H_{\chi(\cdot,t)}$ is the generalized mean curvature vector of $\operatorname{supp}|\nabla\chi(\cdot,t)|\llcorner\Omega$ in the sense of Röger [51, Definition 1.1], it follows from Röger's result [51, Proposition 3.1] that

$$H_{\partial\mathscr{A}(t)} = H_{\chi(\cdot,t)} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\mathscr{A}(t)\cap\Omega. \tag{161}$$

Hence, we deduce from (26) that

$$w(\cdot,t)n_{\partial\mathscr{A}(t)} = c_0 H_{\partial\mathscr{A}(t)} \qquad \mathcal{H}^{d-1}\text{-a.e. on } \partial\mathscr{A}(t)\cap\Omega. \tag{162}$$

Recalling $w\in\mathcal{G}_\chi\subset\mathcal{H}_\chi$ and (49) shows that w is harmonic in $\mathscr{A}$ and in the interior of $\Omega\setminus\mathscr{A}$. One may then apply standard elliptic regularity theory for the Dirichlet problem [21] to obtain a continuous representative for w and further conclude that (162) holds everywhere on $\partial\mathscr{A}(t)\cap\Omega$.

Next, we take care of the contact angle condition (1e). To this end, we denote by $\tau_{\partial\mathscr{A}(t)\cap\Omega}$ a vector field on the contact points manifold, $\partial(\overline{\partial\mathscr{A}(t)\cap\Omega})\subset\partial\Omega$, that is tangent to the interface $\partial\mathscr{A}(t)\cap\Omega$, normal to the contact points manifold, and which points away from $\partial\mathscr{A}(t)\cap\Omega$. We further denote by $\tau_{\partial\mathscr{A}(t)\cap\partial\Omega}$ a vector field along the contact points manifold which now is tangent to $\partial\Omega$, again normal to the contact points manifold, and which this time points towards $\partial\mathscr{A}(t)\cap\partial\Omega$. Note that by these choices, at each point of the contact points manifold the vector fields $\tau_{\partial\mathscr{A}(t)\cap\Omega}$, $n_{\partial\mathscr{A}(t)}$, $\tau_{\partial\mathscr{A}(t)\cap\partial\Omega}$ and $n_{\partial\Omega}$ lie in the normal space of the contact points manifold, and that the orientations were precisely chosen such that $\tau_{\partial\mathscr{A}(t)\cap\Omega}\cdot\tau_{\partial\mathscr{A}(t)\cap\partial\Omega} = n_{\partial\mathscr{A}(t)}\cdot n_{\partial\Omega}$. With these constructions in place, we obtain from the classical first variation formula and an integration by parts along $\partial\mathscr{A}(t)\cap\Omega$ and $\partial\mathscr{A}(t)\cap\partial\Omega$ that

$$\begin{aligned}
&\delta E[\chi(\cdot,t)](B)\\
&= c_0\int_{\partial\mathscr{A}(t)\cap\Omega}\nabla^{\tan}\cdot B\,d\mathcal{H}^{d-1} + c_0(\cos\alpha)\int_{\partial\mathscr{A}(t)\cap\partial\Omega}\nabla^{\tan}\cdot B\,d\mathcal{H}^{d-1}\\
&= -c_0\int_{\partial\mathscr{A}(t)\cap\Omega} H_{\partial\mathscr{A}(t)}\cdot B\,d\mathcal{H}^{d-1}\\
&\quad + c_0\int_{\partial(\partial\mathscr{A}(t)\cap\Omega)}(\tau_{\partial\mathscr{A}(t)\cap\Omega} - (\cos\alpha)\tau_{\partial\mathscr{A}(t)\cap\partial\Omega})\cdot B\,d\mathcal{H}^{d-2}
\end{aligned}$$

$$\begin{aligned} &= -c_0 \int_{\partial\mathscr{A}(t)\cap\Omega} H_{\partial\mathscr{A}(t)} \cdot B \, d\mathcal{H}^{d-1} \\ &\quad + c_0 \int_{\partial(\partial\mathscr{A}(t)\cap\Omega)} (\tau_{\partial\mathscr{A}(t)\cap\Omega} \cdot \tau_{\partial\mathscr{A}(t)\cap\partial\Omega} - \cos\alpha)(\tau_{\partial\mathscr{A}(t)\cap\partial\Omega} \cdot B) \, d\mathcal{H}^{d-2} \end{aligned} \tag{163}$$

for all tangential variations $B \in C^1(\overline{\Omega}; \mathbb{R}^d)$.

Recall now that we assume that (17h) even holds with $\delta\mu_t$ replaced on the left hand side by $\delta E[\chi(\cdot,t)]$. In particular, $\delta\mu_t(B) = \delta E[\chi(\cdot,t)](B)$ for all $B \in \mathcal{S}_{\chi(\cdot,t)}$ so that the argument from the proof of Lemma 3, item ii), shows

$$\delta E[\chi(\cdot,t)](B) = -\int_{\partial^*\mathscr{A}(t)\cap\Omega} w(\cdot,t) n_{\partial\mathscr{A}(t)} \cdot B \, d\mathcal{H}^{d-1}$$

for all tangential $B \in C^1(\overline{\Omega}; \mathbb{R}^d)$. Because of (162), we thus infer from (163) that the contact angle condition (1e) indeed holds true.

Note that for each t in $(0, T^*)$, the potential u satisfies

$$\partial_t \chi(\cdot,t) = -V_{\partial\mathscr{A}(t)} \cdot (\nabla\chi \llcorner \Omega)(\cdot,t) = \Delta_N u.$$

In particular, by the assumed regularity of $\operatorname{supp} |\nabla\chi(\cdot,t)| \llcorner \Omega = \partial\mathscr{A}(t) \cap \Omega$,

$$\Delta u(\cdot,t) = 0 \qquad \text{in } \Omega \setminus \partial\mathscr{A}(t). \tag{164}$$

Furthermore, we claim that

$$\begin{aligned} V_{\partial\mathscr{A}(t)} &= -\big(n_{\partial\mathscr{A}(t)} \cdot [\![\nabla u(\cdot,t)]\!]\big) n_{\partial\mathscr{A}(t)} && \text{on } \partial\mathscr{A}(t) \cap \Omega, \\ (n_{\partial\Omega} \cdot \nabla) u(\cdot,t) &= 0 && \text{on } \partial\Omega \setminus \overline{\partial\mathscr{A}(t) \cap \Omega}. \end{aligned} \tag{165}$$

To prove (165), we suppress for notational convenience the time variable and show for any open set $\mathcal{O} \subset \Omega$ which does not contain the contact points manifold $\partial(\overline{\partial\mathscr{A}(t) \cap \Omega}) \subset \partial\Omega$ that

$$u \in H^3(\mathcal{O} \cap \mathscr{A}(t)) \cap H^3(\mathcal{O} \cap (\Omega \setminus \overline{\mathscr{A}(t)})). \tag{166}$$

With this, ∇u will have continuous representatives in $\overline{\mathscr{A}(t)}$ and $\overline{\Omega \setminus \mathscr{A}(t)}$ excluding contact points, from which (165) will follow by applying the integration by parts formula (2). Note that typical estimates apply for the Neumann problem if $\mathcal{O}$ does not intersect $\partial\mathscr{A}(t)$, and consequently to conclude (166), it suffices to prove regularity in the case of a flattened and translated interface $\partial\mathscr{A}(t)$ with u truncated, that is, for u satisfying

$$\begin{aligned} \Delta u &= -V \mathcal{H}^{d-1} \llcorner \{x_d = 0\} && \text{in } B(0,1), \\ u &= 0 && \text{on } \partial B(0,1), \end{aligned} \tag{167}$$

with V smooth. The above equation can be differentiated for all multi-indices $\beta \subset \mathbb{N}^{d-1}$ representing tangential directions, showing that $\partial^\beta u \in H^1(B(0,1))$. Rearranging (164) to extract $\partial_d^2 u$ from the Laplacian, we have that u belongs to $H^2(B(0,1) \setminus \{x_d = 0\})$. To control the higher derivatives, note by the comment regarding multi-indices, we already have $\partial_i \partial_j \partial_d u \in L^2(\Omega)$ for all $i, j \neq d$. Furthermore, differentiating (167) with respect to the i-th direction, where $i \in \{1, \ldots, d-1\}$, and repeating the previous argument shows $\partial_d^2 \partial_i u \in L^2(B(0,1) \setminus \{x_d = 0\})$. Finally, differentiating (164) with respect to the d-th direction away from both $\partial\mathscr{A}$ and $\partial\Omega$ and then extracting $\partial_d^3 u$ from $\Delta \partial_d u$, we get $\partial_d^3 u \in L^2(B(0,1) \setminus \{x_d = 0\})$, finishing the proof of (166).

Finally, we show

$$\|(\nabla u - \nabla w)(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{168}$$

Note that (168) is indeed sufficient to conclude that the smooth BV solution χ is a classical solution because we already established (162), (165), (164) and (1e). For a proof of (168), we simply exploit smoothness of the evolution in combination with (156), (162), (164), (165), (1e) and (2) to obtain

$$\frac{d}{dt}E[\chi(\cdot,t)] = -\int_\Omega \nabla u(\cdot,t)\cdot\nabla w(\cdot,t)\,dx.$$

Subtracting the previous identity (in integrated form) from (25) and noting that $E[\chi(\cdot,T)] \leq E[\mu_T]$ (due to the definitions (10) and (17l) as well as the compatibility conditions (17c) and (17d)), we get

$$0 \leq \int_0^T \int_\Omega \frac{1}{2}|\nabla u - \nabla w|^2\,dxdt \leq E[\chi(\cdot,T)] - E[\mu_T] \leq 0 \tag{169}$$

for a.e. $T\in(0,T_*)$, which in turn proves (168).

Note from (169) that $E[\chi(\cdot,T)] = E[\mu_T]$ for a.e. $T\in(0,T_*)$. For general constants $a+b=c+d$, $a\leq c$, and $b\leq d$ implies $a=c$ and $b=d$, and we use this with the coincidence of the varifold and BV energies to find that $|\nabla\chi(\cdot,t)| = |\mu_t^\Omega|\llcorner\Omega$ by (17c) and $(\cos\alpha)\chi(\cdot,t)\,\mathcal{H}^{d-1}\llcorner\partial\Omega = (\cos\alpha)|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}} + |\mu_t^\Omega|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega$ by (17d). This gives the first identity in (28). As $\mu_t^\Omega \in \mathrm{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ is an integer rectifiable oriented varifold, we consider the Radon–Nikodỳm derivative of both sides of (17d) with respect to $\mathcal{H}^{d-1}\llcorner\partial\Omega$ to find

$$(\cos\alpha)\chi(\cdot,t) = (\cos\alpha)\frac{|\mu_t^{\partial\Omega}|_{\mathbb{S}^{d-1}}}{\mathcal{H}^{d-1}\llcorner\partial\Omega}(\cdot,t) + m(\cdot,t),$$

for some integer-valued function $m\colon \partial\Omega\to\mathbb{N}\cup\{0\}$. Necessarily, $m\equiv 0$, concluding the second identity of (28) and (29).

Step 3: Compactness of solution space. It is again convenient to work explicitly with potentials. More precisely, for each $k\in\mathbb{N}$ we fix a potential w_k subject to item *ii)* of Lemma 3 with respect to the varifold solution (χ_k,μ_k). By virtue of (18) and (23), we have for all $k\in\mathbb{N}$ and all $0<s<T<T_*$

$$E[(\mu_k)_T] + \frac{1}{2}\int_s^T \big\|(\partial_t\chi_k)(\cdot,t)\big\|^2_{H^{-1}_{(0)}} + \|w_k(\cdot,t)\|^2_{L^2(\Omega)}\,dt \leq E[(\mu_k)_s] \leq E[\chi_{k,0}].$$

By assumption, we may select a subsequence $k\to\infty$ such that $\chi_{k,0} \overset{*}{\rightharpoonup} \chi_0$ in $BV(\Omega;\{0,1\})$ to some $\chi_0\in BV(\Omega;\{0,1\})$. Since we also assumed tightness of the sequence $(|\nabla\chi_{k,0}|\llcorner\Omega)_{k\in\mathbb{N}}$, it follows that along the previous subsequence we also have $|\nabla\chi_{k,0}|(\Omega)\to|\nabla\chi_0|(\Omega)$. In other words, $\chi_{k,0}$ converges strictly in $BV(\Omega;\{0,1\})$ along $k\to\infty$ to χ_0, which in turn implies convergence of the associated traces in $L^1(\partial\Omega;d\mathcal{H}^{d-1})$. In summary, we may deduce $E[\mu_{k,0}] = E[\chi_{k,0}] \to E[\chi_0] = E[\mu_0]$ for the subsequence $k\to\infty$.

For the rest of the argument, a close inspection reveals that one may simply follow the reasoning from *Step 2* to *Step 11* of the proof of Theorem 1 as these steps do not rely on the actual procedure generating the sequence of (approximate) solutions but only on consequences derived from the validity of the associated sharp energy dissipation inequalities. □

Proof of Proposition 5. We split the proof into three steps. In the first and second step, we develop estimates for an approximation of the $(d-1)$-density of the varifold using ideas introduced by Grüter and Jost [27, Proof of Theorem 3.1] (see also Kagaya and Tonegawa [33, Proof of Theorem 3.2]), that were originally used to derive monotonicity formula for varifolds with integrable curvature near a domain boundary. In the third step, we combine this approach with Schätzle's [55] work, which derived a monotonicity formula in the interior, to obtain a montonicity formula up to the boundary.

Step 1: Preliminaries. Since $\partial\Omega$ is compact and of class C^2, we may choose a localization scale $r = r(\partial\Omega) \in (0,1)$ such that $\partial\Omega$ admits a regular tubular neighborhood of width $2r$. More precisely, the map

$$\Psi_{\partial\Omega}\colon \partial\Omega \times (-2r, 2r) \to \{y \in \mathbb{R}^d\colon \operatorname{dist}(y, \partial\Omega) < 2r\}, \quad (x,s) \mapsto x + s\mathrm{n}_{\partial\Omega}(x)$$

is a C^1-diffeomorphism such that $\|\nabla\Psi_{\partial\Omega}\|_{L^\infty}, \|\nabla\Psi^{-1}_{\partial\Omega}\|_{L^\infty} \le C$. The inverse splits in form of $\Psi^{-1}_{\partial\Omega} = (P_{\partial\Omega}, s_{\partial\Omega})$, where $s_{\partial\Omega}$ represents the signed distance to $\partial\Omega$ oriented with respect to $\mathrm{n}_{\partial\Omega}$ and $P_{\partial\Omega}$ represents the nearest point projection onto $\partial\Omega$: $P_{\partial\Omega}(x) = x - s_{\partial\Omega}(x)\mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x)) = x - s_{\partial\Omega}(x)(\nabla s_{\partial\Omega})(x)$ for all $x \in \mathbb{R}^d$ such that $\operatorname{dist}(x, \partial\Omega) < 2r$.

In order to extend the argument of Schätzle [55, Proof of Lemma 2.1] up to the boundary of $\partial\Omega$, we employ the reflection technique of Grüter and Jost [27, Proof of Theorem 3.1]. To this end, we introduce further notation. First, we denote by

$$\tilde{x} := 2P_{\partial\Omega}(x) - x, \quad x \in \mathbb{R}^d \text{ such that } \operatorname{dist}(x, \partial\Omega) < 2r \tag{170}$$

the reflection of the point x across $\partial\Omega$ in normal direction. Further, we define the "reflected ball"

$$\begin{aligned}\tilde{B}_\rho(x_0) := \{x \in \mathbb{R}^d\colon \operatorname{dist}(x, \partial\Omega) < 2r,\, |\tilde{x} - x_0| < \rho\}, \\ \rho \in (0,r),\, x_0 \in \overline{\Omega}\colon \operatorname{dist}(x_0, \partial\Omega) < r.\end{aligned} \tag{171}$$

Finally, we set

$$\begin{aligned}\iota_x(y) := \big(\mathrm{Id} - \mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x)) \otimes \mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x))\big)y& \\ - \big(y \cdot \mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x))\big)\mathrm{n}_{\partial\Omega}(P_{\partial\Omega}(x)),& \\ y \in \mathbb{R}^d,\, x \in \mathbb{R}^d\colon \operatorname{dist}(x, \partial\Omega) < 2r,&\end{aligned} \tag{172}$$

which reflects a vector across the tangent space at $P_{\partial\Omega}(x)$ on $\partial\Omega$.

Step 2: A preliminary monotonicity formula. Let $\rho \in (0,r)$ and $x_0 \in \overline{\Omega}$ such that $\operatorname{dist}(x_0, \partial\Omega) < r$. Let $\eta\colon [0,\infty) \to [0,1]$ be smooth and nonincreasing such that $\eta \equiv 1$ on $[0, \frac{1}{2}]$, $\eta \equiv 0$ on $[1,\infty)$, and $|\eta'| \le 4$ on $[0,\infty)$. Consider then the functional

$$I_{x_0}(\rho) := \int_{\overline{\Omega}} \eta\left(\frac{|x - x_0|}{\rho}\right) + \tilde{\eta}_{x_0,\rho}(x)\, d|\mu|_{\mathbb{S}^{d-1}} \ge 0,$$

where $\tilde{\eta}_{x_0,\rho}\colon \overline{\Omega} \to [0,1]$ represents the C^1-function

$$\tilde{\eta}_{x_0,\rho}(x) := \begin{cases} \eta\big(\frac{|\tilde{x} - x_0|}{\rho}\big) & \text{if } \operatorname{dist}(x, \partial\Omega) < 2r, \\ 0 & \text{else.}\end{cases} \tag{173}$$

A close inspection of the argument given by Kagaya and Tonegawa [33, Estimate (4.11) and top of page 151] reveals that we have the bound

$$\begin{aligned} \frac{d}{d\rho}\big(\rho^{-(d-1)}I_{x_0}(\rho)\big) \geq -C\big(\rho^{1-(d-1)}I'_{x_0}(\rho) + \rho^{-(d-1)}I_{x_0}(\rho)\big) \\ - \rho^{-d}\int_{\overline{\Omega}} (\mathrm{Id} - s\otimes s) : \nabla\Phi_{x_0,\rho}\,d\mu, \end{aligned} \tag{174}$$

where the test vector field $\Phi_{x_0,\rho}$ is given by (recall the definition (173) of $\tilde\eta_{x_0,\rho}$)

$$\Phi_{x_0,\rho}(x) := \eta\left(\frac{|x-x_0|}{\rho}\right)(x-x_0) + \tilde\eta_{x_0,\rho}(x)\iota_x(\tilde x - x_0). \tag{175}$$

For any $q \in [d,\infty)$, we may further post-process (174) by an application of the chain rule to the effect of

$$\begin{aligned} &\frac{d}{d\rho}\big(\rho^{-(d-1)}I_{x_0}(\rho)\big)^{\frac{1}{q}} \\ &\geq -\frac{C}{q}\big(\rho^{-(d-1)}I_{x_0}(\rho)\big)^{\frac{1}{q}-1}\big(\rho^{1-(d-1)}I'_{x_0}(\rho) + \rho^{-(d-1)}I_{x_0}(\rho)\big) \\ &\quad - \frac{1}{q}\big(\rho^{-(d-1)}I_{x_0}(\rho)\big)^{\frac{1}{q}-1}\rho^{-d}\int_{\overline{\Omega}} (\mathrm{Id} - s\otimes s) : \nabla\Phi_{x_0,\rho}\,d\mu. \end{aligned} \tag{176}$$

Since we do not yet know that the generalized mean curvature vector field of the interface $\operatorname{supp}|\nabla\chi|\llcorner\Omega$ is q-integrable, we can not simply proceed as in [27, Proof of Theorem 3.1] or [33, Proof of Theorem 3.2] (at least with respect to the second right hand side term of the previous display).

To circumvent this technicality, define

$$f(\rho) := \max\{(\rho^{-(d-1)}I_{x_0}(\rho))^{1/q}, 1\}.$$

Noting that by choice of $q \geq d$ the first right-hand side term of (176) is bounded from below by the derivative of the product $\rho^{1-\frac{d-1}{q}}(I_{x_0}(\rho))^{1/q}$, we integrate (176) over an interval (σ,τ) where $(\rho^{-(d-1)}I_{x_0}(\rho))^{1/q} \geq 1$ to find

$$(1+\frac{C}{q}\tau)f(\tau) - (1+\frac{C}{q}\sigma)f(\sigma) \geq -\frac{1}{q}\int_\sigma^\tau \left|\rho^{-d}\int_{\overline{\Omega}} (\mathrm{Id} - s\otimes s) : \nabla\Phi_{x_0,\rho}\,d\mu\right|. \tag{177}$$

The same bound trivially holds, over intervals (σ,τ) where $f(\rho) = 1$, and consequently telescoping, we have the montonicity formula (177) for all $0 < \sigma < \tau \ll 1$.

Step 3: Local trace estimate for the chemical potential. We first post-process the preliminary monotonicity formula (177) by estimating the associated second right-hand side term involving the first variation. To this end, we recall for instance from [33, p. 147] that the test vector field $\Phi_{x_0,\rho}$ from (175) is tangential along $\partial\Omega$. In particular, it represents an admissible choice for testing (30):

$$\int_{\overline{\Omega}} (\mathrm{Id} - s\otimes s) : \nabla\Phi_{x_0,\rho}\,d\mu = \int_\Omega \chi\big(w(\nabla\cdot\Phi_{x_0,\rho}) + \Phi_{x_0,\rho}\cdot\nabla w\big)\,dx.$$

We distinguish between two cases. If $\rho < \operatorname{dist}(x_0,\partial\Omega)$, then $\tilde\eta_{x_0,\rho} \equiv 0$ and by plugging in (175) as well as the bounds for η

$$\left|\rho^{-d}\int_\Omega \chi\big(w(\nabla\cdot\Phi_{x_0,\rho}) + \Phi_{x_0,\rho}\cdot\nabla w\big)\,dx\right| \leq C\rho^{-d}\int_{\Omega\cap B_\rho(x_0)} \rho|\nabla w| + |w|\,dx.$$

If instead $\rho \geq \operatorname{dist}(x_0,\partial\Omega)$, straightforward arguments show

$$|\tilde\eta_{x_0,\rho}\iota_x(\tilde x - x_0)| \leq C\rho,$$

$$
\begin{aligned}
|\nabla\tilde{\eta}_{x_0,\rho}||\iota_x(\tilde{x}-x_0)| &\leq C,\\
|\nabla\iota_x(\tilde{x}-x_0)| &\leq C,\\
\operatorname{supp}\tilde{\eta}_{x_0,\rho} \subset \tilde{B}_\rho(x_0) &\subset B_{5\rho}(x_0),
\end{aligned}
$$

and therefore for $\rho \geq \operatorname{dist}(x_0,\partial\Omega)$

$$
\left|\rho^{-d}\int_\Omega \chi\big(w(\nabla\cdot\Phi_{x_0,\rho}) + \Phi_{x_0,\rho}\cdot\nabla w\big)\,dx\right| \leq C\rho^{-d}\int_{\Omega\cap B_{5\rho}(x_0)} \rho|\nabla w| + |w|\,dx.
$$

To control the right-hand side of the display we argue for dimension $d = 3$ and note that embeddings are stronger in dimension $d = 2$ (see also [55, Proof of Lemma 2.1]). Extending w in $H^1(\Omega)$ to a function in $H^1(\{x\colon \operatorname{dist}(x,\partial\Omega) < 2r\}\cup\Omega)$, we let $2^* = 6$ be the dimension dependent Sobolev exponent and apply Hölder's inequality to find

$$
\begin{aligned}
\rho^{-d}\int_{\Omega\cap B_{5\rho}(x_0)} \rho|\nabla w| + |w|\,dx &\leq \rho^{1-d/2}\|\nabla w\|_{L^2(B_{5\rho}(x_0))} + \rho^{-d/2^*}\|w\|_{L^{2^*}(B_{5\rho}(x_0))}\\
&\leq C\rho^{-1/2}\|w\|_{H^1(\Omega)},
\end{aligned}
\tag{178}
$$

as the exponents on ρ coincide.

Thus for all $0 < \rho < \frac{r}{5}$ due to the previous case study and estimate above

$$
\left|\rho^{-d}\int_{\overline{\Omega}} (\mathrm{Id} - s\otimes s) : \nabla\Phi_{x_0,\rho}\,d\mu\right| \leq C\|w\|_{H^1(\Omega)}\rho^{\beta-1}
\tag{179}
$$

for some $\beta \in (0,1)$ (accounting also for $d = 2$). Inserting (179) back into (177) finally yields that the function

$$
\rho \mapsto (1 + \frac{C}{q}\rho)\max\Big\{\big(\rho^{-(d-1)}I_{x_0}(\rho)\big)^{\frac{1}{q}}, 1\Big\} + C\beta^{-1}\|w\|_{H^1(\Omega)}\rho^\beta
$$

is nondecreasing in $(0, \frac{r}{5})$. In particular, since $\eta \equiv 1$ on $[0, \frac{1}{2}]$, we obtain one-sided Alhfor's regularity for the varifold as

$$
\begin{aligned}
\sup_{x_0\in\overline{\Omega}\colon \operatorname{dist}(x_0,\partial\Omega)<r}\ \sup_{0<\rho<\frac{r}{5}} \Big(\frac{\rho}{2}\Big)^{-(d-1)} &|\mu|_{\mathbb{S}^{d-1}}\big(\overline{\Omega}\cap B_{\frac{\rho}{2}}(x_0)\big)\\
&\leq C_{q,r,d}\big(1 + \max\big\{|\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}), \|w\|^q_{H^1(\Omega)}\big\}\big)
\end{aligned}
\tag{180}
$$

for some $C_{q,r,d} \geq 1$ for all $q \geq d$. The estimate (180) is sufficient to apply the trace theory (as in [45]) for the BV function $|w|^s$, and the asserted local estimate for the L^s-norm of the trace of the potential w on $\operatorname{supp}|\mu|_{\mathbb{S}^{d-1}}$ now follows as in Schätzle [55, Proof of Theorem 1.3]. □

Proof of Corollary 6. In view of the Gibbs–Thomson law (30) and by defining the Radon–Nikodỳm derivative $\rho^\Omega := \frac{c_0|\nabla\chi|\llcorner\Omega}{|\mu|_{\mathbb{S}^{d-1}}\llcorner\Omega} \in [0,1]$, we recall the fact that $H^\Omega := \rho^\Omega \frac{w}{c_0}\frac{\nabla\chi}{|\nabla\chi|} \in L^1(\overline{\Omega}, d|\mu|_{\mathbb{S}^{d-1}})$ represents the generalized mean curvature vector of μ with respect to tangential variations, i.e.,

$$
\delta\mu(B) = \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla B\,d\mu = -\int_{\overline{\Omega}} H^\Omega\cdot B\,d|\mu|_{\mathbb{S}^{d-1}}
$$

for all $B \in C^1(\overline{\Omega})$, $(B\cdot n_{\partial\Omega})|_{\partial\Omega} \equiv 0$. A recent result of De Masi [19, Theorem 1.1] therefore ensures that there exists $H^{\partial\Omega} \in L^\infty(\partial\Omega, d|\mu|_{\mathbb{S}^{d-1}})$ with the property that

$H^{\partial\Omega}(x) \perp \mathrm{Tan}_x\partial\Omega$ for $|\mu|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega$-a.e. $x \in \overline{\Omega}$, and a bounded Radon measure $\sigma_\mu \in \mathrm{M}(\partial\Omega)$ such that

$$\delta\mu(B) = -\int_{\overline{\Omega}} (H^{\Omega}+H^{\partial\Omega}) \cdot B \, d|\mu|_{\mathbb{S}^{d-1}} + \int_{\partial\Omega} B \cdot n_{\partial\Omega} \, d\sigma_\mu$$

for all "normal variations" $B \in C^1(\overline{\Omega})$ in the sense that $B(x) \perp \mathrm{Tan}_x\partial\Omega$ for all $x \in \partial\Omega$. There moreover exists a constant $C = C(\Omega) > 0$ (depending only on the second fundamental form of the domain boundary $\partial\Omega$) such that

$$\|H^{\partial\Omega}\|_{L^\infty(\partial\Omega, d|\mu|_{\mathbb{S}^{d-1}})} \leq C,$$
$$\sigma_\mu(\partial\Omega) \leq C|\mu|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \|H^{\Omega}\|_{L^1(\overline{\Omega}, d|\mu|_{\mathbb{S}^{d-1}})}.$$

In particular, by a splitting argument into "tangential" and "normal components" of a general variation $B \in C^1(\overline{\Omega})$, we deduce that the varifold μ is of bounded first variation in $\overline{\Omega}$ with representation (36). The asserted bounds (37)–(39) are finally consequences of the two bounds from the previous display, the representation of the first variation from (36), and the definition of H^{Ω}. □

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of Interest Statement

The authors have no relevant financial or non-financial interests to disclose.

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