CORRECTOR ESTIMATES FOR HIGHER-ORDER LINEARIZATIONS IN STOCHASTIC HOMOGENIZATION OF NONLINEAR UNIFORMLY ELLIPTIC EQUATIONS

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Abstract. Corrector estimates constitute a key ingredient in the derivation of optimal convergence rates via two-scale expansion techniques in homogenization theory of random uniformly elliptic equations. The present work follows up—in terms of corrector estimates—on the recent work of Fischer and Neukamm (arXiv:1908.02273) which provides a quantitative stochastic homogenization theory of nonlinear uniformly elliptic equations under a spectral gap assumption. We establish optimal-order estimates (with respect to the scaling in the ratio between the microscopic and the macroscopic scale) for higher-order linearized correctors. A rather straightforward consequence of the corrector estimates is the higher-order regularity of the associated homogenized monotone operator.

1. Introduction

Consider the setting of a monotone, uniformly elliptic and bounded PDE

$$-\nabla\cdot A\Big(\frac{x}{\varepsilon},\nabla u_\varepsilon\Big)=\nabla\cdot f\quad\text{in }\mathbb{R}^d,\quad f\in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d),\tag{1}$$

with $\varepsilon\ll 1$ denoting a microscale. We in addition assume that the monotone nonlinearity A is random (see Subsection 1.4 for a precise account on the assumptions of this work). The theory of nonlinear stochastic homogenization is then concerned with the behavior of the solutions to equation (1) in the limit $\varepsilon\to 0$.

If the monotone nonlinearity is sampled according to a stationary and ergodic probability distribution (which we will always assume), the classical qualitative prediction (see, e.g., [12] and [13]) consists of the convergence of u_ε to the solution u_{hom} of an effective nonlinear PDE

$$-\nabla\cdot A_{\mathrm{hom}}(\nabla u_{\mathrm{hom}})=\nabla\cdot f\quad\text{in }\mathbb{R}^d,\tag{2}$$

with A_{hom} being a monotone, uniformly elliptic and bounded operator. In the rigorous transition from the random model (1) to the deterministic effective model (2), next to the purely qualitative questions of convergence or the derivation of a homogenization formula for the effective operator A_{hom}, also quantitative aspects like the validity of convergence rates are obviously of interest.

For all of these questions in homogenization theory, the probably most fundamental concept is the notion of the homogenization corrector ϕ^ε_ξ, which for a constant macroscopic field gradient $\xi\in\mathbb{R}^d$ is given by the almost surely sublinearly growing solution of

$$-\nabla\cdot A\Big(\frac{x}{\varepsilon},\xi+\nabla\phi^\varepsilon_\xi\Big)=0\quad\text{in }\mathbb{R}^d.\tag{3}$$

For instance, by means of the homogenization correctors the homogenization formula for the effective operator reads as

$$A_{\mathrm{hom}}(\xi) = \Big\langle A\Big(\frac{x}{\varepsilon}, \xi+\nabla\phi^{\varepsilon}_{\xi}\Big)\Big\rangle, \tag{4}$$

which is well-defined as a consequence of stationarity of the underlying probability distribution. For quantitatively inclined questions like those concerned with the derivation of convergence rates, it is useful to introduce in addition a notion of flux correctors. For a given constant macroscopic field gradient $\xi \in \mathbb{R}^d$, the associated flux corrector σ_ξ is a random field with almost surely sublinear growth at infinity, taking values in the skew-symmetric matrices $\mathbb{R}^{d\times d}_{\mathrm{skew}}$, and solving

$$\nabla\cdot\sigma^{\varepsilon}_{\xi} = A\Big(\frac{x}{\varepsilon}, \xi+\nabla\phi^{\varepsilon}_{\xi}\Big) - A_{\mathrm{hom}}(\xi) \quad \text{in } \mathbb{R}^d. \tag{5}$$

The merit of the corrector pair $(\phi^{\varepsilon}_{\xi}, \sigma^{\varepsilon}_{\xi})$ is that it allows to represent, at least on a formal level, the error for the two-scale expansion $w_\varepsilon := u_{\mathrm{hom}}(x) + \phi^{\varepsilon}_{\xi}(x)|_{\xi=\nabla u_{\mathrm{hom}}(x)}$ in divergence form by means of first-order linearized correctors

$$\begin{aligned} &-\nabla\cdot A\Big(\frac{x}{\varepsilon}, \nabla w_\varepsilon\Big) \\ &= \nabla\cdot f - \nabla\cdot\Big(\big((a^{\varepsilon}_{\xi}\otimes\partial_\xi\phi^{\varepsilon}_{\xi})|_{\xi=\nabla u_{\mathrm{hom}}(x)} - \partial_\xi\sigma^{\varepsilon}_{\xi}|_{\xi=\nabla u_{\mathrm{hom}}(x)}\big) : \nabla^2 u_{\mathrm{hom}}\Big), \end{aligned} \tag{6}$$

where we also introduced the linearized coefficient field $a^{\varepsilon}_{\xi} := (\partial_\xi A)(\frac{x}{\varepsilon}, \xi+\nabla\phi^{\varepsilon}_{\xi})$. It is clear from the previous display that estimates on the corrector pair $(\phi^{\varepsilon}_{\xi}, \sigma^{\varepsilon}_{\xi})$ (and its first-order linearization) constitute a key ingredient in quantifying the convergence $u_\varepsilon \to u_{\mathrm{hom}}$. In the present nonlinear setting, we refer to the recent work of Fischer and Neukamm [18] where this program was carried out in the regime of a spectral gap assumption, resulting in homogenization error estimates being optimal in terms of scaling with respect to ε.

We establish in the present work optimal-order estimates (with respect to the scaling in ε) for higher-order linearized homogenization and flux correctors. Given a linearization order $L \in \mathbb{N}_0$ and a family of vectors $w_1, \ldots, w_L \in \mathbb{R}^d$, the Lth order linearized homogenization corrector is formally given by the directional derivative $\phi^{\varepsilon}_{\xi,w_1\odot\cdots\odot w_L} = (\partial_\xi\phi^{\varepsilon}_{\xi})[w_1\odot\cdots\odot w_L]$. Its defining PDE may be obtained by differentiating the nonlinear corrector problem (3) in the macroscopic variable $\xi \in \mathbb{R}^d$. In particular, note that $\phi^{\varepsilon}_{\xi,w_1\odot\cdots\odot w_L} = \varepsilon\phi_{\xi,w_1\odot\cdots\odot w_L}(\frac{\cdot}{\varepsilon})$ where $\phi_{\xi,w_1\odot\cdots\odot w_L}$ formally represents the Lth order directional derivative (in direction of $w_1\odot\cdots\odot w_L$) of the almost surely sublinearly growing solution of

$$-\nabla\cdot A(x, \xi+\nabla\phi_\xi) = 0 \quad \text{in } \mathbb{R}^d. \tag{7}$$

We then derive on the level of $\phi_{\xi,w_1\odot\cdots\odot w_L}$, amongst other things (cf. Theorem 1 for a more precise statement), corrector estimates of the form

$$\Big\langle\Big|\fint_{B_1(x_0)} \big|\phi_{\xi,w_1\odot\cdots\odot w_L}\big|^2\Big|^q\Big\rangle^{\frac{1}{q}} \lesssim_{L,q,|\xi|} |w_1|^2\cdots|w_L|^2\mu_*^2\big(1+|x_0|\big) \tag{8}$$

with the scaling function $\mu_* : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined by (20). This in turn implies

$$\Big\langle\Big|\fint_{B_\varepsilon(x_0)} \big|\phi^{\varepsilon}_{\xi,w_1\odot\cdots\odot w_L}\big|^2\Big|^q\Big\rangle^{\frac{1}{q}} \lesssim_{L,q,|\xi|} \varepsilon^2\mu_*^2\Big(\frac{1}{\varepsilon}\Big)|w_1|^2\cdots|w_L|^2\mu_*^2\big(1+|x_0|\big) \tag{9}$$

as is immediate from the scaling relation $\phi^{\varepsilon}_{\xi,w_1\odot\cdots\odot w_L} = \varepsilon\phi_{\xi,w_1\odot\cdots\odot w_L}(\frac{\cdot}{\varepsilon})$, a change of variables as well as (20). In the case $L = 1$, this recovers the optimal-order corrector estimates of [18]. As properties of $\phi_{\xi,w_1\odot\cdots\odot w_L}$ may always be translated into properties of $\phi^{\varepsilon}_{\xi,w_1\odot\cdots\odot w_L}$ based on their scaling relation, from Section 1.4 onwards we set $\varepsilon = 1$ and study higher-order linearizations of (7).

For a proof of corrector estimates of the form (8) in terms of higher-order linearized correctors, we devise a suitable inductive scheme to propagate corrector estimates from one linearization order to the next. The actual implementation of this inductive scheme, cf. Subsections 3.4–3.7 below, is in large parts directly inspired by the methods of Gloria, Neukamm and Otto [22]–[21], Fischer and Neukamm [18] as well as Josien and Otto [29]. Similar to the latter two works, we also employ a small-scale regularity assumption (see Assumption 3 below).

1.1. **Applications for corrector estimates of higher-order linearizations.**

The motivation for the present work derives from the expectation that estimates for higher-order linearized correctors constitute one of the important ingredients for open questions of interest in nonlinear stochastic homogenization, e.g., *i)* an optimal quantification of the commutability of homogenization and linearization (cf. [2] and [1] for suboptimal algebraic rates in the regime of finite range of dependence), or *ii)* the development of a nonlinear analogue of the theory of fluctuations as worked out for the linear case in [16], [15] and [14].

The former for instance concerns the study of the homogenization of the first-order linearized problem

$$-\nabla\cdot(\partial_\xi A)\Big(\frac{x}{\varepsilon},\nabla u_\varepsilon\Big)\nabla U^{(1)}_\varepsilon = \nabla\cdot f^{(1)} \quad \text{in } \mathbb{R}^d, \quad f^{(1)} \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d) \tag{10}$$

towards the linearized effective equation

$$-\nabla\cdot(\partial_\xi A_{\mathrm{hom}})(\nabla u_{\mathrm{hom}})\nabla U^{(1)}_{\mathrm{hom}} = \nabla\cdot f^{(1)} \quad \text{in } \mathbb{R}^d; \tag{11}$$

of course under appropriate regularity assumption for the nonlinearity. It is natural to define a two-scale expansion of $U^{(1)}_\varepsilon$ in terms of first-order linearized homogenization correctors $W^{(1)}_\varepsilon := U^{(1)}_{\mathrm{hom}} + (\partial_\xi\phi^\varepsilon_\xi)[\nabla U^{(1)}_{\mathrm{hom}}]\big|_{\xi=\nabla u_{\mathrm{hom}}}$, so that the difference $\nabla U^{(1)}_\varepsilon - \nabla W^{(1)}_\varepsilon$ formally satisfies a uniformly elliptic equation with fluctuating coefficient $(\partial_\xi A)(\frac{x}{\varepsilon},\nabla u_\varepsilon)$ and a right hand side, which—amongst other terms—in particular features second-order linearized homogenization (and flux) correctors. The estimates obtained in the present work therefore represent a key ingredient if one aims for a derivation of optimal-order convergence rates of the homogenization of (10) towards (11).

The second topic mentioned above concerns the study of the random fluctuations of several macroscopic observables of interest in homogenization theory, e.g.,

$$\int_{\mathbb{R}^d} F\cdot\nabla u_\varepsilon, \quad \int_{\mathbb{R}^d} F\cdot\nabla\phi^\varepsilon_\xi, \quad F\in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d). \tag{12}$$

In the works [16] and [15], Duerinckx, Gloria and Otto identified in the framework of linear stochastic homogenization $A(\cdot,\xi) = a(\cdot)\xi$ an object, the so-called standard homogenization corrector

$$\Xi_\xi := (a - a_{\mathrm{hom}})(\xi + \nabla\phi_\xi), \tag{13}$$

which relates the fluctuations of the corrector gradients with the fluctuations of the field ∇u_ε. That fluctuations are related in terms of a single object is by no means obvious as substituting naively a two-scale expansion for ∇u_ε in $\int_{\mathbb{R}^d} F \cdot \nabla u_\varepsilon$ does not characterize the fluctuations of $\int_{\mathbb{R}^d} F \cdot \nabla u_\varepsilon$ to leading order as observed in [27].

In a forthcoming work [28], we perform an intermediate step towards understanding the fluctuations of random variables of the form (12) in nonlinear settings. To this end, we introduce a nonlinear counterpart of the standard homogenization commutator (13) and derive a scaling limit result in a Gaussian setting (cf. [14]). As in the linear regime, this nonlinear counterpart of (13) also dictates the fluctuations of linear functionals of the corrector gradients $\nabla\phi_\xi$ (and their (higher-order) linearized descendants in terms of (higher-order) linearized homogenization commutators). The results of the work [28] are based, amongst other things, on estimates for higher-order linearized homogenization and flux correctors of the dual linearized operator $-\nabla \cdot a_\xi^* \nabla$ (cf. Section 2.4 below), where a_ξ^* denotes the transpose of the linearized coefficient field $a_\xi := (\partial_\xi A)(\omega, \xi + \nabla\phi_\xi)$.

1.2. **Stochastic homogenization of linear uniformly elliptic equations and systems.** Before we give a precise account of the underlying assumptions for the present work in Subsection 1.4, let us first briefly review the by-now substantial literature on the subject. The classical results in qualitative stochastic homogenization are due to Papanicolaou and Varadhan [37] and Kozlov [30], who studied heat conduction in a randomly heterogeneous medium under the assumption of stationarity and ergodicity (for the discrete setting, see [31] and [32]). The first result in quantitative stochastic homogenization is due to Yurinskii [39], who derived a suboptimal quantitative result for linear elliptic PDEs under a uniform mixing condition. Naddaf and Spencer [36] expressed mixing for the first time in the form of a spectral gap inequality, and as a result obtained optimal results for the fluctuations of the energy density of the corrector. Their work is however limited to small ellipticity contrast, see also Conlon and Naddaf [10] or Conlon and Fahim [11].

Extensions to the non-perturbative regime in the discrete setting were established through a series of articles by Gloria and Otto [23], [24] and [26], see also Gloria, Neukamm and Otto [20]. These works contain optimal estimates for the approximation error of the homogenized coefficients, the approximation error for the solutions, the corrector as well as the fluctuation of the energy density of the corrector under the assumption of i.i.d. conductivities. In the continuum setting and under spectral gap type assumptions, we refer to the works [22] and [21] of Gloria, Neukamm and Otto for optimal-order estimates in linear stochastic homogenization. Armstrong, Mourrat and Kuusi [3] establish these results in the finite range of dependence regime including also optimal stochastic integrability, see to this end also Gloria and Otto [25].

1.3. **Stochastic homogenization in nonlinear settings.** In the context of qualitative nonlinear stochastic homogenization, the first results are due to Dal Maso and Modica [12] and [13] in the setting of convex integral functionals. Lions and Souganidis [33] studied the homogenization of Hamilton–Jacobi equations under the qualitative assumptions of stationarity and ergodicity. Caffarelli, Souganidis and Wang [9] obtained stochastic homogenization in the context of nonlinear, uniformly elliptic equations in divergence form (see also Armstrong and Smart [5]). A homogenization result in the same framework but without assuming uniform ellipticity is due to Armstrong and Smart [6]. For an example of stochastic homogenization for nonlinear nonlocal equations, we refer to Schwab [38].

A first quantitative result in the context of nonlinear stochastic homogenization was established by Caffarelli and Souganidis [8], who succeeded in the derivation of a logarithmic-type convergence rate under strong mixing conditions. Substantial progress in the nonlinear setting was later provided by the works of Armstrong and Smart [7] on uniformly convex integral functionals, and Armstrong and Mourrat [4] on elliptic equations in divergence form with monotone coefficient fields. In the two recent works [2] resp. [1], Armstrong, Ferguson and Kuusi succeeded in proving that the processes of homogenization and (first-order resp. higher-order) linearization commute. Moreover, as it is also the case in the previously mentioned works of Armstrong et al., they derive quantitative estimates in terms of a suboptimal algebraic rate of convergence with respect to the ratio in the microscopic and macroscopic scale, assuming finite range of dependence for the underlying probability space. The established estimates, however, are optimal in terms of stochastic integrability. Under a spectral gap assumption, Fischer and Neukamm [18] recently provided quantitative homogenization estimates for monotone uniformly elliptic coefficient fields, which on one side are the first being optimal in the ratio between the microscopic and macroscopic scale, but which on the other side are non-optimal in terms of stochastic integrability.

1.4. **Assumptions and setting.** In this section, we give a precise account of the underlying assumptions for the present work. They represent the natural higher-order analogues of the assumptions from [18]. We start with the deterministic requirements on the family of monotone operators (cf. [18, Section 2.1]).

Assumption 1 (Family of monotone operators)**.** Let $d \in \mathbb{N}$ be the spatial dimension, and let $0 < \lambda \leq \Lambda < \infty$ be two constants (playing the role of ellipticity constants in the sequel). Let $n \in \mathbb{N}$ and $L \in \mathbb{N}_0$ be given. We then assume that we are equipped with a family of operators indexed by elements of $\mathbb{R}^n$

$$A\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$$

which is subject to the following three conditions:

(A1) The map A gives rise to a family of monotone operators in the second variable with lower bound λ. More precisely, for all $\tilde{\omega} \in \mathbb{R}^n$ we require

$$\big(A(\tilde{\omega},\xi_1)-A(\tilde{\omega},\xi_2)\big)\cdot(\xi_1-\xi_2) \geq \lambda|\xi_1-\xi_2|^2$$

for all $\xi_1,\xi_2 \in \mathbb{R}^d$. Furthermore, $A(\tilde{\omega},0)=0$ for all $\tilde{\omega} \in \mathbb{R}^n$.

$(A2)_L$ Each operator $A(\tilde{\omega},\cdot)$, $\tilde{\omega} \in \mathbb{R}^n$, is $L{+}1$ times continuously differentiable in the second variable. In quantitative terms, we assume that for all $\tilde{\omega} \in \mathbb{R}^n$

$$\sup_{k\in\{1,\ldots,L+1\}} \sup_{\xi\in\mathbb{R}^d} \big|(\partial_\xi^k A)(\tilde{\omega},\xi)\big| \leq \Lambda,$$

$$\sup_{k\in\{1,\ldots,L+1\}} \sup_{\xi\in\mathbb{R}^d} \sup_{\tilde{\omega}_1,\tilde{\omega}_2\in\mathbb{R}^n,\,\tilde{\omega}_1\neq\tilde{\omega}_2} \frac{|(\partial_\xi^k A)(\tilde{\omega}_1,\xi)-(\partial_\xi^k A)(\tilde{\omega}_2,\xi)|}{|\tilde{\omega}_1-\tilde{\omega}_2|} \leq \Lambda.$$

In particular, we have $|A(\tilde{\omega},\xi)| \leq \Lambda|\xi|$ for all $\tilde{\omega} \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^d$.

$(A3)_L$ For each $\xi \in \mathbb{R}^d$ and $k \in \{0,\ldots,L\}$, the map $\tilde{\omega} \mapsto (\partial_\xi^k A)(\tilde{\omega},\xi)$ is differentiable with uniformly Lipschitz continuous derivative. In quantitative terms, the following bounds are required to hold true for all $\tilde\omega \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^d$

$$|(\partial_\omega A)(\tilde\omega,\xi)| \leq \Lambda|\xi|, \qquad \sup_{k\in\{1,\ldots,L\}} |(\partial_\omega\partial_\xi^k A)(\tilde\omega,\xi)| \leq \Lambda,$$

$$\sup_{\tilde\omega_1,\tilde\omega_2\in\mathbb{R}^n,\,\tilde\omega_1\neq\tilde\omega_2} \frac{|(\partial_\omega A)(\tilde\omega_1,\xi) - (\partial_\omega A)(\tilde\omega_2,\xi)|}{|\tilde\omega_1-\tilde\omega_2|} \leq \Lambda|\xi|,$$

$$\sup_{k\in\{1,\ldots,L\}} \sup_{\tilde\omega_1,\tilde\omega_2\in\mathbb{R}^n,\,\tilde\omega_1\neq\tilde\omega_2} \frac{|(\partial_\omega\partial_\xi^k A)(\tilde\omega_1,\xi) - (\partial_\omega\partial_\xi^k A)(\tilde\omega_2,\xi)|}{|\tilde\omega_1-\tilde\omega_2|} \leq \Lambda.$$

For some results, we in addition require the following condition to be true.

(A4)$_L$ For each $\tilde\omega \in \mathbb{R}^n$, the maps $\xi \mapsto (\partial_\xi^{L+1} A)(\tilde\omega,\xi)$ and $\xi \mapsto (\partial_\omega\partial_\xi^L A)(\tilde\omega,\xi)$ are uniformly Lipschitz continuous. More precisely, for all $\tilde\omega \in \mathbb{R}^n$ we are equipped with bounds

$$\sup_{\xi_1,\xi_2\in\mathbb{R}^d,\,\xi_1\neq\xi_2} \frac{|(\partial_\xi^{L+1} A)(\tilde\omega,\xi_1) - (\partial_\xi^{L+1} A)(\tilde\omega,\xi_2)|}{|\xi_1-\xi_2|} \leq \Lambda.$$

$$\sup_{\xi_1,\xi_2\in\mathbb{R}^d,\,\xi_1\neq\xi_2} \frac{|(\partial_\omega\partial_\xi^L A)(\tilde\omega,\xi_1) - (\partial_\omega\partial_\xi^L A)(\tilde\omega,\xi_2)|}{|\xi_1-\xi_2|} \leq \Lambda.$$

Having the deterministic requirements on the family of monotone operators in place, we next turn to the probabilistic assumptions.

Assumption 2 (Stationarity and quantified ergodicity for probability distribution of parameter fields)**.** We call a measurable function $\omega\colon \mathbb{R}^d \to \mathbb{R}^n$ a *parameter field*, and denote by Ω the *space of parameter fields* with the $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$ topology. We assume that we are equipped with a probability measure $\mathbf{P}$ on Ω so that $\mathbf{P}[\{\omega\colon [\omega]_{C^\eta(B_1(x_0))} < \infty \text{ for all } x_0 \in \mathbb{R}^d\}] = 1$ for some $\eta \in (0,1)$, and which is subject to the following two further conditions:

(P1) The probability measure $\mathbf{P}$ on Ω is $\mathbb{R}^d$-stationary. In other words, the probability distributions of $\omega(\cdot)$ and $\omega(\cdot+z)$ coincide for all shifts $z \in \mathbb{R}^d$.

(P2) The probability measure $\mathbf{P}$ on Ω is ergodic. We in fact require a stronger condition in form of a *spectral gap inequality* as follows: Denoting with $\langle\cdot\rangle$ the expectation with respect to $\mathbf{P}$, there exists a constant $C_{\mathrm{sg}} > 0$ such that for all random variables $X \in \bigcap_{q\geq1} L^{2q}_{\langle\cdot\rangle}$ for which there exists $\partial^{\mathrm{fct}}X$ satisfying $[\partial^{\mathrm{fct}}X]_1 \in \bigcap_{q\geq1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n)$ as well as

$$\lim_{t\downarrow0} \frac{X(\omega+t\delta\omega) - X(\omega)}{t} = \int \delta\omega\cdot\partial^{\mathrm{fct}}X \quad \text{in } \bigcap_{q\geq1} L^{2q}_{\langle\cdot\rangle} \tag{14}$$

for all perturbations $\delta\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$, we then have the estimate

$$\Big\langle \big|X - \langle X\rangle\big|^2 \Big\rangle \leq C_{\mathrm{sg}}^2 \Big\langle \int \Big(\fint_{B_1(x)} \big|\partial^{\mathrm{fct}}X\big| \Big)^2 \Big\rangle. \tag{15}$$

We finally require a stronger form of the already stated small-scale regularity assumption $\mathbf{P}[\{\omega\colon [\omega]_{C^\eta(B_1(x_0))} < \infty \text{ for all } x_0 \in \mathbb{R}^d\}] = 1$, which in turn is essential to obtain optimal-order estimates (i.e., with respect to the ratio of the microscopic and macroscopic scale) for linearized homogenization and flux correctors, as well as their higher-order analogues.

Assumption 3 (Annealed small-scale regularity condition)**.** Let the conditions and notation of Assumption 2 be in place; in particular, let $\eta \in (0,1)$ be the associated Hölder continuity exponent. We then in addition require that

(R) There exist constants $C_{\mathrm{reg}}, C'_{\mathrm{reg}} > 0$ such that for all $q \in [1,\infty)$ it holds

$$\Big\langle \Big| \sup_{x,y\in B_1,\, x\neq y} \frac{|\omega(x)-\omega(y)|}{|x-y|^\eta} \Big|^{2q} \Big\rangle^{\frac{1}{q}} \leq C_{\mathrm{reg}}^2 q^{2C'_{\mathrm{reg}}}.$$

Note that our small-scale regularity condition is slightly weaker than the corresponding assumption in [18]. For this reason we provide a proof in Appendix A concerning the small-scale Hölder regularity of the (massive) corrector solving the nonlinear corrector problem (44a), see Lemma 23, which in turn implies small-scale Hölder regularity of the linearized coefficient field, see Lemma 24.

1.5. **Example.** We give an example for a random parameter field subject to Assumption 2 and Assumption 3. To this end, consider first a (for notational convenience 1-dimensional) white noise $W\colon L^2(\mathbb{R}^d) \to L^2(\Omega,\mathcal{F},\mathbf{P})$ constructed over some probability space $(\Omega,\mathcal{F},\mathbf{P})$. (We may assume without loss of generality that the σ-algebra $\mathcal{F}$ is the one generated by $W(h)$, $h \in L^2(\mathbb{R}^d)$.) In other words, $(W(h))_{h\in L^2(\mathbb{R}^d)}$ is a family of centered, real-valued Gaussian random variables such that $\langle W(h)W(g)\rangle = \int hg$.

Given some $c_0 \in L^2(\mathbb{R}^d)$ satisfying the following decay assumption on its (non-negative) Fourier transform $\widehat{c_0}$ for some $\alpha \in (0,1)$

$$0 \leq \widehat{c_0}(k) \leq C(1+|k|)^{-\frac{d+2\alpha}{2}}, \quad k \in \mathbb{R}^d,$$

we then define a stationary and centered Gaussian random field with bounded covariance function $c := c_0 * c_0^- : \mathbb{R}^d \to \mathbb{R}$, $c_0^-(x) := c_0(-x)$, by means of

$$\omega(x) := (c_0 * W)(x) := W\big(c_0(x-\cdot)\big).$$

Since the decay of $\widehat{c_0}$ translates into $0 \leq \widehat{c}(k) = \widehat{c_0}(k)\widehat{c_0}(-k) \leq C(1+|k|)^{-(d-2\alpha)}$, it is a well-known fact that, for any $\eta \in (0,\alpha)$, the Gaussian random field is η-Hölder continuous with probability one. In fact, one can show that Assumption 3 holds true. For a proof, see, e.g., [29, Lemma 3.1, Appendix A.3.1].

Moreover, the spectral gap inequality (15) is a consequence of Malliavin calculus associated with the underlying white noise W which can be seen as follows. Fix a square integrable $h \in L^2(\mathbb{R}^d)$, and define $\delta\omega := c_0 * h$. We have

$$\int \sup_{y\in B_1(x)} (\delta\omega)^2(y) \lesssim \int (1+|k|)^{d+2\alpha} |\widehat{\delta\omega}|^2 \,\mathrm{d}k \lesssim \int |\widehat{h}|^2 \,\mathrm{d}k < \infty,$$

where the first inequality is a consequence of a Sobolev embedding (see, e.g., [29, Appendix A.2]), whereas the second follows from $\widehat{\delta\omega} = \widehat{c_0}\,\widehat{h}$ together with the decay assumption on $\widehat{c_0}$. Furthermore, for any $x,y \in \mathbb{R}^d$ with $x \neq y$ we may estimate

$$\begin{aligned}
|\delta\omega(x)-\delta\omega(y)| \leq \int \big|e^{ikx}-e^{iky}\big|\big|\widehat{c_0}(-k)\big|\big|\widehat{h}\big| \,\mathrm{d}k &\lesssim \bigg(\int (1\wedge |k(x-y)|^2)\big|\widehat{c_0}(-k)\big|^2 \,\mathrm{d}k\bigg)^{\frac{1}{2}} \\
&\lesssim |x-y|^\alpha \bigg(\int (1\wedge |k|^2)|k|^{-d-2\alpha} \,\mathrm{d}k\bigg)^{\frac{1}{2}} \\
&\lesssim |x-y|^\alpha.
\end{aligned}$$

Hence, $\delta\omega = c_0 * h$ is an admissible test function for the condition of (14) for all $h \in L^2(\mathbb{R}^d)$. In other words, for any random variable $X \in \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle}$ satisfying (14) one recognizes the field $c_0^- * \partial^{\mathrm{fct}} X$ as its Malliavin derivative DX (since the fields $c_0 * h$, $h \in L^2(\mathbb{R}^d)$, represent precisely the elements of the Cameron–Martin space associated with the Gaussian measure on $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ induced by the Gaussian random field ω; cf. [34] for Malliavin calculus), which then also satisfies the estimate

$$\langle \|DX\|^2_{L^2(\mathbb{R}^d)}\rangle \lesssim \bigg\langle \int (1+|k|)^{-d-2\alpha} |\widehat{\partial^{\mathrm{fct}} X}|^2 \,\mathrm{d}k \bigg\rangle \lesssim \big\langle \big\|[\partial^{\mathrm{fct}} X]_1\big\|^2_{L^2(\mathbb{R}^d)}\big\rangle.$$

We thus arrived at the right hand side of (15). That the left hand side from the previous display is bounded from below by $\big\langle \big|X - \langle X\rangle\big|^2\big\rangle$ is finally nothing else but the well-known first-order Poincaré inequality on probability space (see, e.g., [17, Proposition 4.1, Appendix A]).

1.6. **Notation.** We denote by $\mathbb{N}$ the set of positive integers, and define $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. For given $d \in \mathbb{N}$, the space of real-valued $d\times d$ matrices is denoted by $\mathbb{R}^{d\times d}$. The transpose of a matrix $A \in \mathbb{R}^{d\times d}$ is given by A^*. We write $\mathbb{R}^{d\times d}_{\mathrm{skew}}$ for the space of skew-symmetric matrices $A^* = -A$. For a given $L \in \mathbb{N}$, we define $\mathrm{Par}\{1,\ldots,L\}$ to be the set of all partitions of $\{1,\ldots,L\}$. For any $x_0 \in \mathbb{R}^d$ and $R > 0$, we denote by $B_R(x_0) \subset \mathbb{R}^d$ the d-dimensional open ball of radius R centered at x_0. In case of $x_0 = 0$, we simply write B_R. In the rare occasion that the dimension of the ambient space is not represented by $d \in \mathbb{N}$ but, say, $n \in \mathbb{N}$, we emphasize the dimension of the ambient space by writing $B^n_R(x_0)$ for the n-dimensional open ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^n$.

The tensor product of vectors $v_1,\ldots,v_L \in \mathbb{R}^d$, $L \geq 2$, is denoted by $v_1 \otimes\cdots\otimes v_L$. For the symmetric tensor product, we write $v_1 \odot\cdots\odot v_L$. The L-fold tensor product of a vector $v \in \mathbb{R}^d$ is abbreviated as $v^{\otimes L}$; or $v^{\odot L}$ for the corresponding symmetric version. For a differentiable map $A\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$, $(\omega,\xi) \mapsto A(\omega,\xi)$, we make use of the usual notation $\partial_\omega A, \partial_\xi A$ for the respective partial derivatives. Higher-order (possibly mixed) partial derivatives of a map $A\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ are denoted by $\partial^l_\omega A, \partial^k_\xi A, \partial^l_\omega \partial^k_\xi A$ for any $k,l \in \mathbb{N}$.

Integrals $\int_{\mathbb{R}^d} f \,\mathrm{d}x$ with respect to the d-dimensional Lebesgue measure are abbreviated in the course of the paper as $\int f$. Given a Lebesgue-measurable subset $A \subset \mathbb{R}^d$ with finite and non-trivial Lebesgue measure $|A| \in (0,\infty)$, we denote by $\fint_A f := \frac{1}{|A|}\int \mathbb{1}_A f$ the average integral of f over A. Here, $\mathbb{1}_A$ represents the characteristic function with respect to a set A. For a probability measure $\mathbf{P}$ on a measure space $(\Omega,\mathcal{A})$, we write $\langle\cdot\rangle$ for the expectation with respect to $\mathbf{P}$.

We make use of the usual notation of Lebesgue and Sobolev spaces on $\mathbb{R}^d$ (with respect to the Lebesgue measure), e.g., $L^p(\mathbb{R}^d)$, $W^{1,p}(\mathbb{R}^d)$, $H^1(\mathbb{R}^d) := W^{1,2}(\mathbb{R}^d)$ and so on. For a probability measure $\mathbf{P}$ on a measure space $(\Omega,\mathcal{A})$, we instead use the notation $L^p_{\langle\cdot\rangle}$. If we want to emphasize the target space, say, a finite-dimensional real vector space V, we do so by writing $L^p(\mathbb{R}^d;V)$. For the Lebesgue resp. Sobolev spaces on $\mathbb{R}^d$ with only locally finite norm, we write $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and so on. Furthermore, in the case of uniformly locally finite norm, i.e.,

$$\sup_{x_0\in\mathbb{R}^d} \|f\|_{L^p(B_1(x_0))} < \infty \quad \text{resp.} \quad \sup_{x_0\in\mathbb{R}^d} \|f\|_{H^1(B_1(x_0))} < \infty$$

we reserve the notation $L^p_{\mathrm{uloc}}(\mathbb{R}^d)$ resp. $H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ for the corresponding subspaces of $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ resp. $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. The subspace of locally p-integrable functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$

satisfying

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}|f|^p\,\mathrm{d}x<\infty$$

is in turn denoted by $L^p_{\mathrm{erg}}(\mathbb{R}^d)$. Local L^p averages (with the obvious modification for $p=\infty$) will also be abbreviated as $[f]_p(x):=(\fint_{B_1(x)}|f|^p)^{\frac{1}{p}}$.

The space of all compactly supported and smooth functions on $\mathbb{R}^d$ is denoted by $C^\infty_{\mathrm{cpt}}(\mathbb{R}^d)$. For $\eta\in(0,1)$ and $x_0\in\mathbb{R}^d$, we further define the local Hölder seminorm $[f]_{C^\eta(B_1(x_0))}:=\sup_{x,y\in B_1(x_0),x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta}$ and the norm $\|f\|_{C^\eta(B_1(x_0))}:=\|f\|_{L^\infty(B_1(x_0))}+[f]_{C^\eta(B_1(x_0))}$. We say $f\in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d)$ if $\sup_{x_0\in\mathbb{R}^d}\|f\|_{C^\eta(B_1(x_0))}<\infty$. Finally, for an exponent $q\in[1,\infty]$, we write $q_*\in[1,\infty]$ for its dual Hölder exponent: $\frac{1}{q}+\frac{1}{q_*}=1$.

1.7. **Structure of the paper.** In the upcoming Section 2, we formulate the main results of the present work and provide definitions for the underlying key objects. Section 3 is devoted to a discussion of the strategy for the proof of the main results. In the course of it, we also collect several auxiliary results representing the main steps in the proof. Section 4 contains the proofs of all the main and auxiliary results as stated in the previous two sections. The paper finishes with three appendices. In Appendix A we list (and partly prove) several results from elliptic regularity theory. Most of them are classical results from deterministic theory. In addition, we also rely on some annealed regularity theory; however, only in a perturbative regime à la Meyers. Appendix B deals with existence of higher-order linearized correctors for a suitable class of parameter fields. Finally, as the proof of the main results proceeds by an induction over the linearization order, we formulate and prove in Appendix C the corresponding statements taking care of the base case of the induction.

2. Main results

This section collects the statements of the main results of this work which are twofold: *i)* corrector estimates for higher-order linearizations of the nonlinear problem, and *ii)* higher-order regularity of the homogenized monotone operator.

2.1. **Corrector bounds for higher-order linearized correctors.** The first main result constitutes the analogue (and slight extension) of [18, Corollary 15] for the higher-order linearized correctors of Definition 4.

Theorem 1 (Corrector estimates for higher-order linearizations)**.** *Let $L\in\mathbb{N}$ and $M>0$ be fixed. Let the requirements and notation of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Fix a set of vectors $w_1,\ldots,w_L\in\mathbb{R}^d$ and define $B:=w_1\odot\cdots\odot w_L$. Let*

$$\phi_{\xi,B}\in H^1_{\mathrm{loc}}(\mathbb{R}^d)\quad\textit{and}\quad\sigma_{\xi,B}\in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}})$$

be the linearized homogenization and flux corrector from Definition 4.

There exist $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$, $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ and $\alpha=\alpha(d,\lambda,\Lambda)\in(0,\eta)$ such that for all $|\xi|\le M$, all $q\in[1,\infty)$, all $x_0\in\mathbb{R}^d$, and *all compactly supported and square-integrable deterministic fields g_ϕ, g_σ it holds*

$$\Big\langle \Big| \Big(\int g_\phi \cdot \nabla\phi_{\xi,B}, \int g_\sigma^{kl} \cdot \nabla\sigma_{\xi,B,kl} \Big) \Big|^{2q} \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2 \int \big|(g_\phi, g_\sigma)\big|^2, \tag{16}$$

$$\big\langle \big\| (\nabla\phi_{\xi,B}, \nabla\sigma_{\xi,B}) \big\|_{L^2(B_1)}^{2q} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2, \tag{17}$$

$$\big\langle \big\| (\nabla\phi_{\xi,B}, \nabla\sigma_{\xi,B}) \big\|_{C^\alpha(B_1)}^{2q} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2, \tag{18}$$

$$\Big\langle \Big| \fint_{B_1(x_0)} \big|(\phi_{\xi,B}, \sigma_{\xi,B})\big|^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2 \mu_*^2\big(1+|x_0|\big), \tag{19}$$

with the scaling function $\mu_*\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined by

$$\mu_*(\ell) := \begin{cases} \ell^{\frac{1}{2}}, & d = 1, \\ \log^{\frac{1}{2}}(1+\ell), & d = 2, \\ 1, & d \geq 3. \end{cases} \tag{20}$$

Let $\xi_0 \in \mathbb{R}^d$ and $K \in \mathbb{N}_0$ be fixed, and assume in addition to the previous requirements that (A2)$_{L+K}$ and (A3)$_{L+K}$ from Assumption 1 hold true. We may then define $\mathbf{P}$-almost surely Kth-order Taylor expansions for the linearized homogenization and flux correctors with base point ξ_0 by means of

$$\Phi_{\xi_0,B}^K(\xi) := \phi_{\xi,B} - \sum_{k=0}^K \frac{1}{k!} (\partial_\xi \phi_{\xi_0,B})\big[(\xi-\xi_0)^{\odot k}\big] \in H^1_{\mathrm{loc}}(\mathbb{R}^d), \tag{21}$$

$$\Sigma_{\xi_0,B}^K(\xi) := \sigma_{\xi,B} - \sum_{k=0}^K \frac{1}{k!} (\partial_\xi \sigma_{\xi_0,B})\big[(\xi-\xi_0)^{\odot k}\big] \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{skew}}). \tag{22}$$

Under the stronger assumptions of (A2)$_{L+K+1}$ and (A3)$_{L+K+1}$ from Assumption 1, there exist constants $C = C(d, \lambda, \Lambda, C_{\mathrm{sg}}, C_{\mathrm{reg}}, \eta, M, L, K)$, $C' = C'(d, \lambda, \Lambda, C'_{\mathrm{reg}}, \eta, L)$ and $\alpha = \alpha(d, \lambda, \Lambda) \in (0, \eta)$ such that for all $|(\xi_0, \xi)| \leq M$, all $q \in [1, \infty)$, and all $x_0 \in \mathbb{R}^d$ it holds

$$\big\langle \big\| (\nabla\Phi_{\xi_0,B}^K(\xi), \nabla\Sigma_{\xi_0,B}^K(\xi)) \big\|_{L^2(B_1)}^{2q} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2 |\xi-\xi_0|^{2(K+1)}, \tag{23}$$

$$\big\langle \big\| (\nabla\Phi_{\xi_0,B}^K(\xi), \nabla\Sigma_{\xi_0,B}^K(\xi)) \big\|_{C^\alpha(B_1)}^{2q} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2 |\xi-\xi_0|^{2(K+1)}, \tag{24}$$

$$\Big\langle \Big| \fint_{B_1(x_0)} \big|(\Phi_{\xi_0,B}^K(\xi), \Sigma_{\xi_0,B}^K(\xi))\big|^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |B|^2 |\xi-\xi_0|^{2(K+1)} \mu_*^2\big(1+|x_0|\big). \tag{25}$$

2.2. **Differentiability of the homogenized operator.** A rather straightforward consequence of the estimates for higher-order linearized correctors is the higher-order regularity of the associated homogenized monotone operator.

Theorem 2 (Higher-order regularity of the homogenized operator)**.** *Let $L \in \mathbb{N}$ and $M > 0$ be fixed. Let the requirements and notation of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Fix next a set of vectors $w_1, \ldots, w_L \in \mathbb{R}^d$ and define $B := w_1 \odot \cdots \odot w_L$. Let finally*

$$\mathbb{R}^d \ni \xi \mapsto \bar{A}(\xi) := \langle q_\xi \rangle \tag{26}$$

be the homogenized operator, with the flux q_ξ being defined in (29b).

The homogenized operator is L times differentiable as a map $\xi \mapsto \bar{A}(\xi)$. There exists a constant $C = C(d, \lambda, \Lambda, C_{\mathrm{sg}}, C_{\mathrm{reg}}, C'_{\mathrm{reg}}, \eta, M, L)$ such that for all $|\xi| \leq M$ its Lth Gâteaux derivative in direction B admits the bound

$$\big|(\partial_\xi^L \bar{A})(\xi)[B]\big| \leq C|B|. \tag{27}$$

Finally, we have the following representation for the Lth order Gâteaux derivative in direction B

$$(\partial_\xi^L \bar{A})(\xi)[B] = \langle q_{\xi,B} \rangle. \tag{28}$$

Here, $q_{\xi,B}$ denotes the linearized flux from (30b).

2.3. **Basic definitions.** We introduce the precise definitions of the homogenization and flux correctors and their (higher-order) linearized analogues. We start by recalling these notions on the level of the nonlinear problem.

Given $\xi \in \mathbb{R}^d$, the equation for the homogenization corrector is given by

$$-\nabla \cdot A(\omega, \xi + \nabla\phi_\xi) = 0. \tag{29a}$$

Abbreviating the flux by means of

$$q_\xi := A(\omega, \xi + \nabla\phi_\xi), \tag{29b}$$

the equation for the corresponding flux corrector is given by

$$-\Delta\sigma_{\xi,kl} = (e_l \otimes e_k - e_k \otimes e_l) : \nabla q_\xi. \tag{29c}$$

Sublinear growth of the flux corrector gives rise to

$$q_\xi - \langle q_\xi \rangle = \nabla \cdot \sigma_\xi. \tag{29d}$$

Definition 3 (Homogenization correctors and flux correctors of the nonlinear problem)**.** Let the requirements and notation of (A1), $(A2)_0$ and $(A3)_0$ of Assumption 1, as well as (P1) and (P2) of Assumption 2 be in place. Let $\xi \in \mathbb{R}^d$ be given. The corresponding *homogenization corrector* ϕ_ξ and *flux corrector* σ_ξ are two random fields

$$(\phi_\xi, \sigma_\xi) \colon \Omega \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^{d\times d}_{\mathrm{skew}}$$

subject to the following list of requirements:

(i) It holds $\mathbf{P}$-almost surely that $\phi_\xi \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\sigma_\xi \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{skew}})$ as well as $\fint_{B_1} (\phi_\xi, \sigma_\xi) \,\mathrm{d}x = 0$. In addition, the associated PDEs (29a) resp. (29c) and (29d) are satisfied in the distributional sense $\mathbf{P}$-almost surely.

(ii) The gradients $\nabla\phi_\xi$ and $\nabla\sigma_\xi$ are stationary random fields. Moreover, it holds

$$\big\langle (\nabla\phi_\xi, \nabla\sigma_\xi) \big\rangle = 0, \quad \big\langle |\nabla\phi_\xi|^2 \big\rangle + \big\langle |\nabla\sigma_\xi|^2 \big\rangle < \infty.$$

(iii) The two random fields ϕ_ξ and σ_ξ feature $\mathbf{P}$-almost surely sublinear growth at infinity

$$\lim_{R\to\infty} \frac{1}{R^2} \fint_{B_R} \big|(\phi_\xi, \sigma_\xi)\big|^2 \,\mathrm{d}x = 0.$$

We next introduce the (higher-order) linearized analogues of the corrector equations (29a)–(29d) by formally differentiating in the macroscopic variable. To this end, let a linearization order $L \in \mathbb{N}$ be fixed. We also fix vectors $w_1, \ldots, w_L \in \mathbb{R}^d$ and let $B := w_1 \odot \cdots \odot w_L$. Finally, fix $\xi \in \mathbb{R}^d$ and denote by a_ξ the coefficient field $(\partial_\xi A)(\omega, \xi + \nabla\phi_\xi)$. Due to (A1) and $(A2)_0$ from Assumption 1, this coefficient field is uniformly elliptic and bounded with respect to the same constants (λ, Λ) from Assumption 1.

As suggested by the Faà di Bruno formula, the equation for the *Lth-order linearized homogenization corrector in direction B* shall be given by

$$\begin{aligned} &-\nabla \cdot a_\xi(\mathbb{1}_{L=1}B + \nabla\phi_{\xi,B}) \\ &= \nabla \cdot \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\} \\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi_{\xi,B'_\pi})\Big], \end{aligned} \tag{30a}$$

where we also introduced the notational convention

$$B'_\pi := \bigodot_{m\in\pi} v_m, \quad \forall\pi\in\Pi,\, \Pi\in\mathrm{Par}\{1,\ldots,L\}.$$

Note that the right hand side only features linearized correctors of order $\leq L-1$, if any. Motivated by this observation, existence of solutions to the linearized corrector problem (30a) with stationary gradient and (almost sure) sublinear growth at infinity will be given *inductively* through approximation with an additional massive term, see (49a) for the associated corrector problem. For the latter, solutions may be constructed—again inductively—on purely deterministic grounds (under suitable assumptions which are in particular modeled on the small-scale regularity condition (R) from Assumption 3). For more details, we refer the reader to the discussion in Section 3.2 below.

To state the equation for the linearized flux corrector, we first define the linearized flux by means of

$$\begin{aligned} q_{\xi,B} &:= a_\xi(\mathbb{1}_{L=1}B + \nabla\phi_{\xi,B}) \\ &\quad+ \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\} \\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi_{\xi,B'_\pi})\Big]. \end{aligned} \tag{30b}$$

The associated flux corrector shall then be a solution of

$$-\Delta\sigma_{\xi,B,kl} = (e_l\otimes e_k - e_k\otimes e_l) : \nabla q_{\xi,B}. \tag{30c}$$

Due to the sublinear growth of the correctors, the previous relations entail that

$$q_{\xi,B} - \langle q_{\xi,B}\rangle = \nabla\cdot\sigma_{\xi,B}. \tag{30d}$$

Definition 4 (Higher-order linearized homogenization correctors and flux correctors)**.** Let $L\in\mathbb{N}$ and $\xi\in\mathbb{R}^d$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_L$ and $(\mathrm{A3})_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. We also fix a set of vectors $w_1,\ldots,w_L\in\mathbb{R}^d$ and define $B := w_1\odot\cdots\odot w_L$. The corresponding *linearized homogenization corrector* $\phi_{\xi,B}$ and *flux corrector* $\sigma_{\xi,B}$ are two random fields

$$(\phi_{\xi,B},\sigma_{\xi,B})\colon \Omega\times\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^{d\times d}_{\mathrm{skew}}$$

subject to the following list of requirements:

(i) It holds $\mathbf{P}$-almost surely that $\phi_{\xi,B}\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\sigma_{\xi,B}\in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}})$ as well as $\fint_{B_1}(\phi_{\xi,B},\sigma_{\xi,B})\,\mathrm{d}x = 0$. In addition, the associated PDEs (30a) resp. (30c) and (30d) are satisfied in the distributional sense $\mathbf{P}$-almost surely.

(ii) The gradients $\nabla\phi_{\xi,B}$ and $\nabla\sigma_{\xi,B}$ are stationary random fields. Moreover, it holds

$$\big\langle(\nabla\phi_{\xi,B},\nabla\sigma_{\xi,B})\big\rangle = 0, \quad \big\langle|\nabla\phi_{\xi,B}|^2\big\rangle + \big\langle|\nabla\sigma_{\xi,B}|^2\big\rangle < \infty.$$

(iii) The two random fields $\phi_{\xi,B}$ and $\sigma_{\xi,B}$ feature **P**-almost surely sublinear growth at infinity

$$\lim_{R\to\infty}\frac{1}{R^2}\fint_{B_R}\big|(\phi_{\xi,B},\sigma_{\xi,B})\big|^2\,\mathrm{d}x = 0.$$

2.4. **Linearized correctors for the dual linearized operator.** It is an immediate consequence of the proofs that analogous results hold true for the (higher-order) correctors of the dual linearized operator $-\nabla\cdot a_\xi^*\nabla$, where a_ξ^* denotes the transpose of the linearized coefficient field $a_\xi := (\partial_\xi A)(\omega,\xi{+}\nabla\phi_\xi)$. We state these corrector results for the dual linearized operator for ease of reference for future works.

Let $L\in\mathbb{N}$ and $M>0$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_L$ and $(\mathrm{A3})_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Fix moreover a set of vectors $w_1,w_2,\dots,w_L\in\mathbb{R}^d$ and define $B := w_1\otimes(w_2\odot\cdots\odot w_L)$. For a partition $\Pi\in\mathrm{Par}\{1,\dots,L\}$ with $\Pi\neq\{\{1,\dots,L\}\}$, denote by π_1^* the unique element $\pi\in\Pi$ such that $1\in\pi$. The equation for the *Lth-order linearized homogenization corrector in direction B of the dual linearized operator* is then given by

$$\begin{aligned} &-\nabla\cdot a_\xi^*(\mathbb{1}_{L=1}B+\nabla\phi^*_{\xi,B}) && \text{(31a)}\\ &=\nabla\cdot\sum_{\substack{\Pi\in\mathrm{Par}\{1,\dots,L\}\\ \Pi\neq\{\{1,\dots,L\}\}}}\partial_\xi^{|\Pi|-1}\,\,(a_\xi^* v)\big|_{v=\mathbb{1}_{|\pi_1^*|=1}B'_{\pi_1^*}+\nabla\phi^*_{\xi,B'_{\pi_1^*}}}\Big[\bigodot_{\substack{\pi\in\Pi\\1\notin\pi}}\big(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi_{\xi,B'_\pi}\big)\Big], \end{aligned}$$

where we also relied, for each $\Pi\in\mathrm{Par}\{1,\dots,L\}$, on the notational convention

$$B'_\pi := \begin{cases} v_1\otimes(v_{l_2}\odot\cdots\odot v_{l_{|\pi|}}), & \text{if } \pi=\pi_1^*=\{1,l_2,\dots,l_{|\pi|}\} \text{ with } l_2<\cdots<l_{|\pi|},\\ \bigodot_{m\in\pi}v_m, & \text{else.}\end{cases}$$

With the dual linearized flux given by

$$\begin{aligned} q^*_{\xi,B} &:= a_\xi^*(\mathbb{1}_{L=1}B+\nabla\phi^*_{\xi,B}) && \text{(31b)}\\ &\quad+\sum_{\substack{\Pi\in\mathrm{Par}\{1,\dots,L\}\\ \Pi\neq\{\{1,\dots,L\}\}}}\partial_\xi^{|\Pi|-1}\,\,(a_\xi^* v)\big|_{v=\mathbb{1}_{|\pi_1^*|=1}B'_{\pi_1^*}+\nabla\phi^*_{\xi,B'_{\pi_1^*}}}\Big[\bigodot_{\substack{\pi\in\Pi\\1\notin\pi}}\big(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi_{\xi,B'_\pi}\big)\Big], \end{aligned}$$

the *Lth order linearized flux corrector in direction B of the dual linearized operator* is in turn a solution of

$$-\Delta\sigma^*_{\xi,B,kl} = (e_l\otimes e_k - e_k\otimes e_l):\nabla q^*_{\xi,B}, \tag{31c}$$

as well as

$$q^*_{\xi,B}-\langle q^*_{\xi,B}\rangle = \nabla\cdot\sigma^*_{\xi,B}. \tag{31d}$$

(More precisely, the notion of linearized homogenization and flux correctors of the dual linearized problem are understood in the precise sense of Definition 4, with the equations (30a)–(30d) replaced by the equations (31a)–(31d)).

Under the above assumptions, the following analogous results to Theorem 1 then hold true. First, there exists constants $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$ and

$C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ such that for all $|\xi|\leq M$, all $q\in[1,\infty)$, all $x_0\in\mathbb{R}^d$, and all compactly supported and square-integrable g_ϕ, g_σ it holds

$$\bigg\langle\bigg|\bigg(\int g_\phi\cdot\nabla\phi^*_{\xi,B},\int g^{kl}_\sigma\cdot\nabla\sigma^*_{\xi,B,kl}\bigg)\bigg|^{2q}\bigg\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}|B|^2\int\big|(g_\phi,g_\sigma)\big|^2, \tag{32}$$

$$\big\langle\big\|(\nabla\phi^*_{\xi,B},\nabla\sigma^*_{\xi,B})\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}} \leq C^2q^{2C'}|B|^2, \tag{33}$$

$$\big\langle\big\|(\nabla\phi^*_{\xi,B},\nabla\sigma^*_{\xi,B})\big\|^{2q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q}} \leq C^2q^{2C'}|B|^2, \tag{34}$$

$$\bigg\langle\bigg|\fint_{B_1(x_0)}\big|(\phi^*_{\xi,B},\sigma^*_{\xi,B})\big|^2\bigg|^q\bigg\rangle^{\frac{1}{q}} \leq C^2q^{2C'}|B|^2\mu_*^2\big(1+|x_0|\big), \tag{35}$$

with the scaling function $\mu_*\colon\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ defined in (20).

Fix $K\in\mathbb{N}$, and assume that (A2)$_{L+K}$ and (A3)$_{L+K}$ from Assumption 1 hold true on top of the previous assumptions of this section. Then, both the maps $\xi\mapsto\nabla\phi^*_{\xi,B}$ and $\xi\mapsto\nabla\sigma^*_{\xi,B}$ are $\mathbf{P}$-almost surely K times Gâteaux differentiable with values in the Fréchet space $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, for any collection of vectors $w_{L+1},\ldots,w_{L+K}\in\mathbb{R}^d$ and any $k\in\{1,\ldots,K\}$ we have the following representations of the kth order Gâteaux derivatives in direction $\hat{B}_k := w_{L+1}\odot\cdots\odot w_{L+k}$:

$$(\partial^k_\xi\phi^*_{\xi,B})[\hat{B}_k] = \phi^*_{\xi,w_1\otimes(w_2\odot\cdots\odot w_{L+k})},\quad (\partial^k_\xi\sigma^*_{\xi,B})[\hat{B}_k] = \sigma^*_{\xi,w_1\otimes(w_2\odot\cdots\odot w_{L+k})}. \tag{36}$$

Denote by $\overline{a^*_\xi}\in\mathbb{R}^{d\times d}$ the *homogenized coefficient of the dual linearized operator* characterized by

$$\overline{a^*_\xi}w = \langle q^*_{\xi,w}\rangle,\quad w\in\mathbb{R}^d.$$

Then the following version of Theorem 2 holds true for $L\geq 1$. The map $\xi\mapsto\overline{a^*_\xi}$ is $L{-}1$ times differentiable. There exists $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},C'_{\mathrm{reg}},\eta,M,L)$ such that for all $|\xi|\leq M$ its $(L{-}1)$th Gâteaux derivative in the direction of $B' = w_2\odot\cdots\odot w_L$ admits the bound

$$\big|\big(\partial^{L-1}_\xi\overline{a^*_\xi}\big)[B']\big|\leq C|B'|. \tag{37}$$

We finally have for all $w\in\mathbb{R}^d$ the following representation for the $(L{-}1)$th order Gâteaux derivative in direction B'

$$\big(\partial^{L-1}_\xi\big(\overline{a^*_\xi}w\big)\big)[B'] = \langle q^*_{\xi,w\otimes B'}\rangle. \tag{38}$$

3. Outline of strategy

The proof of the corrector bounds from Theorem 1 is based on the massive approximation of the operator $-\nabla\cdot(\partial_\xi A)(\omega,\xi+\nabla\phi_\xi)$. For the problem with an additional massive term, we will argue by an induction with respect to the order of the linearization. This will entail the following analogue of Theorem 1 in terms of the massive approximation.

Theorem 5 (Estimates for massive correctors)**.** *Let $L\in\mathbb{N}$, $M>0$ as well as $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Fix a set of unit vectors $v_1,\ldots,v_L\in\mathbb{R}^d$ and define $B := v_1\odot\cdots\odot v_L$. Let*

$$\phi^T_{\xi,B}\in H^1_{\mathrm{uloc}}(\mathbb{R}^d),\quad \sigma^T_{\xi,B}\in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}})\quad\textit{and}\quad \psi^T_{\xi,B}\in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^d)$$

denote the unique solutions of the linearized corrector problems (49a)–(49d)*, which* $\mathbf{P}$*-almost surely exist by means of Lemma 8 below.*

There exist $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$*,* $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ *and* $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ *such that for all* $|\xi| \leq M$*, all* $q \in [1,\infty)$*, and all compactly supported and square-integrable fields* g_ϕ, g_σ, g_ψ *resp.* f_ϕ, f_σ, f_ψ *it holds*

$$\begin{aligned}&\Big\langle\Big|\Big(\int g_\phi\cdot\nabla\phi^T_{\xi,B},\int g^{kl}_\sigma\cdot\nabla\sigma^T_{\xi,B,kl},\int g^k_\psi\cdot\frac{\nabla\psi^T_{\xi,B,k}}{\sqrt{T}}\Big)\Big|^{2q}\Big\rangle^{\frac{1}{q}}\\&\leq C^2q^{2C'}\int\big|(g_\phi,g_\sigma,g_\psi)\big|^2,\end{aligned}\tag{39}$$

and

$$\begin{aligned}&\Big\langle\Big|\Big(\int \frac{1}{T}f_\phi\phi^T_{\xi,B},\int \frac{1}{T}f^{kl}_\sigma\sigma^T_{\xi,B,kl},\int \frac{1}{T}f^k_\psi\frac{\psi^T_{\xi,B,k}}{\sqrt{T}}\Big)\Big|^{2q}\Big\rangle^{\frac{1}{q}}\\&\leq C^2q^{2C'}\int\frac{1}{T}\big|(f_\phi,f_\sigma,f_\psi)\big|^2,\end{aligned}\tag{40}$$

as well as

$$\Big\langle\Big\|\Big(\nabla\phi^T_{\xi,B},\nabla\sigma^T_{\xi,B},\frac{\nabla\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} \leq C^2q^{2C'},\tag{41}$$

$$\Big\langle\Big\|\Big(\nabla\phi^T_{\xi,B},\nabla\sigma^T_{\xi,B},\frac{\nabla\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \leq C^2q^{2C'},\tag{42}$$

$$\Big\langle\Big|\fint_{B_1}\Big(\phi^T_{\xi,B},\sigma^T_{\xi,B},\frac{\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} \leq C^2q^{2C'}\mu_*^2(\sqrt{T}),\tag{43}$$

with the scaling function $\mu_*\colon\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ *given in* (20)*. Moreover, the relation* (49e) *holds true.*

As an input for the base case of the induction we will take the localized corrector of the nonlinear problem. So let us start by quickly reviewing the corresponding results from [18].

3.1. **Corrector estimates for the nonlinear PDE: A brief review.**

Given $\xi \in \mathbb{R}^d$ and $T \in [1,\infty)$, the equation for the localized homogenization corrector is given by

$$\frac{1}{T}\phi^T_\xi - \nabla\cdot A(\omega,\xi{+}\nabla\phi^T_\xi) = 0.\tag{44a}$$

Abbreviating the flux by means of

$$q^T_\xi := A(\omega,\xi{+}\nabla\phi^T_\xi),\tag{44b}$$

the equation for the corresponding localized flux corrector is given by

$$\frac{1}{T}\sigma^T_{\xi,kl} - \Delta\sigma^T_{\xi,kl} = (e_l\otimes e_k - e_k\otimes e_l) : \nabla q^T_\xi.\tag{44c}$$

Moreover, we introduce an auxiliary localized corrector by means of

$$\frac{1}{T}\psi^T_\xi - \Delta\psi^T_\xi = q^T_\xi - \langle q^T_\xi\rangle - \nabla\phi^T_\xi.\tag{44d}$$

The motivation behind the introduction of the auxiliary corrector ψ_ξ^T is to mimic equation (29d) for the flux correction at the level of the massive approximation:

$$q_\xi^T - \langle q_\xi^T \rangle = \nabla \cdot \sigma_\xi^T + \frac{1}{T}\psi_\xi^T. \tag{44e}$$

We then have the following result, which was essentially proven by Fischer and Neukamm [18]. For a proof of those facts which are not explicitly spelled out in [18], we refer to the beginning of Appendix C.

Proposition 6 (Estimates for localized homogenization correctors of the nonlinear problem)**.** *Let the requirements and notation of (A1), (A2)$_0$ and (A3)$_0$ of Assumption 1, as well as (P1) and (P2) of Assumption 2 be in place. Let $T \in [1,\infty)$ be fixed, and for any $\xi \in \mathbb{R}^d$ let*

$$\Big(\frac{\phi_\xi^T}{\sqrt{T}}, \nabla\phi_\xi^T\Big) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$$

denote the unique solution of the localized corrector problem (44a). The localized homogenization corrector ϕ_ξ^T then admits the following list of estimates:

- *There exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}}) > 0$ and $C' = C'(d,\lambda,\Lambda) > 0$ such that for all $q \in [1,\infty)$, and all compactly supported and square-integrable f, g we have* corrector estimates

$$\begin{aligned} \Big\langle \Big| \Big(\int g \cdot \nabla\phi_\xi^T, \int \frac{1}{T} f \phi_\xi^T \Big) \Big|^{2q} \Big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'} |\xi|^2 \int \Big| \Big(g, \frac{f}{\sqrt{T}} \Big) \Big|^2, \\ \Big\langle \Big\| \Big(\frac{\phi_\xi^T}{\sqrt{T}}, \nabla\phi_\xi^T \Big) \Big\|_{L^2(B_1)}^{2q} \Big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'} |\xi|^2. \end{aligned} \tag{45}$$

- *Let $g, f \in \bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$ be two compactly supported random fields. There then exists a random field G_ξ^T satisfying $\big[G_\xi^T\big]_1 \in \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n)$, which in addition is related to (g,f) via ϕ_ξ^T in the sense that, $\mathbf{P}$-almost surely, it holds for all perturbations $\delta\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$*

$$\int g \cdot \nabla\delta\phi_\xi^T - \int f \frac{1}{T} \delta\phi_\xi^T = \int G_\xi^T \cdot \delta\omega. \tag{46a}$$

Here, $(\frac{\delta\phi_\xi^T}{\sqrt{T}}, \nabla\delta\phi_\xi^T) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$ denotes the Gâteaux derivative of the corrector ϕ_ξ^T and its gradient in direction $\delta\omega$, cf. the proof of Lemma 26.

Moreover, there exists $c_0 = c_0(d,\lambda,\Lambda) \in (1,\infty)$ such that for any $\kappa \in (0,1]$ there exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\kappa) > 0$ and $C' = C'(d,\lambda,\Lambda) > 0$ such that for all $q \in [1,\infty)$ and all $q_0 \in [\frac{c_0}{c_0-1},\infty)$ the random field G_ξ^T gives rise to a sensitivity estimate

$$\begin{aligned} &\Big\langle \Big| \int \Big(\fint_{B_1(x)} |G_\xi^T| \Big)^2 \Big|^q \Big\rangle^{\frac{1}{q}} \\ &\leq C^2 (q\vee q_0)^{2C'} |\xi|^2 \sup_{\langle F^{2q_*}\rangle = 1} \int \Big\langle \Big| \Big(Fg, \frac{Ff}{\sqrt{T}} \Big) \Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}. \end{aligned} \tag{46b}$$

If $(g_r, f_r)_{r\geq 1}$ is a sequence in $\bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$ of compactly supported random fields, denote by $G_\xi^{T,r}$, $r \geq 1$, the random field associated to (g_r, f_r), $r \geq 1$, *in the sense of* (46a)*. Let* (g,f) *be two random fields such that* $(g_r,f_r)\to(g,f)$ *as* $r\to\infty$ *in* $\bigcap_{q\geq 1}L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$*. Then there exists a random field* G^T_ξ *such that*

$$\big[G^{T,r}_\xi - G^T_\xi\big]_1 \to 0 \text{ as } r\to\infty \text{ in } \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle}L^2(\mathbb{R}^d;\mathbb{R}^n), \tag{46c}$$

and the limit random field G^T_ξ *satisfies the sensitivity estimate* (46b)*.*

- *Let in addition to the above requirements the condition (R) of Assumption 3 be in place, and let* $M>0$ *be fixed. There exist* $\alpha=\alpha(d,\lambda,\Lambda)\in(0,\eta)$ *as well as constants* $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)>0$ *and* $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta)>0$ *such that for all* $q\in[1,\infty)$ *and all* $|\xi|\leq M$ *we have a* small-scale annealed Schauder estimate *in form of*

$$\Big\langle\big\|\nabla\phi^T_\xi\big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}. \tag{47}$$

Before we move on with the statement of the induction hypotheses for corrector bounds of higher-order linearizations, we register for reference purposes the following standard consequence of the spectral gap inequality (15). It constitutes the key ingredient for the reduction of stochastic moment bounds to sensitivity estimates.

Lemma 7. *Let the conditions and notation of Assumption 2 be in place, and let* $X\in\bigcap_{q\geq 1}L^{2q}_{\langle\cdot\rangle}$ *be subject to* (14)*. Then, there exists* $C_{\mathrm{sg}}>0$ *such that for all* $q\geq 1$

$$\Big\langle\big|X-\langle X\rangle\big|^{2q}\Big\rangle^{\frac{1}{q}} \leq C^2_{\mathrm{sg}}q^2\bigg\langle\bigg|\int\bigg(\fint_{B_1(x)}\big|\partial^{\mathrm{fct}}X\big|\bigg)^2\bigg|^q\bigg\rangle^{\frac{1}{q}}. \tag{48}$$

3.2. **Corrector bounds for higher-order linearizations: the induction hypotheses.** Let $L\in\mathbb{N}$ and $T\in[1,\infty)$ be fixed. If not otherwise explicitly stated, let the requirements and notation of (A1), $(\mathrm{A2})_L$ and $(\mathrm{A3})_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Before we can formulate the induction hypothesis, we first have to introduce the analogues for the higher-order linearized homogenization correctors on the level of the massive approximation. To this end, we fix a set of unit vectors $v_1,\ldots,v_L\in\mathbb{R}^d$ and define $B:=v_1\odot\cdots\odot v_L$. Next, fix $\xi\in\mathbb{R}^d$ and denote by a^T_ξ the coefficient field $(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$. Due to Assumption 1, the coefficient field is uniformly elliptic and bounded with respect to the constants (λ,Λ) from Assumption 1.

In anticipation of the higher-order differentiability of the localized corrector for the nonlinear PDE, we introduce the equation for the *localized Lth-order linearized homogenization corrector in direction B* by means of the Faà di Bruno formula in form of

$$\begin{aligned}
&\frac{1}{T}\phi^T_{\xi,B} - \nabla\cdot a^T_\xi\big(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi,B}\big)\\
&= \nabla\cdot\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}(\partial^{|\Pi|}_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})\Big],
\end{aligned} \tag{49a}$$

where we also introduced the notational convention

$$B'_\pi := \bigodot_{m\in\pi} v_m, \quad \forall\pi\in\Pi,\ \Pi\in\mathrm{Par}\{1,\ldots,L\}.$$

Note that the right hand side of (49a) only features linearized homogenization correctors up to order $L-1$, if any. Hence, it turns out that we may argue inductively using standard (and, in particular, only deterministic) arguments, that the corrector problem (49a) admits for every random parameter field $\omega \in \Omega$ a unique solution

$$\phi^T_{\xi,B} = \phi^T_{\xi,B}(\cdot,\omega) \in H^1_{\mathrm{uloc}}(\mathbb{R}^d).$$

In particular, the uniqueness part of this statement entails stationarity of the linearized corrector $\phi^T_{\xi,B}$ in the sense that for each $z \in \mathbb{R}^d$ and each random $\omega \in \Omega$ it holds

$$\phi^T_{\xi,B}(\cdot + z,\omega) = \phi^T_{\xi,B}(\cdot,\omega(\cdot + z)) \quad \text{almost everywhere in } \mathbb{R}^d.$$

An analogous statement holds true for the linearized flux correctors

$$\sigma^T_{\xi,B} \in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}}) \quad \text{and} \quad \psi^T_{\xi,B} \in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^d).$$

These are more precisely the unique solutions of

$$\frac{1}{T}\sigma^T_{\xi,B,kl} - \Delta\sigma^T_{\xi,B,kl} = (e_l \otimes e_k - e_k \otimes e_l) : \nabla q^T_{\xi,B}, \tag{49b}$$

respectively

$$\frac{1}{T}\psi^T_{\xi,B} - \Delta\psi^T_{\xi,B} = q^T_{\xi,B} - \langle q^T_{\xi,B}\rangle - \nabla\phi^T_{\xi,B}, \tag{49c}$$

with the linearized flux being defined by

$$\begin{aligned} q^T_{\xi,B} &:= a^T_\xi(\mathbb{1}_{L=1}B + \nabla\phi^T_{\xi,B}) \\ &\quad + \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial^{|\Pi|}_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big]. \end{aligned} \tag{49d}$$

As in the case of the corrector for the nonlinear PDE with an additional massive term, the relations (49a)–(49d) will give rise to the equation

$$q^T_{\xi,B} - \langle q^T_{\xi,B}\rangle = \nabla\cdot\sigma^T_{\xi,B} + \frac{1}{T}\psi^T_{\xi,B}. \tag{49e}$$

With all of this notation in place, we can state the following result on existence of (higher-order) linearized correctors. For a proof, we refer the reader to Appendix B where we also formulate and proof a corresponding result on the differentiability of (higher-order) linearized correctors with respect to the parameter field.

Lemma 8 (Existence of localized correctors)**.** *Let $L \in \mathbb{N}$ and $T \in [1,\infty)$ be fixed. Let the requirements and notation of (A1), (A2)$_{L-1}$ and (A3)$_{L-1}$ of Assumption 1 be in place. Fix $\eta \in (0,1)$, and consider a parameter field $\tilde\omega\colon \mathbb{R}^d \to \mathbb{R}^n$ such that for all $p \geq 2$ it holds*

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty} \fint_{B_R(x_0)} \bigg| \sup_{y,z\in B_1(x),\, y\neq z} \frac{|\tilde\omega(y)-\tilde\omega(z)|}{|y-z|^\eta}\bigg|^p < \infty. \tag{50}$$

Under these assumptions, one obtains inductively that for all $l \in \{1,\ldots,L\}$ and all $B := v_1 \odot \cdots \odot v_l$ formed by unit vectors $v_1,\ldots,v_l \in \mathbb{R}^d$, there exists a unique solution

$$\phi^T_{\xi,B} = \phi^T_{\xi,B}(\cdot,\tilde\omega) \in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$$

of the linearized corrector problem (49a) *with* ω *replaced by* $\tilde\omega$*. Under the set of conditions (A1), (A2)*$_{1\vee(L-1)}$ *and (A3)*$_{1\vee(L-1)}$ *of Assumption* 1*, the linearized corrector* $\phi^T_{\xi,B}(\cdot,\tilde\omega)$ *moreover satisfies for all* $p\in[2,\infty)$

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\Big|\Big(\frac{\phi^T_{\xi,B}(\cdot,\tilde\omega)}{\sqrt{T}},\nabla\phi^T_{\xi,B}(\cdot,\tilde\omega)\Big)\Big|^p<\infty. \tag{51}$$

There also exist unique solutions

$$\sigma^T_{\xi,B}=\sigma^T_{\xi,B}(\cdot,\tilde\omega)\in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}}),$$
$$\psi^T_{\xi,B}=\psi^T_{\xi,B}(\cdot,\tilde\omega)\in H^1_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^d)$$

of the linearized flux corrector problems (49b) *resp.* (49c) *with* ω *replaced by* $\tilde\omega$*. The analogue of* (51) *holds true for these flux correctors.*

In particular, under the requirements of (A1), (A2)$_{L-1}$ *and (A3)*$_{L-1}$ *of Assumption* 1*, (P1) and (P2) of Assumption* 2*, and (R) of Assumption* 3*, there exists a set* $\Omega'\subset\Omega$ *of full* $\mathbf{P}$*-measure on which the existence of (higher-order) linearized correctors is guaranteed in the above sense for all random parameter fields* $\omega\in\Omega'$*.*

We have by now everything in place to proceed with the statement of the

Induction hypothesis. Let $L\in\mathbb{N}$, $M>0$ and $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. For any $l\le L-1$ and any collection of unit vectors $v'_1,\ldots,v'_l\in\mathbb{R}^d$ we assume that under the above conditions the associated localized lth-order linearized homogenization corrector $\phi^T_{\xi,B'}$ in direction $B':=v'_1\odot\cdots\odot v'_l$ satisfies the following list of conditions (if $l=0$—and thus B' being an empty symmetric tensor product—$\phi^T_{\xi,B'}$ is understood to denote the localized homogenization corrector of the nonlinear PDE):

- There exist $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$ and $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ such that for all $|\xi|\le M$, all $q\in[1,\infty)$, and all compactly supported and square-integrable f,g we have *corrector estimates*

$$\begin{aligned}\Big\langle\Big|\Big(\int g\cdot\nabla\phi^T_{\xi,B'},\int\frac{1}{T}f\phi^T_{\xi,B'}\Big)\Big|^{2q}\Big\rangle^{\frac1q}&\le C^2q^{2C'}\int\Big|\Big(g,\frac{f}{\sqrt{T}}\Big)\Big|^2,\\ \Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B'}}{\sqrt{T}},\nabla\phi^T_{\xi,B'}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}&\le C^2q^{2C'}.\end{aligned}\tag{H1}$$

- Fix $p\in(2,\infty)$, and let $g,f\in\bigcap_{q\ge1}L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$ be two compactly supported and $L^p(\mathbb{R}^d)$-valued random fields. There then exists a random field $G^T_{\xi,B'}$ satisfying $\big[G^T_{\xi,B'}\big]_1\in\bigcap_{q\ge1}L^{2q}_{\langle\cdot\rangle}L^2(\mathbb{R}^d;\mathbb{R}^n)$, which in addition is related to (g,f) via ϕ^T_ξ in the sense that, $\mathbf{P}$-almost surely, it holds for all perturbations $\delta\omega\in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)}<\infty$

$$\int g\cdot\nabla\delta\phi^T_{\xi,B'}-\int f\frac{1}{T}\delta\phi^T_{\xi,B'}=\int G^T_{\xi,B'}\cdot\delta\omega. \tag{H2a}$$

  Here, $(\frac{\delta\phi^T_{\xi,B'}}{\sqrt{T}},\nabla\delta\phi^T_{\xi,B'})\in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$ denotes the Gâteaux derivative of the linearized corrector $\phi^T_{\xi,B'}$ and its gradient in direction $\delta\omega$, cf. Lemma 26.

  Moreover, there exists $c_0=c_0(d,\lambda,\Lambda)\in(1,\infty)$ such that for any $\kappa\in(0,1]$ there exist $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\kappa)$ and $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$

such that for all $|\xi| \leq M$, all $q \in [1,\infty)$ and all $q_0 \in [\frac{c_0}{c_0-1},\infty)$ the random field $G^T_{\xi,B'}$ gives rise to a *sensitivity estimate* of the form

$$\begin{aligned} &\Big\langle \Big| \int \Big(\fint_{B_1(x)} |G^T_{\xi,B'}| \Big)^2 \Big|^q \Big\rangle^{\frac{1}{q}} \\ &\leq C^2 (q \vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle = 1} \int \Big\langle \Big| \Big(Fg, \frac{Ff}{\sqrt{T}} \Big) \Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}. \end{aligned} \tag{H2b}$$

If $(g_r,f_r)_{r\geq 1}$ is a sequence in $\bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$ of compactly supported and $L^p(\mathbb{R}^d)$-valued random fields, denote by $G^{T,r}_{\xi,B'}$, $r \geq 1$, the random field associated to (g_r,f_r), $r\geq 1$, in the sense of (46a). Let (g,f) be two $L^p_{\mathrm{loc}}(\mathbb{R}^d)$-valued random fields such that $(g_r,f_r) \to (g,f)$ as $r\to\infty$ in $\bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$. Then there exists a random field $G^T_{\xi,B'}$ with

$$\big[G^{T,r}_{\xi,B'} - G^T_{\xi,B'}\big]_1 \to 0 \text{ as } r\to\infty \text{ in } \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n), \tag{H2c}$$

and the sensitivity estimate (46b) is satisfied in terms of $(G^T_{\xi,B'}, g)$.

- There exist $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$, $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ and $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $|\xi|\leq M$ and all $q\in[1,\infty)$ we have a *small-scale annealed Schauder estimate* of the form

$$\big\langle \big\|\nabla\phi^T_{\xi,B'}\big\|^{2q}_{C^\alpha(B_1)} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}. \tag{H3}$$

3.3. **Corrector bounds for higher-order linearizations: the base case.** The first step in the proof of Theorem 1—on the level of the massive approximation in form of Theorem 5—is of course to verify the induction hypotheses (H1)–(H3) for the corrector of the nonlinear problem (44a). This is covered by Proposition 6 which constitutes one of the main results of [18]. We briefly summarize at the beginning of Appendix C how to obtain the assertions of Proposition 6.

3.4. **Corrector bounds for higher-order linearizations: the induction step.** The main step in the proof of Theorem 5 consists of lifting the induction hypotheses (H1)–(H3) to the Lth-order linearized homogenization corrector $\phi^T_{\xi,B}$ satisfying (49a). This task is performed by means of several auxiliary results where we are guided by the well-established literature on quantitative stochastic homogenization, cf. for instance [22], [21], [18] and [29]. We start with the concept of a minimal radius for the (higher-order) linearized corrector equation (49a).

Definition 9 (Minimal radius for linearized corrector problem)**.** Let the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). For a given constant $\underline{\gamma} > 0$ we then define a random variable

$$\begin{aligned} r_{*,T,\xi,B} := \inf \Big\{ 2^k \colon k\in\mathbb{N}_0, \text{ and for all } R = 2^l,\, l\geq k, \text{ it holds:} \\ \inf_{b\in\mathbb{R}} \Big\{ \frac{1}{R^2} \fint_{B_R} \big|\phi^T_{\xi,B} - b\big|^2 + \frac{1}{T}|b|^2 \Big\} \leq 1, \\ \sup_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\} \\ \Pi\neq\{\{1,\ldots,L\}\}}} \sup_{\pi\in\Pi} \fint_{B_R} \big|\mathbb{1}_{|\pi|=1} B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^{4|\Pi|} \leq R^{4\underline{\gamma}} \Big\}. \end{aligned}$$

The stationary extension $r_{*,T,\xi,B}(x,\omega) := r_{*,T,\xi,B}\big(\omega(\cdot+x)\big)$ is called the *minimal radius for the linearized corrector problem* (49a).

Stochastic moments of the linearized homogenization correctors are related to stochastic moments of the minimal radius in the following way.

Lemma 10 (Annealed small-scale energy estimate). *Let the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). Let $r_{*,T,\xi,B}$ denote the minimal radius for the linearized corrector problem (49a) from Definition 9. Then, there exists a constant $C = C(d,\lambda,\Lambda,L)$ and an exponent $\delta = \delta(d,\lambda,\Lambda)$ such that for all $\xi \in \mathbb{R}^d$ and all $q \in [1,\infty)$ it holds*

$$\Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla \phi^T_{\xi,B} \Big) \Big\|^{2q}_{L^2(B_1)} \Big\rangle^{\frac{1}{q}} \leq C^2 \big\langle r_{*,T,\xi,B}^{(d-\delta+2\underline{\gamma})q} \big\rangle^{\frac{1}{q}}. \tag{52}$$

Moreover, we have the suboptimal estimate (for which we do not have to specify the form of the implicit constant)

$$\Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla \phi^T_{\xi,B} \Big) \Big\|^{2q}_{L^2(B_1)} \Big\rangle^{\frac{1}{q}} \lesssim \sqrt{T}^d. \tag{53}$$

As it is already the case for the proof of corrector estimates with respect to first-order linearizations, cf. [18], the argument for the realization of the induction step relies on a small-scale regularity estimate for the linearized correctors.

Lemma 11 (Annealed small-scale Schauder estimate). *Let again the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). There exists $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$, and for every $\tau \in (0,1)$ constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\tau)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$, such that for all $|\xi| \leq M$ and all $q \in [1,\infty)$ it holds*

$$\Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla \phi^T_{\xi,B} \Big) \Big\|^{2q}_{C^\alpha(B_1)} \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} \bigg\{ 1 + \Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla \phi^T_{\xi,B} \Big) \Big\|^{\frac{2q}{1-\tau}}_{L^2(B_1)} \Big\rangle^{\frac{1-\tau}{q}} \bigg\}. \tag{54}$$

Lemma 10 shifts the task of establishing stretched exponential moment bounds for the linearized corrector to the task of proving stretched exponential moments for the associated minimal radius. For the latter, a key input are stochastic moment bounds for linear functionals of the linearized corrector gradient. This in turn is the content of the following result.

Lemma 12 (Annealed estimates for linear functionals of the homogenization corrector and its gradient). *Let the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). Let g,f be two square-integrable and compactly supported deterministic fields. For every $\tau \in (0,1)$ there exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\tau)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ such that for all $|\xi| \leq M$ and all $q \in [C,\infty)$ it holds*

$$\begin{aligned} &\bigg\langle \bigg| \Big(\int g \cdot \nabla \phi^T_{\xi,B}, \int \frac{1}{T} f \phi^T_{\xi,B} \Big) \bigg|^{2q} \bigg\rangle^{\frac{1}{q}} \\ &\leq C^2 q^{2C'} \bigg\{ 1 + \Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla \phi^T_{\xi,B} \Big) \Big\|^{\frac{2q}{(1-\tau)^2}}_{L^2(B_1)} \Big\rangle^{\frac{(1-\tau)^2}{q}} \bigg\} \int \Big| \Big(g, \frac{f}{\sqrt{T}} \Big) \Big|^2. \end{aligned} \tag{55}$$

We have everything in place to prove a stretched exponential moment bound for the minimal radius $r_{*,T,\xi,B}$. The key ingredient of the proof is a buckling argument based on the annealed estimates (52)–(55).

Lemma 13 (Stretched exponential moment bound for minimal radius)**.** *Let the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). Let $r_{*,T,\xi,B}$ denote the minimal radius for the linearized corrector problem (49a) from Definition (9). Then, there exists $\underline{\gamma} = \underline{\gamma}(d,\lambda,\Lambda)$, as well as constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ such that for all $|\xi| \leq M$ and all $q \geq 1$ it holds*

$$\big\langle r_{*,T,\xi,B}^{2q} \big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}. \tag{56}$$

A rather straightforward post-processing of Lemma 10, Lemma 11 and Lemma 12 based on the stretched exponential moment bounds for the minimal radius from Lemma 13 now allows to conclude the induction step.

Lemma 14. *Let the assumptions and notation of Section 3.2 be in place; in particular, the induction hypotheses (H1)–(H3). Then the Lth-order linearized homogenization corrector $\phi^T_{\xi,B}$ also satisfies (H1)–(H3).*

3.5. **Estimates for higher-order linearized flux correctors.** In view of the defining equations (49b) and (49c) for the linearized flux correctors, it is natural to establish first the analogues of the estimates (39)–(43) for the linearized flux $q^T_{\xi,B}$. Actually, it suffices to establish the pendant of induction hypothesis (H2b). This is captured in the following result.

Lemma 15 (Sensitivity estimate for the linearized flux)**.** *Let the assumptions and notation of Section 3.2 be in place. In particular, let $q^T_{\xi,B}$ be the linearized flux as defined by (49d). Fix $p \in (2,\infty)$, and let $g \in \bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$ be a compactly supported and $L^p(\mathbb{R}^d)$-valued random field. Then there exists a random field $Q^T_{\xi,B}$ satisfying $\big[Q^T_{\xi,B}\big]_1 \in \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n)$, and which is related to g via $q^T_{\xi,B}$ in the sense that, $\mathbf{P}$-almost surely, for all perturbations $\delta\omega \in C^{\eta}_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$ it holds*

$$\int g \cdot \delta q^T_{\xi,B} = \int Q^T_{\xi,B} \cdot \delta\omega. \tag{57}$$

In addition, there exist $c_0 = c_0(d,\lambda,\Lambda) \in (1,\infty)$, $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$ and $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$ such that for all $|\xi| \leq M$, all $q \in [1,\infty)$ and all $q_0 \in [\frac{c_0}{c_0-1},\infty)$ the random field $Q^T_{\xi,B}$ gives rise to a sensitivity estimate

$$\Big\langle \Big| \int \Big(\fint_{B_1(x)} |Q^T_{\xi,B}| \Big)^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 (q \vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle = 1} \int \big\langle |Fg|^{2(q\vee q_0)_*} \big\rangle^{\frac{1}{(q\vee q_0)_*}}. \tag{58}$$

If $(g_r)_{r\geq 1}$ represents a sequence in $\bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$ of compactly supported and $L^p(\mathbb{R}^d)$-valued random fields, denote by $Q^{T,r}_{\xi,B}$, $r \geq 1$, the random field associated to g_r, $r \geq 1$, in the sense of (57). Let g be an $L^p_{\mathrm{loc}}(\mathbb{R}^d)$-valued random field such that $g_r \to g$ as $r \to \infty$ in $\bigcap_{q\geq 1} L^2(\mathbb{R}^d; L^{2q}_{\langle\cdot\rangle})$. Then there exists a random field $Q^T_{\xi,B}$ with

$$\big[Q^{T,r}_{\xi,B} - Q^T_{\xi,B}\big]_1 \to 0 \text{ as } r \to \infty \text{ in } \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n), \tag{59}$$

and the sensitivity estimate (58) holds true in terms of $(Q^T_{\xi,B}, g)$.

Once this result is established, the asserted estimates in Theorem 5 for the massive linearized flux correctors $(\sigma^T_{\xi,B},\psi^T_{\xi,B})$ follow readily.

3.6. **Differentiability of the massive correctors and the massive approximation of the homogenized operator.** As a preparation for the proof of the estimates (23)–(25), which in particular contain estimates for differences of linearized correctors, and the higher-order differentiability of the homogenized operator in form of Theorem 2, we establish the desired differentiability properties on the level of the massive approximation. A first step in this direction are the following estimates for differences of linearized correctors.

Lemma 16 (Estimates for differences of linearized correctors)**.** *Let $L\in\mathbb{N}$, $M>0$ as well as $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), $(A2)_L$, $(A3)_L$ and $(A4)_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. We also fix a set of unit vectors $v_1,\ldots,v_L\in\mathbb{R}^d$ and define $B:=v_1\odot\cdots\odot v_L$.*

For every $\beta\in(0,1)$, there exist constants $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\beta)$ and $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L,\beta)$ such that for all $|\xi|\leq M$, all $q\in[1,\infty)$, all unit vectors $e\in\mathbb{R}^d$ and all $|h|\leq 1$ it holds

$$\Big\langle\big\|\big(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B},\nabla\sigma^T_{\xi+he,B}-\nabla\sigma^T_{\xi,B}\big)\big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}\leq C^2q^{2C'}|h|^{2(1-\beta)},\tag{60}$$

$$\Big\langle\big\|\big(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B},\nabla\sigma^T_{\xi+he,B}-\nabla\sigma^T_{\xi,B}\big)\big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac1q}\leq C^2q^{2C'}|h|^{2(1-\beta)},\tag{61}$$

as well as for all compactly supported and square-integrable g_ϕ,g_σ

$$\begin{aligned}&\bigg\langle\bigg|\bigg(\int g_\phi\cdot(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}),\int g^{kl}_\sigma\cdot(\nabla\sigma^T_{\xi+he,B,kl}-\nabla\sigma^T_{\xi,B,kl})\bigg)\bigg|^{2q}\bigg\rangle^{\frac1q}\\&\leq C^2q^{2C'}|h|^{2(1-\beta)}\int\big|\big(g_\phi,g_\sigma\big)\big|^2.\end{aligned}\tag{62}$$

The already mentioned differentiability result for the massive linearized correctors and the massive approximation of the homogenized operator now reads as follows.

Lemma 17 (Differentiability of massive correctors and the massive version of the homogenized operator)**.** *Let $L\in\mathbb{N}$, $M>0$ and $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), $(A2)_L$, $(A3)_L$ and $(A4)_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. We also fix a set of unit vectors $v_1,\ldots,v_L\in\mathbb{R}^d$ and define $B:=v_1\odot\cdots\odot v_L$. Then, both the maps $\xi\mapsto\nabla\phi^T_{\xi,B}$ and $\xi\mapsto\nabla\sigma^T_{\xi,B}$ are Fréchet differentiable with values in the Fréchet space $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$.*

Given a vector $\xi\in\mathbb{R}^d$, a unit vector $e\in\mathbb{R}^d$ and some $|h|\leq1$ we define

$$\bar{A}^T_{B,e,h}(\xi):=\big\langle q^T_{\xi+he,B}\big\rangle-\big\langle q^T_{\xi,B}\big\rangle-\big\langle q^T_{\xi,B\odot e}\big\rangle h.$$

For every $\beta\in(0,1)$, there then exists $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},C'_{\mathrm{reg}},\eta,M,L,\beta)$ such that for all $|\xi|\leq M$, all unit vectors $e\in\mathbb{R}^d$, and all $|h|\leq1$ it holds

$$\big|\bar{A}^T_{B,e,h}(\xi)\big|^2\leq C^2h^{4(1-\beta)}.\tag{63}$$

Assume in addition to the above conditions that the stronger forms of $(A2)_{L+1}$, $(A3)_{L+1}$ and $(A4)_{L+1}$ from Assumption 1 hold true. We then have the following *quantitative estimates on first-order Taylor expansions of the linearized correctors* $\phi^T_{\xi,B}$ *and* $\sigma^T_{\xi,B}$. *Given a vector* $\xi \in \mathbb{R}^d$, *a unit vector* $e \in \mathbb{R}^d$ *and some* $|h| \leq 1$, *let*

$$\begin{aligned} \phi^T_{\xi,B,e,h} &:= \phi^T_{\xi+he,B} - \phi^T_{\xi,B} - \phi^T_{\xi,B\odot e}h, \\ \sigma^T_{\xi,B,e,h} &:= \sigma^T_{\xi+he,B} - \sigma^T_{\xi,B} - \sigma^T_{\xi,B\odot e}h. \end{aligned}$$

For every $\beta \in (0,1)$, *there then exist constants* $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\beta)$ *and* $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L,\beta)$ *such that for all* $|\xi| \leq M$, *all unit vectors* $e \in \mathbb{R}^d$, *all* $q \in [1,\infty)$, *all* $|h| \leq 1$ *and all compactly supported and square-integrable* g_ϕ, g_σ *it holds*

$$\big\langle \big\| \big(\nabla\phi^T_{\xi,B,e,h}, \nabla\sigma^T_{\xi,B,e,h}\big)\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} h^{4(1-\beta)}, \tag{64}$$

$$\bigg\langle \bigg| \bigg(\int g_\phi \cdot \nabla\phi^T_{\xi,B,e,h}, \int g^{kl}_\sigma \cdot \nabla\sigma^T_{\xi,B,e,h} \bigg) \bigg|^{2q} \bigg\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{4(1-\beta)} \int \big|(g_\phi, g_\sigma)\big|^2. \tag{65}$$

Remark 18. *In case of* $q = 1$, *the estimates* (64) *and* (65) *actually hold true requiring only (A2)*$_L$, *(A3)*$_L$ *and (A4)*$_L$ *from Assumption 1. This in turn represents exactly the form of Assumption 1 for which qualitative differentiability of the linearized corrector gradients is established in Lemma 17. A proof of this claim is contained in the proof of Lemma 17.*

3.7. **The limit passage in the massive approximation.** The last main ingredient in the proof of Theorem 1 and Theorem 2 consists of studying the limit $T \to \infty$ in the massive approximation. More precisely, we establish the following result.

Lemma 19 (Limit passage in the massive approximation). *Let* $L \in \mathbb{N}$ *and* $M > 0$ *be fixed. Let the requirements and notation of (A1), (A2)*$_L$ *and (A3)*$_L$ *of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. We also fix a set of unit vectors* $v_1, \ldots, v_L \in \mathbb{R}^d$ *and define* $B := v_1 \odot \cdots \odot v_L$.

Then the sequence

$$\Big(\nabla\phi^T_{\xi,B},\, \nabla\sigma^T_{\xi,B}\Big)_{T\in[1,\infty)}$$

is Cauchy in $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$ *(with respect to the strong topology). Moreover, there exists* $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},C'_{\mathrm{reg}},\eta,M,L)$ *such that for all* $|\xi| \leq M$ *and all* $T \in [1,\infty)$ *we have the estimates*

$$\big\langle \big\| \big(\nabla\phi^{2T}_{\xi,B} - \nabla\phi^T_{\xi,B}, \nabla\sigma^{2T}_{\xi,B} - \nabla\sigma^T_{\xi,B}\big)\big\|^2_{L^2(B_1)}\big\rangle \leq C^2 \Big(\frac{\mu^2_*(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}, \tag{66}$$

$$\big|\big\langle q^{2T}_{\xi,B}\big\rangle - \big\langle q^T_{\xi,B}\big\rangle\big|^2 \leq C^2 \Big(\frac{\mu^2_*(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}. \tag{67}$$

The corresponding limits give rise to higher-order linearized homogenization correctors and flux correctors in the sense of Definition 4. Moreover,

$$\big\langle \big|q^T_{\xi,B} - q_{\xi,B}\big|^2\big\rangle \to 0, \tag{68}$$

with the limiting linearized flux $q_{\xi,B}$ *defined in* (30b).

4. Proofs

4.1. **Proof of Lemma 10** (Annealed small-scale energy estimate)**.** Applying the hole filling estimate (T2) to equation (49a) for the linearized homogenization corrector (putting the term $-\nabla\cdot a_\xi^T \mathbb{1}_{L=1}B$ on the right hand side) yields in combination with $(\mathrm{A2})_L$ from Assumption 1

$$\begin{aligned}&\Big\|\Big(\frac{\phi_{\xi,B}^T}{\sqrt{T}},\nabla\phi_{\xi,B}^T\Big)\Big\|_{L^2(B_1)}^2\\&\lesssim_{d,\lambda,\Lambda,L} r_{*,T,\xi,B}^{d-\delta}\mathbb{1}_{L=1}+r_{*,T,\xi,B}^{d-\delta}\fint_{B_{r_{*,T,\xi,B}}}\Big|\Big(\frac{\phi_{\xi,B}^T}{\sqrt{T}},\nabla\phi_{\xi,B}^T\Big)\Big|^2\\&\quad+r_{*,T,\xi,B}^{d}\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}\fint_{B_{r_{*,T,\xi,B}}}\frac{1}{|x|^\delta}\prod_{\pi\in\Pi}\big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^T\big|^2.\end{aligned}$$

For the second right hand side term, we proceed by making use of Caccioppoli's inequality (T1) with respect to equation (49a); and in the course of this we again rely on $(\mathrm{A2})_L$ from Assumption 1 in order to bound the right hand side term appearing in equation (49a). For the third right hand side term, we simply argue by Hölder's inequality. In total, we obtain the estimate

$$\begin{aligned}&\Big\|\Big(\frac{\phi_{\xi,B}^T}{\sqrt{T}},\nabla\phi_{\xi,B}^T\Big)\Big\|_{L^2(B_1)}^2\\&\lesssim_{d,\lambda,\Lambda,L} r_{*,T,\xi,B}^{d-\delta}\mathbb{1}_{L=1}\\&\quad+r_{*,T,\xi,B}^{d-\delta}\inf_{b\in\mathbb{R}}\bigg\{\frac{1}{(2r_{*,T,\xi,B})^2}\fint_{B_{2r_{*,T,\xi,B}}}\big|\phi_{\xi,B}^T-b\big|^2+\frac{1}{T}|b|^2\bigg\}\\&\quad+r_{*,T,\xi,B}^{d-\delta}\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}\fint_{B_{2r_{*,T,\xi,B}}}\prod_{\pi\in\Pi}\big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^T\big|^2\\&\quad+r_{*,T,\xi,B}^{d-\delta}\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}\prod_{\pi\in\Pi}\bigg(\fint_{B_{r_{*,T,\xi,B}}}\big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^T\big|^{4|\Pi|}\bigg)^{\frac{1}{2|\Pi|}}.\end{aligned}$$

Taking into account Definition 9 of the minimal radius, the fact that $r_{*,T,\xi,B}\geq 1$, as well as Hölder's and Jensen's inequality (to deal with the second right hand side term in the previous display) we deduce from this

$$\Big\|\Big(\frac{\phi_{\xi,B}^T}{\sqrt{T}},\nabla\phi_{\xi,B}^T\Big)\Big\|_{L^2(B_1)}^2\lesssim_{d,\lambda,\Lambda,L} r_{*,T,\xi,B}^{d-\delta+2\gamma}.$$

Taking stochastic moments thus entails the asserted estimate (52).

For a proof of (53), we rely on the weighted energy estimate (T3). In order to apply it, we first have to check the polynomial growth at infinity (in the precise sense of the statement of Lemma 21) of the constituents in the linearized corrector problem (49a). For the solution $(\phi_{\xi,B}^T,\nabla\phi_{\xi,B}^T)$ itself, this is a consequence of the fact that $\phi_{\xi,B}^T\in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$. For the right hand side term in equation (49a) we argue as follows. First, thanks to the ergodic theorem we may choose almost surely a radius $R_0 > 0$ such that

$$\begin{aligned} &\fint_{B_R} \prod_{\pi\in\Pi} \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^2 \\ &\leq 1 + \Big\langle \prod_{\pi\in\Pi} \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^2 \Big\rangle \quad \text{for all } R \geq R_0, \end{aligned} \tag{69}$$

uniformly over all partitions $\Pi \in \mathrm{Par}\{1,\ldots,L\}$ with $\Pi \neq \{\{1,\ldots,L\}\}$. Thanks to the induction hypothesis (H1) and (H3), we may then smuggle in spatial averages over the unit ball B_1 followed by an application of Hölder's inequality to deduce that

$$\begin{aligned} &\Big\langle \prod_{\pi\in\Pi} \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^2 \Big\rangle \\ &\lesssim \Big\langle \prod_{\pi\in\Pi} \big\|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big\|^2_{C^\alpha(B_1)} \Big\rangle + \Big\langle \prod_{\pi\in\Pi} \int_{B_1} \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^2 \Big\rangle \\ &\lesssim \prod_{\pi\in\Pi} \Big\langle \big\|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big\|^{2|\Pi|}_{C^\alpha(B_1)} \Big\rangle^{\frac{1}{|\Pi|}} + \prod_{\pi\in\Pi} \Big\langle \big\|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big\|^{2|\Pi|}_{L^2(B_1)} \Big\rangle^{\frac{1}{|\Pi|}} \\ &\lesssim 1, \end{aligned}$$

uniformly over all partitions $\Pi \in \mathrm{Par}\{1,\ldots,L\}$ with $\Pi \neq \{\{1,\ldots,L\}\}$. Inserting this back into (69) shows that also the right hand side terms of equation (49a) feature at most polynomial growth at infinity.

Hence, we may apply the weighted energy estimate (T3) to equation (49a) which entails in combination with Jensen's and Hölder's inequality

$$\begin{aligned} &\Big\langle \Big\| \Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big) \Big\|^{2q}_{L^2(B_1)} \Big\rangle^{\frac{1}{q}} \\ &\lesssim \sqrt{T}^d \mathbb{1}_{L=1} + \sqrt{T}^d \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\} \\ \Pi\neq\{\{1,\ldots,L\}\}}} \int \ell_{\gamma,\sqrt{T}} \Big\langle \prod_{\pi\in\Pi} \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^{2q} \Big\rangle^{\frac{1}{q}} \\ &\lesssim \sqrt{T}^d \mathbb{1}_{L=1} + \sqrt{T}^d \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\} \\ \Pi\neq\{\{1,\ldots,L\}\}}} \int \ell_{\gamma,\sqrt{T}} \prod_{\pi\in\Pi} \Big\langle \big|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}\big|^{2q|\Pi|} \Big\rangle^{\frac{1}{q|\Pi|}}. \end{aligned}$$

Taking into account the induction hypothesis (H1) and (H3)—the latter in particular allowing us to smuggle in a spatial average over unit balls $B_1(x)$— and making use of stationarity of the linearized homogenization correctors, we thus infer from the previous display the asserted estimate (53). □

4.2. **Proof of Lemma 11** (Annealed small-scale Schauder estimate)**.** We aim to apply the local Schauder estimate (T5) to equation (49a) for the linearized homogenization corrector (putting to this end the term $-\nabla\cdot a^T_\xi \mathbb{1}_{L=1}B$ on the right hand side). This is facilitated by the annealed Hölder regularity of the linearized coefficient field $a^T_\xi = (\partial_\xi A)(\omega, \xi + \nabla\phi^T_\xi)$, cf. Lemma 24 in Appendix A. Hence, in view of the local Schauder estimate (T5) we may estimate by an application of Hölder's inequality with respect to the exponents $(\frac{1}{\tau}, \frac{1}{1-\tau})$, $\tau \in (0,1)$,

$$\begin{aligned}
&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \\
&\leq C^2\big\langle\|a^T_\xi\|^{\frac{2q}{\tau}\frac{d}{\alpha}(\frac{1}{2}+\frac{1}{d})}_{C^\alpha(B_2)}\big\rangle^{\frac{\tau}{q}}\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2q}{1-\tau}}_{L^2(B_2)}\Big\rangle^{\frac{1-\tau}{q}} \\
&\quad + C^2\big\langle\|a^T_\xi\|^{4q(\frac{1}{\alpha}-1)}_{C^\alpha(B_2)}\big\rangle^{\frac{1}{2q}} \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} \Big\langle\Big\|\prod_{\pi\in\Pi}|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}|\Big\|^{4q}_{C^\alpha(B_2)}\Big\rangle^{\frac{1}{2q}} \\
&\quad + C^2\big\langle\|a^T_\xi\|^{\frac{2q}{\alpha}}_{C^\alpha(B_2)}\big\rangle^{\frac{1}{q}}\mathbb{1}_{L=1}.
\end{aligned}$$

A combination of the annealed estimate (T7) for the Hölder norm of the linearized coefficient, the small-scale annealed Schauder estimate from induction hypothesis (H3), the stationarity of the linearized coefficient field and of the linearized homogenization correctors, and Hölder's inequality updates the previous display to

$$\begin{aligned}
&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \\
&\leq C^2q^{2C'}\Big\{1 + \Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2q}{1-\tau}}_{L^2(B_1)}\Big\rangle^{\frac{1-\tau}{q}}\Big\} \\
&\quad + C^2q^{2C'}\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}\prod_{\pi\in\Pi}\Big\langle\Big\||\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi}|\Big\|^{4q|\Pi|}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{2q|\Pi|}} \\
&\leq C^2q^{2C'}\Big\{1 + \Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2q}{1-\tau}}_{L^2(B_1)}\Big\rangle^{\frac{1-\tau}{q}}\Big\}.
\end{aligned}$$

This concludes the proof of Lemma 11. □

4.3. **Proof of Lemma 12** (Annealed estimates for linear functionals of the homogenization corrector and its gradient)**.** The proof proceeds in three steps. In the course of it, we will make use of the abbreviation $\sum_\Pi := \sum_{\Pi\in\mathrm{Par}\{1,\ldots,L\},\,\Pi\neq\{\{1,\ldots,L\}\}}$. Since $(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi,B}) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$, we may assume for the proof of (55) without loss of generality through an approximation argument that $\mathbf{P}$-almost surely

$$(g,f) \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}). \tag{70}$$

Step 1 (Computation of functional derivative): We start by computing the functional derivative of

$$F_\phi := \int g\cdot\nabla\phi^T_{\xi,B} - \int \frac{1}{T}f\phi^T_{\xi,B}. \tag{71}$$

To this end, consider a perturbation $\delta\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$. Based on Lemma 26, we may $\mathbf{P}$-almost surely differentiate the defining equation (49a) for the linearized homogenization corrector with respect to the parameter field in the direction of $\delta\omega$. This yields $\mathbf{P}$-almost surely the following PDE for the variation $(\frac{\delta\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\delta\phi^T_{\xi,B}) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d) \cap L^p_{\mathrm{erg}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$, $p \geq 2$, of the linearized corrector $\phi^T_{\xi,B}$ with massive term:

$$\begin{aligned}
&\frac{1}{T}\delta\phi^T_{\xi,B} - \nabla\cdot a^T_\xi \nabla\delta\phi^T_{\xi,B} \\
&= \nabla\cdot(\partial_\omega\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\delta\omega\odot\nabla\phi^T_{\xi,B}\big] \\
&\quad + \nabla\cdot(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\nabla\delta\phi^T_\xi\odot\nabla\phi^T_{\xi,B}\big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial_\omega\partial^{|\Pi|}_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\delta\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi^T_{\xi,B'_\pi})\Big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial^{1+|\Pi|}_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\nabla\delta\phi^T_\xi\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi^T_{\xi,B'_\pi})\Big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial^{|\Pi|}_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}{+}\nabla\phi^T_{\xi,B'_{\pi'}})\Big] \\
&=: \nabla\cdot R^{T,(1)}_{\xi,B}\delta\omega + \nabla\cdot R^{T,(2)}_{\xi,B}\nabla\delta\phi^T_\xi + \nabla\cdot R^{T,(3)}_{\xi,B}\delta\omega + \nabla\cdot R^{T,(4)}_{\xi,B}\nabla\delta\phi^T_\xi \\
&\quad + \sum_\Pi\sum_{\pi\in\Pi}\nabla\cdot R^{T,(5),\pi}_{\xi,B}\nabla\delta\phi^T_{\xi,B'_\pi}.
\end{aligned} \tag{72}$$

Observe that as a consequence of (51) and (A2)$_L$ resp. (A3)$_L$ of Assumption 1 we have **P**-almost surely

$$R^{T,(1)}_{\xi,B}, R^{T,(3)}_{\xi,B} \in L^p_{\mathrm{erg}}(\mathbb{R}^d;\mathbb{R}^{d\times n}),\ R^{T,(2)}_{\xi,B}, R^{T,(4)}_{\xi,B}, R^{T,(5),\pi}_{\xi,B} \in L^p_{\mathrm{erg}}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \tag{73}$$

for all $p \geq 2$. For $r \geq 1$, let $(\frac{\delta\phi^{T,r}_{\xi,B}}{\sqrt{T}}, \nabla\delta\phi^{T,r}_{\xi,B}) \in L^2(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$ be the unique Lax–Milgram solution of

$$\begin{aligned}
&\frac{1}{T}\delta\phi^{T,r}_{\xi,B} - \nabla\cdot a^T_\xi\nabla\delta\phi^{T,r}_{\xi,B} \\
&=: \nabla\cdot\mathbb{1}_{B_r}R^{T,(1)}_{\xi,B}\delta\omega + \nabla\cdot\mathbb{1}_{B_r}R^{T,(2)}_{\xi,B}\nabla\delta\phi^T_\xi + \nabla\cdot\mathbb{1}_{B_r}R^{T,(3)}_{\xi,B}\delta\omega \\
&\quad + \nabla\cdot\mathbb{1}_{B_r}R^{T,(4)}_{\xi,B}\nabla\delta\phi^T_\xi + \sum_\Pi\sum_{\pi\in\Pi}\nabla\cdot\mathbb{1}_{B_r}R^{T,(5),\pi}_{\xi,B}\nabla\delta\phi^T_{\xi,B'_\pi}.
\end{aligned} \tag{74}$$

Note that **P**-almost surely

$$\Big(\frac{\delta\phi^{T,r}_{\xi,B}}{\sqrt{T}}, \nabla\delta\phi^{T,r}_{\xi,B}\Big) \to \Big(\frac{\delta\phi^T_{\xi,B}}{\sqrt{T}}, \nabla\delta\phi^T_{\xi,B}\Big) \text{ as } r\to\infty \text{ in } L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}) \tag{75}$$

by means of applying the weighted energy estimate (T3) to the difference of the equations (72) and (74).

Denoting the transpose of a_ξ^T by $a_\xi^{T,*}$, we may now compute by means of Lemma 26, (73), (74), (75) and (189)

$$
\begin{aligned}
\delta F_\phi &= \int g\cdot\nabla\delta\phi_{\xi,B}^T - \int \frac{1}{T} f\delta\phi_{\xi,B}^T \\
&= \lim_{r\to\infty}\Big\{\int g\cdot\nabla\delta\phi_{\xi,B}^{T,r} - \int\frac{1}{T}f\delta\phi_{\xi,B}^{T,r}\Big\} \\
&= -\lim_{r\to\infty}\Big\{\int \nabla\delta\phi_{\xi,B}^{T,r}\cdot a_\xi^{T,*}\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\qquad\qquad + \int \delta\phi_{\xi,B}^{T,r}\,\frac{1}{T}\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\Big\} \\
&= \lim_{r\to\infty}\Big\{\int \mathbb{1}_{B_r}R_{\xi,B}^{T,(1)}\delta\omega\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r}R_{\xi,B}^{T,(2)}\nabla\delta\phi_\xi^T\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r}R_{\xi,B}^{T,(3)}\delta\omega\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r}R_{\xi,B}^{T,(4)}\nabla\delta\phi_\xi^T\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \sum_{\Pi}\sum_{\pi\in\Pi}\int \mathbb{1}_{B_r}R_{\xi,B}^{T,(5),\pi}\nabla\delta\phi_{\xi,B'_\pi}^T\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\Big\}.
\end{aligned}
\tag{76}
$$

Note that thanks to the approximation argument, the regularity (73) as well as the regularity (189), we may indeed use $\delta\phi_{\xi,B}^{T,r}$ as a test function in the equation of $\big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\big)^{-1}\big(\frac{1}{T}f+\nabla\cdot g\big)$ and vice versa. Moreover, by (73), the assumption (70) ensuring that (g,f) are $L^p(\mathbb{R}^d)$-valued random fields for any $p>2$, and the (local) Meyers estimate for the operator $(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla)$, we obtain that $\mathbf{P}$-almost surely

$$
\begin{aligned}
\big(R_{\xi,B}^{T,(i)}\big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) &\in L^{p'}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n),\ i\in\{1,3\}, \\
\big(R_{\xi,B}^{T,(i)}\big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) &\in L^{p'}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d),\ i\in\{2,4\}, \\
\big(R_{\xi,B}^{T,(5),\pi}\big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) &\in L^{p'}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)
\end{aligned}
\tag{77}
$$

for some suitable Meyers exponent $p'>2$. We have everything in place to proceed with the next step of the proof.

Step 2 (Application of the spectral gap inequality): Note first that $\langle F_\phi\rangle=0$ for the functional F_ϕ from (71). Indeed, $\langle(\phi_{\xi,B}^T,\nabla\phi_{\xi,B}^T)\rangle=0$ is a direct consequence of stationarity and testing the linearized corrector problem (49a). Hence, we may apply the spectral gap inequality in form of (48) and thus obtain in view of $(\mathrm{A2})_L$ and $(\mathrm{A3})_L$ of Assumption 1, (76), (77), induction hypotheses (H2a) and (H2c), and the sensitivity estimates from induction hypothesis (H2b) the estimate (with

$\kappa_1,\kappa_2\in(0,1]$ yet to be determined)

$$
\begin{aligned}
&\big\langle|F_\phi|^{2q}\big\rangle^{\frac1q}\\
&\le C^2q^2\Big\langle\Big|\int\Big(\fint_{B_1(x)}|\nabla\phi^T_{\xi,B}|^2\Big)\Big(\fint_{B_1(x)}\Big|\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1Tf+\nabla\cdot g\Big)\Big|^2\Big)\Big|^q\Big\rangle^{\frac1q}\\
&\quad+C^2q^2\sum_\Pi\Big\langle\Big|\int\Big(\fint_{B_1(x)}\prod_{\pi\in\Pi}|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi}|^2\Big)\\
&\qquad\qquad\times\Big(\fint_{B_1(x)}\Big|\nabla\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1Tf+\nabla\cdot g\Big)\Big|^2\Big)\Big|^q\Big\rangle^{\frac1q}\\
&\quad+C^2(q\vee q_0)^{2C'}\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\phi^T_{\xi,B}\Big|^{2(\frac{q\vee q_0}{\kappa_1})_*}\Big|\nabla\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1TFf+\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{\kappa_1})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa_1})_*}}\\
&\quad+C^2(q\vee q_0)^{2C'}\sum_\Pi\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1TFf+\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{\kappa_2})_*}\\
&\qquad\qquad\times\prod_{\pi\in\Pi}|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi}|^{2(\frac{q\vee q_0}{\kappa_2})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa_2})_*}}\\
&\quad+C^2(q\vee q_0)^{2C'}\sum_\Pi\sum_{\pi\in\Pi}\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1TFf+\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{\kappa_2})_*}\\
&\qquad\qquad\times\prod_{\substack{\pi'\in\Pi\\\pi'\neq\pi}}|\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi,B'_{\pi'}}|^{2(\frac{q\vee q_0}{\kappa_2})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa_2})_*}}\\
&=:I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{78}
$$

For the last three right hand side terms in the previous display, we also exploited the fact that F is purely random so that one can simply multiply with F the equation satisfied by $(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla)^{-1}(\frac1Tf+\nabla\cdot g)$.

Step 3 (Post-processing the right hand side of (78)*):* We estimate each of the right hand side terms of (78) separately. By duality in $L^q_{\langle\cdot\rangle}$, stationarity of the linearized homogenization corrector $\phi^T_{\xi,B}$, and Hölder's inequality with respect to the exponents $(\frac{q\vee q_0}{1-\tau},(\frac{q\vee q_0}{1-\tau})_*)$, $\tau\in(0,1)$, we estimate the contribution from I_1 by

$$
\begin{aligned}
|I_1|\le C^2q^2&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt T},\nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2(q\vee q_0)}{1-\tau}}_{L^2(B_1)}\Big\rangle^{\frac{1-\tau}{q\vee q_0}}\\
&\times\sup_{\langle F^{2q_*}\rangle=1}\int\fint_{B_1(x)}\Big\langle\Big|\nabla\Big(\frac1T-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac1TFf+\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{1-\tau})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{1-\tau})_*}}.
\end{aligned}
$$

As $(\frac{q\vee q_0}{1-\tau})_*=\frac{q\vee q_0}{(q\vee q_0)-(1-\tau)}\le\frac{q\vee q_0}{(q\vee q_0)-1}=(q\vee q_0)_*\le q_*\wedge(q_0)_*$, it follows from Jensen's inequality, the fact that $\int\fint_{B_1(x)}h=\int h$ for all non-negative h, and the annealed Calderón–Zygmund estimate (T8) that

$$
\begin{aligned}
|I_1|&\le C^2q^2\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt T},\nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2(q\vee q_0)}{1-\tau}}_{L^2(B_1)}\Big\rangle^{\frac{1-\tau}{q\vee q_0}}\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(\frac{Ff}{\sqrt T},Fg\Big)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}\\
&\le C^2q^2\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt T},\nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2(q\vee q_0)}{1-\tau}}_{L^2(B_1)}\Big\rangle^{\frac{1-\tau}{q\vee q_0}}\int\Big|\Big(\frac{f}{\sqrt T},g\Big)\Big|^2
\end{aligned}\tag{79}
$$

provided $|(q_0)_*-1|$ is sufficiently small (depending only on d,λ,Λ). In other words, we obtain a bound of required type for all $q\geq q_0$ with $q_0\in[1,\infty)$ sufficiently large.

For the contribution from I_2, we may estimate based on the same ingredients as in the estimate of I_1 (we could actually take $\tau=0$ but prefer to keep the general form for later reference) for all sufficiently large $q_0\in[1,\infty)$ and all $q\in[1,\infty)$

$$\begin{aligned}|I_2| &\leq C^2q^2\sum_{\Pi}\prod_{\pi\in\Pi}\Big\langle\fint_{B_1}\big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi}\big|^{\frac{2(q\vee q_0)|\Pi|}{1-\tau}}\Big\rangle^{\frac{1-\tau}{(q\vee q_0)|\Pi|}} \\ &\quad\times\sup_{\langle F^{2q_*}\rangle=1}\int\fint_{B_1(x)}\Big\langle\Big|\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}Ff+\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{1-\tau})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{1-\tau})_*}} \\ &\leq C^2q^{2C}\int\Big|\Big(\frac{f}{\sqrt{T}},g\Big)\Big|^2, \end{aligned}\tag{80}$$

where in the second step we in addition made use of the small-scale annealed Schauder estimate from induction hypothesis (H3) (by smuggling in a spatial average over the unit ball) and the corrector estimates from induction hypothesis (H1).

We next estimate the contribution from I_3. To this end, we first choose $\kappa_1=\frac{\tau}{2}$ and then estimate via stationarity of the linearized homogenization corrector $\phi^T_{\xi,B}$, the fact that $(\frac{q\vee q_0}{\kappa_1})_*=\frac{q\vee q_0}{(q\vee q_0)-\kappa_1}$, an application of Hölder's inequality with respect to the exponents $(\frac{(q\vee q_0)-\kappa_1}{1-\tau},(\frac{(q\vee q_0)-\kappa_1}{1-\tau})_*)$, and an application of Jensen's inequality based on $(\frac{q\vee q_0}{\kappa_1})_*(\frac{(q\vee q_0)-\kappa_1}{1-\tau})_*=\frac{q\vee q_0}{(q\vee q_0)-\kappa_1-(1-\tau)}\leq\frac{q\vee q_0}{(q\vee q_0)-1}=(q\vee q_0)_*\leq q_*\wedge(q_0)_*$

$$\begin{aligned}|I_3| &\leq C^2(q\vee q_0)^{2C'}\big\langle\|\nabla\phi^T_{\xi,B}\|^{\frac{2(q\vee q_0)}{1-\tau}}_{C^\alpha(B_1)}\big\rangle^{\frac{q\vee q_0}{1-\tau}} \\ &\quad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}Ff+\nabla\cdot Fg\Big)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}.\end{aligned}$$

Hence, by means of the annealed Calderón–Zygmund estimate (T8) and the annealed small-scale Schauder estimate (54) (with q replaced by $\frac{q\vee q_0}{1-\tau}$) we obtain

$$|I_3|\leq C^2(q\vee q_0)^{2C'}\Big\langle\Big\|\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi,B}\Big)\Big\|^{\frac{2(q\vee q_0)}{(1-\tau)^2}}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^2}{q\vee q_0}}\int\Big|\Big(\frac{f}{\sqrt{T}},g\Big)\Big|^2,\tag{81}$$

at least for sufficiently large $q_0\in[1,\infty)$. This in turn is again a bound of required type for all $q\geq q_0$.

We next deal with the contribution from I_4. To this end, we simply choose $\kappa_2=\frac{1}{2}$ and argue based on stationarity of the linearized homogenization correctors, $(\frac{q\vee q_0}{\kappa_2})_*=\frac{q\vee q_0}{(q\vee q_0)-\kappa_2}$, an application of Hölder's inequality with respect to the exponents $(\frac{(q\vee q_0)-\kappa_2}{(q\vee q_0)-1},(\frac{(q\vee q_0)-\kappa_2}{(q\vee q_0)-1})_*)$, and the fact that $(\frac{q\vee q_0}{\kappa_2})_*(\frac{(q\vee q_0)-\kappa_2}{(q\vee q_0)-1})_*=\frac{q\vee q_0}{1-\kappa_2}=2(q\vee q_0)$

$$\begin{aligned}|I_4| &\leq C^2(q\vee q_0)^{2C'}\sum_{\Pi}\prod_{\pi\in\Pi}\Big\langle\big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi}\big|^{4(q\vee q_0)|\Pi|}\Big\rangle^{\frac{1}{2(q\vee q_0)|\Pi|}} \\ &\quad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}Ff+\nabla\cdot Fg\Big)\Big|^{2(q\vee q_0)_*}\Big\rangle^{\frac{1}{(q\vee q_0)_*}}.\end{aligned}$$

As it is by now routine, the second factor of the right hand side term of the previous display is dealt with by appealing to the annealed Calderón–Zygmund estimate (T8) for which we only have to choose $q_0\in[1,\infty)$ sufficiently large. For the first factor, we may smuggle in a spatial average over the ball B_1 and then estimate by means of the induction hypotheses (H1) and (H3). In total, we obtain for all $q \in [1, \infty)$

$$|I_4| \leq C^2 (q \vee q_0)^{2C'} \int \Big| \Big(\frac{f}{\sqrt{T}}, g \Big) \Big|^2 \tag{82}$$

provided $q_0 \in [1, \infty)$ is sufficiently large. As the contribution from I_5 can be treated analogously, the combination of the estimates (79)–(82) establishes the asserted bound (55). This concludes the proof of Lemma 12. □

4.4. **Proof of Lemma 13** (Stretched exponential moment bound for minimal radius of the linearized corrector problem)**.** We start with the estimate

$$\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q} \big\rangle \leq 1 + \sum_{k=1}^{\infty} 2^{k(d-\frac{\delta}{2})2q} \mathbf{P}\big[\{ r_{*,T,\xi,B} = 2^k \} \big]. \tag{83}$$

Fix an integer $k \geq 1$, and let $R := 2^k$. By Definition 9 of the minimal radius $r_{*,T,\xi,B}$, in the event of $\{r_{*,T,\xi,B} = R\}$ we either have

$$\begin{aligned} &\exists \Pi \in \mathrm{Par}\{1, \ldots, L\},\, \Pi \neq \{\{1, \ldots, L\}\} \colon \exists \pi \in \Pi \text{ such that} \\ &\fint_{B_{\frac{R}{2}}} \big| \mathbb{1}_{|\pi|=1} B'_\pi + \nabla \phi^T_{\xi, B'_\pi} \big|^{4|\Pi|} > \Big(\frac{R}{2} \Big)^{4\underline{\gamma}}, \end{aligned} \tag{84}$$

or that

$$\inf_{b \in \mathbb{R}} \bigg\{ \frac{1}{(R/2)^2} \fint_{B_{\frac{R}{2}}} \big| \phi^T_{\xi,B} - b \big|^2 + \frac{1}{T} |b|^2 \bigg\} > 1. \tag{85}$$

We distinguish in the following between these two events, and provide separately an estimate on their probability.

Case 1: (Estimate in the event of (84)*)* Fix a partition $\Pi \in \mathrm{Par}\{1, \ldots, L\}$, $\Pi \neq \{\{1, \ldots, L\}\}$, and some $\pi \in \Pi$ such that the conclusion of (84) holds true. Covering $B_{\frac{R}{2}}$ by a family of $\sim R^d$ many open unit balls, using stationarity of the linearized homogenization correctors, Jensen's inequality, the small-scale annealed Schauder estimate from induction hypothesis (H3) which in particular allows to smuggle in a spatial average over the unit ball, and finally the corrector bounds from induction hypotheses (H1), we infer that

$$\bigg\langle \bigg| \fint_{B_{\frac{R}{2}}} \big| \mathbb{1}_{|\pi|=1} B'_\pi + \nabla \phi^T_{\xi, B'_\pi} \big|^{4|\Pi|} \bigg|^{2q} \bigg\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}$$

for all $q \in [1, \infty)$. It thus follows from Markov's inequality and the previous display (with q replaced by $\underline{\gamma}^{-1} dq$) that

$$\mathbf{P}\big[\{ r_{*,T,\xi,B} = 2^k \} \cap \{ (84) \text{ holds true} \} \big] \leq C^{2q} q^{2C'q} 2^{-k8dq} \tag{86}$$

with constants $C = C(d, \lambda, \Lambda, C_{\mathrm{sg}}, C_{\mathrm{reg}}, \eta, M, L)$ and $C' = C'(d, \lambda, \Lambda, C'_{\mathrm{reg}}, \eta, L)$.

Case 2: (Suboptimal estimate in the event of (85) *but not* (84)*)* Let $0 < R' \leq R$, and abbreviate by $(h)_{R'}(x) := \fint_{B_{R'}(x)} h$ the mollification on scale R' for any locally integrable h. In the event of $\{(85) \text{ holds true}\} \cap \{(84) \text{ holds not true}\}$ we deduce from the triangle inequality that

$$\begin{aligned}1 \lesssim \frac{1}{R^2}\fint_{B_R} \big|\phi_{\xi,B}^T - (\phi_{\xi,B}^T)_{R'}\big|^2 \\ + \frac{1}{R^2}\fint_{B_R} \bigg|(\phi_{\xi,B}^T)_{R'} - \fint_{B_R} (\phi_{\xi,B}^T)_{R'}\bigg|^2 + \frac{1}{T}\bigg|\fint_{B_R} (\phi_{\xi,B}^T)_{R'}\bigg|^2.\end{aligned} \tag{87}$$

The first term on the right hand side of the previous display is estimated by

$$\frac{1}{R^2}\fint_{B_R} \big|\phi_{\xi,B}^T - (\phi_{\xi,B}^T)_{R'}\big|^2 \lesssim \Big(\frac{R'}{R}\Big)^2 \fint_{B_{2R}} \big|\nabla\phi_{\xi,B}^T\big|^2,$$

the second right hand side term based on Poincaré's inequality and the definition of $(\cdot)_{R'}$ by

$$\frac{1}{R^2}\fint_{B_R} \bigg|(\phi_{\xi,B}^T)_{R'} - \fint_{B_R} (\phi_{\xi,B}^T)_{R'}\bigg|^2 \lesssim \fint_{B_{2R}} \bigg|\fint_{B_{R'}(x)} \nabla\phi_{\xi,B}^T\bigg|^2,$$

and the third right hand side term simply by plugging in the definition of $(\cdot)_{R'}$ and Jensen's inequality

$$\frac{1}{T}\bigg|\fint_{B_R} (\phi_{\xi,B}^T)_{R'}\bigg|^2 \lesssim \fint_{B_{2R}} \bigg|\fint_{B_{R'}(x)} \frac{1}{\sqrt{T}}\phi_{\xi,B}^T\bigg|^2.$$

In the event of $\{$(85) holds true$\} \cap \{$(84) holds not true$\}$, the combination of the last four displays therefore entails

$$1 \lesssim \Big(\frac{R'}{R}\Big)^2 \fint_{B_{2R}} \big|\nabla\phi_{\xi,B}^T\big|^2 + \fint_{B_{2R}} \bigg|\fint_{B_{R'}(x)} \nabla\phi_{\xi,B}^T\bigg|^2 + \fint_{B_{2R}} \bigg|\fint_{B_{R'}(x)} \frac{1}{\sqrt{T}}\phi_{\xi,B}^T\bigg|^2. \tag{88}$$

Applying next the Caccioppoli inequality (T1) to equation (49a) for the linearized homogenization corrector (putting the term $-\nabla\cdot a_\xi^T \mathbb{1}_{L=1} B$ on the right hand side), and making use of Definition 9 of the minimal radius $r_{*,T,\xi,B}$ we obtain in the event of $\{$(85) holds true$\} \cap \{$(84) holds not true$\}$

$$\Big(\frac{R'}{R}\Big)^2 \fint_{B_{2R}} \big|\nabla\phi_{\xi,B}^T\big|^2 \leq C(d,\lambda,\Lambda)\Big(\frac{R'}{R}\Big)^2\big(1 + R^{2\bar{\gamma}}\big). \tag{89}$$

Hence, restricting $\underline{\gamma} = \underline{\gamma}(d,\lambda,\Lambda) \in (0,\frac{1}{2})$ (but otherwise yet to be determined), choosing $R' = \theta R^{1-\underline{\gamma}}$ with $\theta = \theta(d,\lambda,\Lambda)$ such that $2\theta^2 C(d,\lambda,\Lambda) = \frac{1}{2}$, we infer from (88) and (89) in the event of $\{$(85) holds true$\} \cap \{$(84) holds not true$\}$

$$1 \lesssim_{d,\lambda,\Lambda} \fint_{B_{2R}} \bigg|\fint_{B_{\theta R^{1-\underline{\gamma}}}(x)} \nabla\phi_{\xi,B}^T\bigg|^2 + \fint_{B_{2R}} \bigg|\fint_{B_{\theta R^{1-\underline{\gamma}}}(x)} \frac{1}{\sqrt{T}}\phi_{\xi,B}^T\bigg|^2. \tag{90}$$

It thus follows from an application of Markov's inequality in combination with stationarity of the linearized homogenization correctors and Jensen's inequality that

$$\begin{aligned}&\mathbf{P}\big[\big\{r_{*,T,\xi,B} = 2^k\big\} \cap \{(85)\text{ holds true}\} \cap \{(84)\text{ holds not true}\}\big] \\ &\leq C(d,\lambda,\Lambda)^{2q}\bigg\langle\bigg|\fint_{B_{\theta 2^{k(1-\underline{\gamma})}}} \nabla\phi_{\xi,B}^T, \fint_{B_{\theta 2^{k(1-\underline{\gamma})}}} \frac{1}{\sqrt{T}}\phi_{\xi,B}^T\bigg|^{4q}\bigg\rangle.\end{aligned} \tag{91}$$

Thanks to the moment bounds (55) for linear functionals of the linearized homogenization corrector and its gradient (applied with, say, $\tau = \frac{1}{2}$) in combination with the suboptimal small-scale energy estimate (53), we get the following update of the previous display

$$\begin{aligned}&\mathbf{P}\big[\big\{r_{*,T,\xi,B}=2^k\big\}\cap\{(85)\text{ holds true}\}\cap\{(84)\text{ holds not true}\}\big]\\&\lesssim_{d,\lambda,\Lambda,q}\sqrt{T}^{2dq}2^{-k(1-\underline{\gamma})2dq},\end{aligned}\tag{92}$$

which is—with respect to the scaling in the stochastic integrability q and the massive approximation T—a highly supobtimal estimate for this probability.

Intermediate summary: (Suboptimal estimate on stochastic moments of the minimal radius) We collect the information provided by the estimates (86) and (92), and combine it with (83) resulting in

$$\begin{aligned}\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q}\big\rangle\leq 1+C^{2q}q^{2C'q}\sum_{k=1}^{\infty}2^{k(d-\frac{\delta}{2})2q}2^{-k8dq}\\+\bar{C}_q\sqrt{T}^{2dq}\sum_{k=1}^{\infty}2^{k(d-\frac{\delta}{2})2q}2^{-k(1-\underline{\gamma})2dq}\end{aligned}\tag{93}$$

with constants $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L)$ and $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$, and a constant $\bar{C}_q$ which in addition depends in a possibly highly suboptimal way on $q\in[1,\infty)$. Choosing $\underline{\gamma}:=\frac{1}{2}\wedge\frac{\delta}{4d}$ thus entails the suboptimal estimate

$$\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q}\big\rangle\leq\bar{C}_q\sqrt{T}^{2dq}.\tag{94}$$

Conclusion: (From suboptimal moment bounds to stretched exponential moments) The merit of (94) is that it at least provides finiteness of arbitrarily high stochastic moments of the minimal radius $r_{*,T,\xi,B}$. We now leverage on that information in a buckling argument. We observe from the previous argument that suboptimality was only a result of using (55) with a non-optimized $\tau\in(0,1)$ and using the suboptimal small-scale energy estimate (53) in order to transition from (91) to (92).

If we instead apply (55) with $\tau\in(0,1)$ yet to be optimized, and then feed in the annealed small-scale energy estimate from Lemma 10 in form of (52) (with q replaced by $\frac{2q}{(1-\tau)^2}$), we may update (92) to

$$\begin{aligned}&\mathbf{P}\big[\big\{r_{*,T,\xi,B}=2^k\big\}\cap\{(85)\text{ holds true}\}\cap\{(84)\text{ holds not true}\}\big]\\&\leq C^{2q}q^{2C'q}2^{-k(1-\underline{\gamma})2dq}\Big\langle r_{*,T,\xi,B}^{\frac{d-\delta+2\gamma}{(1-\tau)^2}2q}\Big\rangle^{(1-\tau)^2},\end{aligned}\tag{95}$$

with constants $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\tau)$ and $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L)$. This in turn provides the following improvement of (93)

$$\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q}\big\rangle\leq C^{2q}q^{2C'q}+C^{2q}q^{2C'q}\sum_{k=1}^{\infty}2^{k(d-\frac{\delta}{2})2q}2^{-k(1-\underline{\gamma})2dq}\Big\langle r_{*,T,\xi,B}^{\frac{d-\delta+2\gamma}{(1-\tau)^2}2q}\Big\rangle^{(1-\tau)^2}.$$

Choosing $\underline{\gamma}:=\frac{1}{2}\wedge\frac{\delta}{8d}$ and then $\tau\in(0,1)$ such that $\frac{d-\delta+2\gamma}{(1-\tau)^2}=d-\frac{\delta}{2}$ we obtain

$$\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q}\big\rangle\leq C^{2q}q^{2C'q}\Big(1+\big\langle r_{*,T,\xi,B}^{(d-\frac{\delta}{2})2q}\big\rangle^{(1-\tau)^2}\Big).$$

Because of (94) and $r_{*,T,\xi,B}\geq 1$, this concludes the proof of Lemma 13. □

4.5. **Proof of Lemma 14** (Conclusion of the induction step)**.** The validity of the corrector bounds from (H1) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$ is an immediate consequence of (52), (55), and (56). The small-scale annealed Schauder estimate from induction hypothesis (H3) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$ follows from combining (52), (54), and (56).

It remains to establish induction hypotheses (H2a)–(H2c) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$. Starting point is the following reminder of (76)

$$
\begin{aligned}
&\int g\cdot\nabla\delta\phi^T_{\xi,B} - \int \frac{1}{T} f\delta\phi^T_{\xi,B} \\
&= -\lim_{r\to\infty}\bigg\{\int \nabla\delta\phi^{T,r}_{\xi,B}\cdot a^{T,*}_\xi\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\qquad\qquad + \int \delta\phi^{T,r}_{\xi,B}\,\frac{1}{T}\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\bigg\} \\
&= \lim_{r\to\infty}\bigg\{\int \mathbb{1}_{B_r} R^{T,(1)}_{\xi,B}\delta\omega\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r} R^{T,(2)}_{\xi,B}\nabla\delta\phi^T_\xi\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r} R^{T,(3)}_{\xi,B}\delta\omega\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \int \mathbb{1}_{B_r} R^{T,(4)}_{\xi,B}\nabla\delta\phi^T_\xi\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \sum_{\Pi}\sum_{\pi\in\Pi}\int \mathbb{1}_{B_r} R^{T,(5),\pi}_{\xi,B}\nabla\delta\phi^T_{\xi,B'_\pi}\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\bigg\},
\end{aligned}
\tag{96}
$$

with the five remainder terms being defined in (72). Since the five right hand side terms of (96) only contain variations of linearized homogenization correctors up to order $L-1$, the representation (H2a) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$ follows from the induction hypotheses (H2a) and (H2c) thanks to (77). In addition, we deduce the following representation of the random field $G^T_{\xi,B}=G^T_{\xi,B}[g,f]$ in form of

$$
\begin{aligned}
G^T_{\xi,B}[g,f] &= \Big(R^{T,(1)}_{\xi,B}+R^{T,(3)}_{\xi,B}\Big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \\
&\quad + \lim_{r\to\infty} G^T_\xi\Big[\mathbb{1}_{B_r}\Big(R^{T,(2)}_{\xi,B}+R^{T,(4)}_{\xi,B}\Big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\Big] \\
&\quad + \sum_{\Pi}\sum_{\pi\in\Pi}\lim_{r\to\infty} G^T_{\xi,B'_\pi}\Big[\mathbb{1}_{B_r}\Big(R^{T,(5),\pi}_{\xi,B}\Big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\Big],
\end{aligned}
\tag{97}
$$

which in particular depends linearly on the input (g,f). Now, the same argument leading to (78) shows

$$
\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^T_{\xi,B}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} \leq I_1+I_2+I_3+I_4+I_5, \tag{98}
$$

with the right hand side terms being identical to those of (78). In order to post-process the right hand side of (98) we may in fact follow very closely the arguments from *Step 3* in the proof of Lemma 12. So let us only mention the minor differences.

For the contribution from I_1, we simply choose $\tau := 1-\kappa$ and avoid the use of Jensen's inequality in the corresponding argument which yields

$$|I_1| \leq C^2(q\vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(Fg, \frac{Ff}{\sqrt{T}}\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.$$

Note that for an application of the annealed Calderón–Zygmund estimate (T8) we have to ensure in this argument that $|(q_0)_*-1|$ is sufficiently small (or equivalently, that q_0 is sufficiently large). Finally, note that we can get rid of the energy term appearing on the right hand side of (79) since we already have in place the corrector bounds from induction hypothesis (H1) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$. As the last two remarks also apply to all of the remaining right hand side terms in (98), we will not mention them anymore from now on.

For the contribution from the second term I_2, the only change concerns taking $\tau := 1-\kappa$ in the argument for (80) in order to deduce

$$|I_2| \leq C^2(q\vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(Fg, \frac{Ff}{\sqrt{T}}\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.$$

With respect to the term I_3, the corresponding argument is the one leading to (81). To adapt it to our needs here, we choose $\kappa_1 := \frac{\kappa}{2}$ and $1-\tau = \frac{\kappa}{2}$ which then entails the estimate

$$|I_3| \leq C^2(q\vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(Fg, \frac{Ff}{\sqrt{T}}\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.$$

Last but not least—as I_5 can again be treated analogously—the adaption of the argument for I_4 leading to (82) consists of taking $\kappa_2 = \frac{\kappa}{2}$ and applying Hölder's inequality with respect to the exponents $(\frac{(q\vee q_0)-\kappa_2}{(q\vee q_0)-\kappa}, (\frac{(q\vee q_0)-\kappa_2}{(q\vee q_0)-\kappa})_*)$. Based on these modifications, we obtain

$$|I_4| \leq C^2(q\vee q_0)^{2C'} \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(Fg, \frac{Ff}{\sqrt{T}}\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*} \Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.$$

In summary, the preliminary estimate (98) updates to the desired bound (H2b) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$.

Finally, the validity of (H2c) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$ is a consequence of following observations. First, note that (77) holds true even for pairs of random fields (g,f) which are merely L^p_{loc}-valued. In particular, the right hand side of (97) makes sense as a definition of the random field $G^T_{\xi,B}[g,f]$ for such pairs. Since the right hand side of (97) depends linearly on the input data, we may then apply the above argument for the estimation of the right hand side terms of (98) and obtain the bound

$$\begin{aligned}\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^T_{\xi,B}-G^{T,r}_{\xi,B}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} &\lesssim_q \sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(F(g-g_r), \frac{F(f-f_r)}{\sqrt{T}}\Big)\Big|^{2(2q)_*}\Big\rangle^{\frac{1}{(2q)_*}}\\ &\lesssim_q \int\Big\langle\Big|(g-g_r), \frac{(f-f_r)}{\sqrt{T}}\Big)\Big|^{2(2q)}\Big\rangle^{\frac{1}{(2q)}}.\end{aligned}$$

Here, the second line is obtained from an application of Hölder's inequality with respect to the exponents $(\frac{2q-1}{2(q-1)}, 2q-1)$. Hence, the conclusions of (H2c) with $\phi^T_{\xi,B'}$ replaced by $\phi^T_{\xi,B}$ follow from the previous display, which in turn concludes the proof of Lemma 14. □

4.6. **Proof of Lemma 15** (Sensitivity estimate for the linearized flux)**.** Consider some $\delta\omega \in C^{\eta}_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$. It is immediate from the definition (49d) of the linearized flux and the computation (72) concerning the variation $\delta\phi^T_{\xi,B}$ for the linearized homogenization corrector that $\mathbf{P}$-almost surely

$$\begin{aligned}
\delta q^T_{\xi,B} &= (\partial_\omega\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\delta\omega\odot\nabla\phi^T_{\xi,B}\big]\\
&\quad+(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\nabla\delta\phi^T_\xi\odot\nabla\phi^T_{\xi,B}\big]\\
&\quad+\sum_{\Pi}(\partial_\omega\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\delta\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})\Big]\\
&\quad+\sum_{\Pi}(\partial_\xi^{1+|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\nabla\delta\phi_\xi\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})\Big]\\
&\quad+\sum_{\Pi}(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi,B'_{\pi'}})\Big]\\
&\quad+(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\nabla\delta\phi^T_{\xi,B}\\
&=: R^{T,(1)}_{\xi,B}\delta\omega+R^{T,(2)}_{\xi,B}\nabla\delta\phi^T_\xi+R^{T,(3)}_{\xi,B}\delta\omega+R^{T,(4)}_{\xi,B}\nabla\delta\phi^T_\xi\\
&\quad+\sum_{\Pi}\sum_{\pi\in\Pi}R^{T,(5),\pi}_{\xi,B}\nabla\delta\phi^T_{\xi,B'_\pi}+(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\nabla\delta\phi^T_{\xi,B}.
\end{aligned}\tag{99}$$

In particular, denoting again by $a^{T,*}_\xi$ the transpose of the uniformly elliptic and bounded coefficient field $a^T_\xi := (\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$ we get

$$\begin{aligned}
\int g\cdot\delta q^T_{\xi,B} &= \int a^{T,*}_\xi g\cdot\nabla\delta\phi^T_{\xi,B}+\int\big(R^{T,(1)}_{\xi,B}+R^{T,(3)}_{\xi,B}\big)^* g\cdot\delta\omega\\
&\quad+\int\big(R^{T,(2)}_{\xi,B}+R^{T,(4)}_{\xi,B}\big)^* g\cdot\nabla\delta\phi^T_\xi+\sum_{\Pi}\sum_{\pi\in\Pi}\int\big(R^{T,(5),\pi}_{\xi,B}\big)^* g\cdot\nabla\delta\phi^T_{\xi,B'_\pi}.
\end{aligned}\tag{100}$$

Hence, we obtain a representation of the asserted form (57) because of (H2a), which as a result of Lemma 14 is even available for linearized homogenization correctors up to order L. By the same argument, we may then derive (58) from (H2b) (applied with $\kappa = 1$). The convergence assertion (59) is in light of (73) and (100) an immediate consequence of (H2c) (which again is already available up to linearization order L thanks to Lemma 14), from which then the validity of (58) for the limit pair $(Q^T_{\xi,B}, g)$ also follows. □

4.7. **Proof of Theorem 5** (Estimates for massive correctors)**.** We split the proof into four parts.

Step 1: (Proof of the estimates (39) *and* (40)*)* In case of the linearized homogenization corrector $\phi^T_{\xi,B}$ this already follows from Lemma 14 in form of (H1). For the linearized flux correctors $(\sigma^T_{\xi,B}, \psi^T_{\xi,B})$, we start by computing the functional derivatives of

$$F_\sigma := \int g^{kl}_\sigma\cdot\nabla\sigma^T_{\xi,B,kl},\qquad F_\psi := \int g^k_\psi\cdot\frac{\nabla\psi^T_{\xi,B,k}}{\sqrt{T}}.\tag{101}$$

To this end, let $\delta\omega \in C^{\eta}_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$. Since it holds $(\nabla\sigma^T_{\xi,B,kl}, \nabla\psi^T_{\xi,B,k}) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$, we may assume for the proof of (39) and (40) without loss of generality through an approximation argument that

$$(g_\sigma^{kl}, g_\psi^{k}) \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d{\times}\mathbb{R}^d). \tag{102}$$

Differentiating the defining equation (49b) for the linearized flux corrector $\sigma^T_{\xi,B,kl}$ with respect to the parameter field in the direction of $\delta\omega$ yields $\mathbf{P}$-almost surely

$$\begin{aligned}\frac{1}{T}\delta\sigma^T_{\xi,B,kl} - \Delta\delta\sigma^T_{\xi,B,kl} &= (e_l\otimes e_k - e_k\otimes e_l) : \nabla\delta q^T_{\xi,B} \\ &= -\nabla\cdot\big((e_l\otimes e_k - e_k\otimes e_l)\delta q^T_{\xi,B}\big).\end{aligned} \tag{103}$$

Moreover, differentiating (49c) for the linearized flux corrector $\psi^T_{\xi,B,k}$ entails

$$\frac{1}{T}\delta\psi^T_{\xi,B,k} - \Delta\delta\psi^T_{\xi,B,k} = e_k\cdot\delta q^T_{\xi,B} - e_k\cdot\nabla\delta\phi^T_{\xi,B}. \tag{104}$$

For $r\geq 1$, denote by $(\frac{\delta\sigma^{T,r}_{\xi,B,kl}}{\sqrt{T}}, \nabla\delta\sigma^{T,r}_{\xi,B,kl}) \in L^2(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d)$ the unique Lax–Milgram solution of

$$\begin{aligned}\frac{1}{T}\delta\sigma^{T,r}_{\xi,B,kl} - \Delta\delta\sigma^{T,r}_{\xi,B,kl} &= (e_l\otimes e_k - e_k\otimes e_l) : \nabla\mathbb{1}_{B_r}\delta q^T_{\xi,B} \\ &= -\nabla\cdot\mathbb{1}_{B_r}\big((e_l\otimes e_k - e_k\otimes e_l)\delta q^T_{\xi,B}\big),\end{aligned} \tag{105}$$

as well as by $(\frac{\delta\psi^{T,r}_{\xi,B,kl}}{\sqrt{T}}, \nabla\delta\psi^{T,r}_{\xi,B,kl}) \in L^2(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d)$ the unique Lax–Milgram solution of

$$\frac{1}{T}\delta\psi^{T,r}_{\xi,B,k} - \Delta\delta\psi^{T,r}_{\xi,B,k} = \mathbb{1}_{B_r}e_k\cdot\delta q^T_{\xi,B} - \mathbb{1}_{B_r}e_k\cdot\nabla\delta\phi^T_{\xi,B}. \tag{106}$$

Note that $\mathbf{P}$-almost surely

$$\Big(\frac{\delta\sigma^{T,r}_{\xi,B,kl}}{\sqrt{T}}, \nabla\delta\sigma^{T,r}_{\xi,B,kl}\Big) \to \Big(\frac{\delta\sigma^{T}_{\xi,B,kl}}{\sqrt{T}}, \nabla\delta\sigma^{T}_{\xi,B,kl}\Big) \text{ as } r\to\infty \text{ in } L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d) \tag{107}$$

as well as $\mathbf{P}$-almost surely

$$\Big(\frac{\delta\psi^{T,r}_{\xi,B,k}}{\sqrt{T}}, \nabla\delta\psi^{T,r}_{\xi,B,k}\Big) \to \Big(\frac{\delta\psi^{T}_{\xi,B,k}}{\sqrt{T}}, \nabla\delta\psi^{T}_{\xi,B,k}\Big) \text{ as } r\to\infty \text{ in } L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d) \tag{108}$$

as a consequence of applying the weighted energy estimate (T3) to the difference of the equations (103) and (105), respectively (104) and (106). We thus deduce from (105), (106), (107) and (108) that

$$\begin{aligned}\delta F_\sigma &= \lim_{r\to\infty}\int g_\sigma^{kl}\cdot\nabla\delta\sigma^{T,r}_{\xi,B,kl} \\ &= -\lim_{r\to\infty}\Big\{\int\nabla\delta\sigma^{T,r}_{\xi,B,kl}\cdot\nabla\Big(\frac{1}{T}{-}\Delta\Big)^{-1}(\nabla\cdot g_\sigma^{kl}) \\ &\qquad\qquad + \int\delta\sigma^{T,r}_{\xi,B,kl}\frac{1}{T}\Big(\frac{1}{T}{-}\Delta\Big)^{-1}(\nabla\cdot g_\sigma^{kl})\Big\} \\ &= \lim_{r\to\infty}\int\mathbb{1}_{B_r}(e_l\otimes e_k - e_k\otimes e_l)\nabla\Big(\frac{1}{T}{-}\Delta\Big)^{-1}(\nabla\cdot g_\sigma^{kl})\cdot\delta q^T_{\xi,B},\end{aligned} \tag{109}$$

as well as

$$\begin{aligned}\delta F_\psi &= \lim_{r\to\infty} \int g_\psi^k \cdot \frac{\nabla \delta\psi_{\xi,B,k}^{T,r}}{\sqrt{T}} \\ &= -\lim_{r\to\infty} \bigg\{ \int \mathbb{1}_{B_r} \Big(\frac{1}{T} - \Delta\Big)^{-1} \Big(\nabla \cdot \frac{g_\psi^k}{\sqrt{T}}\Big) e_k \cdot \delta q_{\xi,B}^T \\ &\qquad - \int \mathbb{1}_{B_r} \Big(\frac{1}{T} - \Delta\Big)^{-1} \Big(\nabla \cdot \frac{g_\psi^k}{\sqrt{T}}\Big) e_k \cdot \nabla \delta\phi_{\xi,B}^T \bigg\}.\end{aligned} \tag{110}$$

Note that thanks to the approximation argument, we may indeed use $\delta\sigma_{\xi,B,kl}^{T,r}$ resp. $\delta\psi_{\xi,B,k}^{T,r}$ as test functions in the weak formulation of the equations satisfied by $\big(\frac{1}{T} - \Delta\big)^{-1}\big(\nabla \cdot g_\sigma^{kl}\big)$ resp. $\big(\frac{1}{T} - \Delta\big)^{-1}\big(\nabla \cdot g_\psi^k\big)$, and vice versa. Moreover, by the Meyers estimate for the operator $(\frac{1}{T} - \Delta)$ in combination with the assumption (102), we obtain that $\mathbf{P}$-almost surely

$$\begin{aligned}(e_l \otimes e_k - e_k \otimes e_l)\nabla \Big(\frac{1}{T} - \Delta\Big)^{-1} (\nabla \cdot g_\sigma^{kl}) &\in L^{p'}(\mathbb{R}^d; \mathbb{R}^d), \\ \Big(\frac{1}{T} - \Delta\Big)^{-1} \Big(\nabla \cdot \frac{g_\psi^k}{\sqrt{T}}\Big) e_k &\in L^{p'}(\mathbb{R}^d; \mathbb{R}^d)\end{aligned} \tag{111}$$

for some suitable Meyers exponents $p' > 2$. Hence, applying the spectral gap inequality in form of (48) with respect to the centered random variable F_σ yields because of (109), (111), (58) resp. (59) (for which we fix $q_0 := \frac{c_0}{c_0 - 1}$), and a simple energy estimate

$$\begin{aligned}\big\langle \big|F_\sigma\big|^{2q} \big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'} \sup_{\langle F^{2q_*}\rangle = 1} \int \Big\langle \Big| F (e_l \otimes e_k - e_k \otimes e_l) \nabla \Big(\frac{1}{T} - \Delta\Big)^{-1} (\nabla \cdot g_\sigma^{kl}) \Big|^{2(q\vee q_0)_*} \Big\rangle^{\frac{1}{(q\vee q_0)_*}} \\ &\leq C^2 q^{2C'} \sup_{kl} \int \Big| \nabla \Big(\frac{1}{T} - \Delta\Big)^{-1} (\nabla \cdot g_\sigma^{kl}) \Big|^2 \\ &\leq C^2 q^{2C'} \int \big|g_\sigma\big|^2.\end{aligned}$$

Moreover, applying the spectral gap inequality (48) with respect to the centered random variable F_ψ entails the estimate

$$\begin{aligned}\big\langle \big|F_\psi\big|^{2q} \big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'} \sup_{\langle F^{2q_*}\rangle = 1} \int \Big\langle \Big| F \Big(\frac{1}{T} - \Delta\Big)^{-1} \Big(\nabla \cdot \frac{g_\psi^k}{\sqrt{T}}\Big) e_k \Big|^{2(q\vee q_0)_*} \Big\rangle^{\frac{1}{(q\vee q_0)_*}} \\ &\leq C^2 q^{2C'} \sup_k \int \frac{1}{T} \Big| \Big(\frac{1}{T} - \Delta\Big)^{-1} (\nabla \cdot g_\psi^k) \Big|^2 \\ &\leq C^2 q^{2C'} \int \big|g_\psi\big|^2.\end{aligned}$$

For the previous display, we relied on a combination of (110), (111), (58), (59), (H2b) resp. (H2c) applied to $\phi_{\xi,B}^T$ with $\kappa = 1$ (which is admissible thanks to Lemma 14) and again a simple energy estimate. This concludes the proof of (39). The proof of (40) in terms of $(\sigma_{\xi,B}^T, \psi_{\xi,B}^T)$ follows along similar lines.

Step 2: (Proof of the estimate (41)) In case of the linearized homogenization corrector $\phi_{\xi,B}^T$, this again already follows from Lemma 14 in form of (H1). Hence, we only have to discuss the case of the linearized flux correctors.

Applying the Caccioppoli estimate (T1) to equation (49b) for the linearized flux corrector $\sigma^T_{\xi,B,kl}$ entails the estimate

$$\begin{aligned}&\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^2_{L^2(B_1)}\\&\lesssim_{d,\lambda,\Lambda}\inf_{b\in\mathbb{R}}\Big\{\frac{1}{R^2}\big\|\sigma^T_{\xi,B,kl}-b\big\|^2_{L^2(B_2)}+\frac{1}{T}|b|^2\Big\}+\big\|q^T_{\xi,B}\big\|^2_{L^2(B_2)}.\end{aligned}$$

By the same argument which starts from the right hand side of (87) and produces the right hand side of (88), we obtain (with R replaced by 2, $R'=\theta 2$, $\theta=\theta(d,\lambda,\Lambda)$ yet to be determined)

$$\begin{aligned}\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^2_{L^2(B_1)}&\lesssim_{d,\lambda,\Lambda}\theta^2\fint_{B_4}\big|\nabla\sigma^T_{\xi,B,kl}\big|^2+\big\|q^T_{\xi,B}\big\|^2_{L^2(B_2)}\\&\quad+\fint_{B_4}\Big|\fint_{B_{\theta 2}(x)}\nabla\sigma^T_{\xi,B,kl}\Big|^2+\fint_{B_4}\Big|\fint_{B_{\theta 2}(x)}\frac{1}{\sqrt{T}}\sigma^T_{\xi,B,kl}\Big|^2.\end{aligned}$$

Hence, taking stochastic moments and exploiting stationarity yields

$$\begin{aligned}&\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}\\&\lesssim_{d,\lambda,\Lambda}\theta^2\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}+\big\langle\big\|q^T_{\xi,B}\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac1q}\\&\quad+\Big\langle\Big|\fint_{B_{\theta 2}}\nabla\sigma^T_{\xi,B,kl}\Big|^{2q}\Big\rangle^{\frac1q}+\Big\langle\Big|\fint_{B_{\theta 2}}\frac{1}{\sqrt{T}}\sigma^T_{\xi,B,kl}\Big|^{2q}\Big\rangle^{\frac1q}.\end{aligned}\tag{112}$$

In principle, we would like to absorb now the first right hand side term of the previous display into the left hand side by choosing θ appropriately. However, we first have to verify finiteness of $\langle\|(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl})\|^{2q}_{L^2(B_1)}\rangle^{\frac1q}$. This is done by appealing to the weighted energy estimate (T3), which with respect to equation (49b) for the linearized flux corrector $\sigma^T_{\xi,B,kl}$ entails

$$\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}\lesssim\sqrt{T}^d\int\ell_{\gamma,\sqrt{T}}\big\langle\big|q^T_{\xi,B}\big|^{2q}\big\rangle^{\frac1q}.$$

Plugging in the definition (49d) for the linearized flux and making use of $(\mathrm{A2})_L$ in Assumption 1 then gives the following update of the previous display

$$\begin{aligned}&\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac1q}\\&\lesssim\sqrt{T}^d\int\ell_{\gamma,\sqrt{T}}\big\langle\big|\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi,B}\big|^{2q}\big\rangle^{\frac1q}\\&\quad+\sqrt{T}^d\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\\Pi\neq\{\{1,\ldots,L\}\}}}\int\ell_{\gamma,\sqrt{T}}\prod_{\pi\in\Pi}\Big\langle\Big|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi}\Big|^{2q|\Pi|}\Big\rangle^{\frac{1}{q|\Pi|}}.\end{aligned}$$

Hence, it now follows from stationarity of the linearized homogenization correctors, smuggling in spatial averages over the unit ball based on (H3) (which is available also for $\phi^T_{\xi,B}$ thanks to Lemma 14), and finally the corrector bounds from (H1) (which again are valid up to linearization order L by Lemma 14) that

$$\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}}\lesssim\sqrt{T}^d.$$

We may now run an absorption argument for the first right hand side term in (112), and then combine this with a bound for $\big\langle\|q^T_{\xi,B}\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}}$ (by plugging in (49d) and using again the corrector bounds from (H1) up to linearization order L as in the preceding discussion) as well as the already established estimates (39) and (40) to infer

$$\Big\langle\Big\|\Big(\frac{\sigma^T_{\xi,B,kl}}{\sqrt{T}},\nabla\sigma^T_{\xi,B,kl}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}.$$

As the argument for $\psi^T_{\xi,B}$ proceeds along the same lines, this time of course based on the defining equation (49c) in form of

$$\frac{1}{T}\frac{\psi^T_{\xi,B,k}}{\sqrt{T}}-\Delta\frac{\psi^T_{\xi,B,k}}{\sqrt{T}}=\frac{1}{T}e_k\cdot\sqrt{T}q^T_{\xi,B}-\frac{1}{T}e_k\cdot\sqrt{T}\nabla\phi^T_{\xi,B},$$

we move on to the next step of the proof.

Step 3: (Proof of the estimate (42)) This is a direct consequence of the corrector bounds (41), the definition (49d) of the linearized flux, the annealed Hölder regularity of the linearized coefficient (T7), and the local Schauder estimate (T5) applied to the localized corrector equations (49a), (49b), and (49c).

Step 4: (Proof of the estimate (43)) Let $\eta_T:\mathbb{R}^d\to[0,1]$ be a smooth and compactly supported cutoff function such that $\eta_T\equiv 1$ throughout B_1 and $|\nabla\eta_T|\leq\frac{2}{\sqrt{T}}$. We may then compute

$$\begin{aligned}\fint_{B_1}\phi^T_{\xi,B}&=\int\frac{1}{T}\phi^T_{\xi,B}\eta_T\Big(\frac{1}{T}-\Delta\Big)^{-1}\Big(\frac{\mathbb{1}_{B_1}}{|B_1|}\Big)+\int\nabla\phi^T_{\xi,B}\cdot\eta_T\nabla\Big(\frac{1}{T}-\Delta\Big)^{-1}\Big(\frac{\mathbb{1}_{B_1}}{|B_1|}\Big)\\&\quad+\int\frac{1}{T}\phi^T_{\xi,B}\Big(T\nabla\eta_T\cdot\nabla\Big(\frac{1}{T}-\Delta\Big)^{-1}\Big(\frac{\mathbb{1}_{B_1}}{|B_1|}\Big)\Big).\end{aligned}$$

As a consequence of (39) and (40), we thus obtain

$$\begin{aligned}\Big\langle\Big|\fint_{B_1}\phi^T_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{q}}&\leq C^2q^{2C'}\int\Big|\Big(\frac{1}{\sqrt{T}}\Big(\frac{1}{T}-\Delta\Big)^{-1}\Big(\frac{\mathbb{1}_{B_1}}{|B_1|}\Big),\nabla\Big(\frac{1}{T}-\Delta\Big)^{-1}\Big(\frac{\mathbb{1}_{B_1}}{|B_1|}\Big)\Big)\Big|^2\\&\leq C^2q^{2C'}\mu_*^2(\sqrt{T}).\end{aligned}$$

The bound for the linearized flux correctors $(\sigma^T_{\xi,B},\psi^T_{\xi,B})$ follows analogously.

Step 5: (Proof of equation (49e)) By the sublinear growth of the flux corrector, it suffices to verify

$$\Big(\frac{1}{T}-\Delta\Big)\Big(\nabla\cdot\sigma^T_{\xi,B}\Big)=\Big(\frac{1}{T}-\Delta\Big)\Big(q^T_{\xi,B}-\langle q^T_{\xi,B}\rangle+\frac{1}{T}\psi^T_{\xi,B}\Big).\tag{113}$$

We obtain from (49a), (49b) and (49d) that

$$\Big(\frac{1}{T}-\Delta\Big)\Big(\nabla\cdot\sigma^T_{\xi,B}\Big)=\partial_l\partial_k\big(q^T_{\xi,B}\big)_k-\partial_k\partial_k\big(q^T_{\xi,B}\big)_l=\frac{1}{T}\partial_l\phi^T_{\xi,B}-\Delta q_l.$$

Hence, plugging in (49c) yields the asserted identity (113). This in turn concludes the proof of Theorem 5. □

4.8. **Proof of Lemma 16** (Estimates for differences of linearized correctors)**.** The proof of (60)–(62) proceeds via an induction over the linearization order.

Step 1: (Induction hypotheses) Let $L \in \mathbb{N}$, $T \in [1,\infty)$ and $M > 0$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_L$, $(\mathrm{A3})_L$ and $(\mathrm{A4})_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place.

For any $0 \leq l \leq L{-}1$, any $|\xi| \leq M$, any $|h| \leq 1$, and any collection of unit vectors $v_1',\ldots,v_l' \in \mathbb{R}^d$ the difference $\phi^T_{\xi+he,B'}{-}\phi^T_{\xi,B'}$ of linearized homogenization correctors in direction $B' := v_1' \odot \cdots \odot v_l'$ is assumed to satisfy—under the above conditions—the following list of estimates (if $l = 0$—and thus B' being an empty symmetric tensor product—$\phi^T_{\xi,B'}$ is understood to denote the localized homogenization corrector ϕ^T_ξ of the nonlinear problem with a massive term):

- For any $\beta \in (0,1)$, there exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\beta)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L,\beta)$ such that for all $q \in [1,\infty)$, and all compactly supported and square-integrable f,g we have *corrector estimates for differences*

$$\begin{aligned}
\Big\langle\Big|\int g\cdot\big(\nabla\phi^T_{\xi+he,B'}{-}\nabla\phi^T_{\xi,B'}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} &\leq C^2q^{2C'}|h|^{2(1-\beta)}\int|g|^2,\\
\Big\langle\Big|\int \frac{1}{T}f\big(\phi^T_{\xi+he,B'}{-}\phi^T_{\xi,B'}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} &\leq C^2q^{2C'}|h|^{2(1-\beta)}\int\Big|\frac{f}{\sqrt{T}}\Big|^2,\\
\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B'}{-}\phi^T_{\xi,B'}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B'}{-}\nabla\phi^T_{\xi,B'}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} &\leq C^2q^{2C'}|h|^{2(1-\beta)}.
\end{aligned} \tag{Hdiff1}$$

- Fix $p \in (2,\infty)$, and let $g \in \bigcap_{q\geq1}L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$ be a compactly supported and $L^p(\mathbb{R}^d)$-valued random field. Then there exists a random field $G^T_{\xi,B',h,e}$ satisfying $\big[G^T_{\xi,B',h,e}\big]_1 \in \bigcap_{q\geq1}L^{2q}_{\langle\cdot\rangle}L^2(\mathbb{R}^d;\mathbb{R}^n)$, and which is related to g via $\phi^T_{\xi+he,B'}{-}\phi^T_{\xi,B'}$ in the sense that, $\mathbf{P}$-almost surely, it holds for all perturbations $\delta\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$

$$\int g\cdot\nabla\big(\delta\phi^T_{\xi+he,B'}{-}\delta\phi^T_{\xi,B'}\big) = \int G^T_{\xi,B',h,e}\cdot\delta\omega; \tag{Hdiff2a}$$

cf. Lemma 26 for the Gâteaux derivative of the linearized corrector and its gradient in direction $\delta\omega$.

There exists $q_0 = q_0(d,\lambda,\Lambda) \in (1,\infty)$ such that for any $\kappa \in (0,1]$ and any $\beta \in (0,1)$, there exist some constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\kappa,\beta)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L,\beta)$ such that for all $q \in [1,\infty)$ and all $|\xi| \leq M$ the random field $G^T_{\xi,B',h,e}$ gives rise to a *sensitivity estimate for differences*

$$\begin{aligned}
&\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^T_{\xi,B',h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\\
&\leq C^2q^{2C'}|h|^{2(1-\beta)}\sup_{\langle F^{2q_*}\rangle=1}\int\big\langle|Fg|^{2(\frac{q\vee q_0}{\kappa})_*}\big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.
\end{aligned} \tag{Hdiff2b}$$

If $(g_r)_{r\geq1}$ is a sequence in $\bigcap_{q\geq1}L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$ of compactly supported and $L^p(\mathbb{R}^d)$-valued random fields, denote by $G^{T,r}_{\xi,B',h,e}$, $r \geq 1$, the random field associated to g_r, $r \geq 1$, in the sense of (Hdiff2a). Let g be an $L^p_{\mathrm{loc}}(\mathbb{R}^d)$-valued random field such that $g_r \to g$ as $r \to \infty$ in $\bigcap_{q\geq1}L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$. Then there exists a random field $G^T_{\xi,B',h,e}$ with

$$\big[G^{T,r}_{\xi,B',h,e} - G^T_{\xi,B',h,e}\big]_1 \to 0 \text{ as } r\to\infty \text{ in } \bigcap_{q\geq 1} L^{2q}_{\langle\cdot\rangle} L^2(\mathbb{R}^d;\mathbb{R}^n), \tag{Hdiff2c}$$

and the limit random field $G^T_{\xi,B',h,e}$ is subject to the sensitivity estimate (Hdiff2b).

- There exists $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $\beta \in (0,1)$ there exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M,L,\beta)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta,L,\beta)$ such that for all $q\in[1,\infty)$ and all $|\xi|\leq M$ we have a *small-scale annealed Schauder estimate for differences*

$$\Big\langle \big\|\nabla\phi^T_{\xi+he,B'} - \nabla\phi^T_{\xi,B'}\big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{2(1-\beta)}. \tag{Hdiff3}$$

Step 2: (Base case of the induction) The base case concerns the correctors of the nonlinear problem with an additional massive term. A proof of the corresponding assertions from the induction hypotheses is given in Appendix C by means of Lemma 27.

Step 3: (Induction step—Reduction to linear functionals) Subtracting the defining equations (49a) for $\phi^T_{\xi+he,B}$ resp. $\phi^T_{\xi,B}$, as well as adding zero yields the following equation for the difference of linearized correctors

$$\begin{aligned}
&\frac{1}{T}\big(\phi^T_{\xi+he,B}-\phi^T_{\xi,B}\big) - \nabla\cdot a^T_\xi\big(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big)\\
&= \nabla\cdot\big(a^T_{\xi+he}-a^T_\xi\big)\big(\mathbb{1}_{L=1}B + \nabla\phi^T_{\xi+he,B}\big)\\
&\quad + \nabla\cdot \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&\quad - \nabla\cdot \sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial^{|\Pi|}_\xi A)(\omega,\xi+\nabla\phi^T_{\xi})\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big]\\
&=: \nabla\cdot R^T_{\xi,B,h,e}.
\end{aligned} \tag{114}$$

Recall that we denote by a^T_ξ the uniformly elliptic and bounded coefficient field $(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$ with respect to the constants (λ,Λ) from Assumption 1. Let a radius $R\in[1,\infty)$ be fixed. Applying first the hole filling estimate (T2) and then Caccioppoli's estimate (T1) to equation (114) yields

$$\begin{aligned}
&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}}\\
&\lesssim_{d,\lambda,\Lambda} R^{d-\delta}\Big\langle\Big|\inf_{b\in\mathbb{R}}\frac{1}{(2R)^2}\fint_{B_{2R}}\big|\phi^T_{\xi+he,B}-\phi^T_{\xi,B}-b\big|^2+\frac{1}{T}|b|^2\Big|^q\Big\rangle^{\frac{1}{q}}\\
&\quad + R^{d-\delta}\Big\langle\Big|\fint_{B_{2R}}\big|R^T_{\xi,B,h,e}\big|^2\Big|^q\Big\rangle^{\frac{1}{q}} + R^d\Big\langle\Big|\fint_{B_R}\frac{1}{|x|^\delta}\big|R^T_{\xi,B,h,e}\big|^2\Big|^q\Big\rangle^{\frac{1}{q}}.
\end{aligned}$$

By the same argument which starts from the right hand side of (87) and produces the right hand side of (88) (with $R' = \theta R$, $\theta = \theta(d,\lambda,\Lambda)$ yet to be determined, and with $\phi^T_{\xi,B}$ replaced by $\phi^T_{\xi+he,B}-\phi^T_{\xi,B}$), and the stationarity of linearized homogenization correctors we obtain

$$
\begin{aligned}
&\bigg\langle\bigg|\inf_{b\in\mathbb{R}}\frac{1}{(2R)^2}\fint_{B_{2R}}\big|\phi^T_{\xi+he,B}-\phi^T_{\xi,B}-b\big|^2+\frac{1}{T}|b|^2\bigg|^q\bigg\rangle^{\frac{1}{q}}\\
&\lesssim_{d,\lambda,\Lambda}\theta^2\bigg\langle\bigg|\fint_{B_R}\big|\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big|^2\bigg|^q\bigg\rangle^{\frac{1}{q}}\\
&\quad+\bigg\langle\bigg|\fint_{B_{\theta R}}\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}+\bigg\langle\bigg|\fint_{B_{\theta R}}\frac{1}{\sqrt{T}}\phi^T_{\xi+he,B}-\frac{1}{\sqrt{T}}\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}\\
&\lesssim_{d,\lambda,\Lambda}\theta^2\bigg\langle\bigg|\inf_{b\in\mathbb{R}}\frac{1}{(2R)^2}\fint_{B_{2R}}\big|\phi^T_{\xi+he,B}-\phi^T_{\xi,B}-b\big|^2+\frac{1}{T}|b|^2\bigg|^q\bigg\rangle^{\frac{1}{q}}\\
&\quad+\theta^2\bigg\langle\bigg|\fint_{B_{2R}}\big|R^T_{\xi,B,h,e}\big|^2\bigg|^q\bigg\rangle^{\frac{1}{q}}\\
&\quad+\bigg\langle\bigg|\fint_{B_{\theta R}}\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}+\bigg\langle\bigg|\fint_{B_{\theta R}}\frac{1}{\sqrt{T}}\phi^T_{\xi+he,B}-\frac{1}{\sqrt{T}}\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}.
\end{aligned}
$$

In the second step, we again used Caccioppoli's inequality with respect to equation (114). Choosing $\theta=\theta(d,\lambda,\Lambda)$ sufficiently small, and exploiting the stationarity of the right hand side term $R^T_{\xi,B,h,e}$ of equation (114), then entails in light of the previous two displays by an absorption argument that

$$
\begin{aligned}
&\bigg\langle\bigg\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big)\bigg\|^{2q}_{L^2(B_1)}\bigg\rangle^{\frac{1}{q}}\\
&\lesssim_{d,\lambda,\Lambda}R^{d-\delta}\bigg\langle\bigg|\fint_{B_{\theta R}}\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}\\
&\quad+R^{d-\delta}\bigg\langle\bigg|\fint_{B_{\theta R}}\frac{1}{\sqrt{T}}\phi^T_{\xi+he,B}-\frac{1}{\sqrt{T}}\phi^T_{\xi,B}\bigg|^{2q}\bigg\rangle^{\frac{1}{q}}\\
&\quad+R^{d-\delta}\bigg\langle\bigg|\fint_{B_1}\big|R^T_{\xi,B,h,e}\big|^4\bigg|^{\frac{q}{2}}\bigg\rangle^{\frac{1}{q}}.
\end{aligned}
\tag{115}
$$

For the remaining parts of the proof, let us make use of the abbreviation $\sum_\Pi:=\sum_{\Pi\in\mathrm{Par}\{1,\ldots,L\},\,\Pi\neq\{\{1,\ldots,L\}\}}$. By adding zero, we may then express the right hand side term $R^T_{\xi,B,h,e}$ in equation (114) in the following equivalent form:

$$
\begin{aligned}
&R^T_{\xi,B,h,e}\\
&=\big((\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big)\big(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B}\big)\\
&\quad+\sum_\Pi\big((\partial_\xi^{|\Pi|}A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\big)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&\quad-\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})-\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big].
\end{aligned}
\tag{116}
$$

As a consequence of an application of Hölder's inequality, stationarity of the linearized homogenization correctors, and $(\mathrm{A2})_L$ from Assumption 1, we deduce from the previous display that

$$
\begin{aligned}
&\Big\langle \Big| \fint_{B_1} \big|R^T_{\xi,B,h,e}\big|^2 \Big|^q \Big\rangle^{\frac{1}{q}} \\
&\lesssim_\Lambda \big\langle \big\|\mathbb{1}_{L=1}B + \nabla\phi^T_{\xi+he,B}\big\|^{4q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q}} \Big\langle \Big| \fint_{B_1} \big|he + \nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big|^2 \Big|^{2q} \Big\rangle^{\frac{1}{2q}} \\
&\quad + \sum_\Pi \prod_{\pi\in\Pi} \big\langle \big\|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi+he,B'_\pi}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}} \Big\langle \Big| \fint_{B_1} \big|he + \nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big|^2 \Big|^{2q} \Big\rangle^{\frac{1}{2q}} \\
&\quad + \sum_\Pi \sup_{\pi\in\Pi} \Big\langle \Big| \fint_{B_1} \big|\nabla\phi^T_{\xi+he,B'_\pi} - \nabla\phi^T_{\xi,B'_\pi}\big|^2 \Big|^{q|\Pi|} \Big\rangle^{\frac{1}{q|\Pi|}} \\
&\qquad\qquad \times \sup_{\substack{\pi'\in\Pi \\ \pi'\neq\pi}} \Big\{ 1 + \big\langle \big\|\nabla\phi^T_{\xi+he,B'_\pi}\big\|^{2q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q|\Pi|}} + \big\langle \big\|\nabla\phi^T_{\xi,B'_\pi}\big\|^{2q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q|\Pi|}} \Big\}.
\end{aligned}
$$

It thus follows from the induction hypothesis (Hdiff1), the small-scale annealed Schauder estimates (H3) (which are available to any linearization order $\leq L$) and the previous display that

$$
\Big\langle \Big| \fint_{B_1} \big|R^T_{\xi,B,h,e}\big|^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})}. \tag{117}
$$

Observe also that based on induction hypothesis (Hdiff3) and the above argument for the right hand side term $R^T_{\xi,B,h,e}$ of equation (114), we also get the estimate

$$
\big\langle \big\|R^T_{\xi,B,h,e}\big\|^{2q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})}. \tag{118}
$$

Smuggling in a spatial average over the unit ball, we deduce from the previous two displays and the stationarity of $R^T_{\xi,B,h,e}$ that

$$
\begin{aligned}
&\Big\langle \Big| \fint_{B_1} \big|R^T_{\xi,B,h,e}\big|^4 \Big|^{\frac{q}{2}} \Big\rangle^{\frac{1}{q}} \\
&\lesssim \big\langle \big\|R^T_{\xi,B,h,e}\big\|^{2q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q}} + \Big\langle \Big| \fint_{B_1} \big|R^T_{\xi,B,h,e}\big|^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})}.
\end{aligned}
$$

This in turn entails the following update of (115):

$$
\begin{aligned}
&\Big\langle \Big\| \Big(\frac{\phi^T_{\xi+he,B} - \phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} \Big) \Big\|^{2q}_{L^2(B_1)} \Big\rangle^{\frac{1}{q}} \\
&\lesssim_{d,\lambda,\Lambda} R^{d-\delta} \Big\langle \Big| \fint_{B_{\theta R}} \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} \Big|^{2q} \Big\rangle^{\frac{1}{q}} \\
&\quad + R^{d-\delta} \Big\langle \Big| \fint_{B_{\theta R}} \frac{1}{\sqrt{T}}\phi^T_{\xi+he,B} - \frac{1}{\sqrt{T}}\phi^T_{\xi,B} \Big|^{2q} \Big\rangle^{\frac{1}{q}} \\
&\quad + R^{d-\delta} C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})}.
\end{aligned} \tag{119}
$$

The upshot of the argument is now the following. We will establish in the next step of the proof that for any $\tau = \tau(d,\lambda,\Lambda,\beta) \in (0,1)$ and any $q \in [q_0,\infty)$ (with

$q_0 = q_0(d,\lambda,\Lambda) \in (1,\infty)$ sufficiently large)

$$\begin{aligned} &\Big\langle \Big| \int g \cdot \big(\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} \\ &\leq C^2 q^{2C} \Big\langle \Big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big\|^{2\frac{q}{(1-\tau)^2}}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^2}{q}} \int |g|^2 \\ &\quad + C^2 q^{2C} |h|^{2(1-\frac{\beta}{2})} \int |g|^2, \end{aligned} \tag{120}$$

as well as

$$\begin{aligned} &\Big\langle \Big| \int \frac{1}{T} f \big(\phi^T_{\xi+he,B'} - \phi^T_{\xi,B'}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} \\ &\leq C^2 q^{2C'} \Big\langle \Big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big\|^{2\frac{q}{(1-\tau)^2}}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^2}{q}} \int \Big|\frac{f}{\sqrt{T}}\Big|^2 \\ &\quad + C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})} \int \Big|\frac{f}{\sqrt{T}}\Big|^2. \end{aligned} \tag{121}$$

The latter two estimates in turn update (119) to

$$\begin{aligned} &\Big\langle \Big\| \Big(\frac{\phi^T_{\xi+he,B} - \phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} \\ &\lesssim_{d,\lambda,\Lambda} R^{-\delta} C^2 q^{2C'} \Big\langle \Big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big\|^{2\frac{q}{(1-\tau)^2}}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^2}{q}} \\ &\quad + R^{d-\delta} C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})} \end{aligned} \tag{122}$$

for all $q \geq q_0$. We are one step away from choosing a suitable radius $R \in [1,\infty)$. Before we do so, we first want to exploit that we already have—to any linearization order $\leq L$—the corrector estimates (41) at our disposal. We leverage on that in form of decomposing $\frac{q}{(1-\tau)^2} = q(1-\tau) + q(\frac{1}{(1-\tau)^2} - (1-\tau))$ and applying Hölder's inequality with respect to the exponents $(\frac{1}{1-\tau}, \frac{1}{\tau})$

$$\Big\langle \Big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big\|^{2\frac{q}{(1-\tau)^2}}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^2}{q}} \leq C^2 q^{2C'} \Big\langle \Big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^3}{q}}$$

with the constant C independent of $|h| \leq 1$. This provides an upgrade of (122) in form of

$$\begin{aligned} &\Big\langle \Big\| \Big(\frac{\phi^T_{\xi+he,B} - \phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} \\ &\lesssim_{d,\lambda,\Lambda} R^{-\delta} C^2 q^{2C'} \Big\langle \Big\| \Big(\frac{\phi^T_{\xi+he,B} - \phi^T_{\xi,B}}{\sqrt{T}}, \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{(1-\tau)^3}{q}} \\ &\quad + R^{d-\delta} C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})} \end{aligned} \tag{123}$$

for all $q \geq q_0$. We may assume without loss of generality that the target term (i.e, the left hand side term of the previous display) is $\geq |h|^2$ (otherwise, there is nothing to prove in the first place). We then choose the radius $R \in [1,\infty)$ and the parameter $\tau=\tau(d,\lambda,\Lambda,\beta)$ in form of

$$\begin{aligned}(1-\tau)^3 &:= 1-\frac{\beta}{2}\frac{\delta}{d-\delta},\\ R^\delta &:= \frac{\Theta(d,\lambda,\Lambda)C^2q^{2C'}}{1\wedge\big\langle\|(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B})\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1-(1-\tau)^3}{q}}},\end{aligned}$$

with $\Theta(d,\lambda,\Lambda)$ sufficiently large in order to allow for an absorption argument in (123). In summary, we obtain

$$\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}|h|^{2(1-\beta)}$$

for all $q\in[1,\infty)$. In other words, under the additional assumption of (120) and (121) we proved that the last estimate in (Hdiff1) with B' replaced by B is satisfied. Plugging this information back into (120) and (121) in turn establishes the first two estimates from (Hdiff1) with B' replaced by B.

Last but not least, applying the local Schauder estimate (T5) to equation (114) (which is admissible based on the annealed Hölder regularity of the linearized coefficient field, see Lemma 24) and making use of Hölder's inequality in combination with the estimates (T7) and (118) moreover yields

$$\begin{aligned}&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big)\Big\|^{2q}_{C^\alpha(B_1)}\Big\rangle^{\frac{1}{q}} \qquad (124)\\ &\leq C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}+C^2q^{2C'}\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big)\Big\|^{4q}_{L^2(B_1)}\Big\rangle^{\frac{1}{2q}}.\end{aligned}$$

In particular, the small-scale annealed Schauder estimate (Hdiff3) with B' replaced by B also holds true by the above reasoning once the validity of the estimates (120) and (121) is established.

Step 4: (Induction step—Estimates (120) *and* (121) *for linear functionals)* For notational convenience, we only discuss in detail the derivation of the estimate (120). The second one follows along the same lines.

We start with the computation of the functional derivative of the centered random variable $F^{\mathrm{diff}}_\phi:=\int g\cdot(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B})$. To this end, let $\delta\omega\in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)}<\infty$. Based on Lemma 26, we may $\mathbf{P}$-almost surely differentiate the equation (114) for differences of linearized homogenization correctors with respect to the parameter field in the direction of $\delta\omega$ (taking already into account the representation (116) of the right hand side term). This yields $\mathbf{P}$-almost surely the following PDE for the variation

$$\Big(\frac{(\delta\phi^T_{\xi+he,B}-\delta\phi^T_{\xi,B})}{\sqrt{T}},\nabla(\delta\phi^T_{\xi+he,B}-\delta\phi^T_{\xi,B})\Big)\in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$$

of differences of linearized homogenization correctors:

$$
\begin{aligned}
&\frac{1}{T}(\delta\phi^T_{\xi+he,B}-\delta\phi^T_{\xi,B})-\nabla\cdot a^T_\xi\nabla(\delta\phi^T_{\xi+he,B}-\delta\phi^T_{\xi,B})\\
&=\nabla\cdot(\partial_\omega\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\delta\omega\odot(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B})\big]\\
&+\nabla\cdot(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\nabla\delta\phi^T_\xi\odot(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B})\big]\\
&+\nabla\cdot\big((\partial_\omega\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\omega\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big)\big[\delta\omega\odot(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B})\big]\\
&+\nabla\cdot\big((\partial^2_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big)\big[\nabla\delta\phi^T_{\xi+he}\odot(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B})\big]\\
&+\nabla\cdot(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[\nabla\delta(\phi^T_{\xi+he}-\phi^T_\xi)\odot(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B})\big]\\
&+\nabla\cdot\big((\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big)\nabla\delta\phi^T_{\xi+he,B}\\
&+\nabla\cdot\sum_\Pi\big((\partial_\omega\partial_\xi^{|\Pi|}A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\omega\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\big)\Big[\delta\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&+\nabla\cdot\sum_\Pi\big((\partial_\xi^{|\Pi|+1}A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi^{|\Pi|+1}A)(\omega,\xi+\nabla\phi^T_\xi)\big)\Big[\nabla\delta\phi^T_{\xi+he}\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&+\nabla\cdot\sum_\Pi(\partial_\xi^{|\Pi|+1}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\nabla\delta(\phi^T_{\xi+he}-\phi^T_\xi)\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&+\nabla\cdot\sum_\Pi\big((\partial_\xi^{|\Pi|}A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\big)\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi+he,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi+he,B'_{\pi'}})\Big]\\
&-\nabla\cdot\sum_\Pi(\partial_\omega\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\delta\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})-\delta\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&-\nabla\cdot\sum_\Pi(\partial_\xi^{|\Pi|+1}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\nabla\delta\phi^T_\xi\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})-\nabla\delta\phi^T_\xi\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&-\nabla\cdot\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\delta(\phi^T_{\xi,B'_\pi}-\phi^T_{\xi+he,B'_\pi})\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi,B'_{\pi'}})\Big]\\
&+\nabla\cdot\bigg\{\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi+he,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi+he,B'_{\pi'}})\Big]\\
&\qquad-\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi+he,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}+\nabla\phi^T_{\xi,B'_{\pi'}})\Big]\bigg\}\\
&=:\nabla\cdot R^{T,(1)}_{\xi,B,h,e}\delta\omega+\nabla\cdot R^{T,(2)}_{\xi,B,h,e}\nabla\delta\phi^T_\xi+\nabla\cdot R^{T,(3)}_{\xi,B,h,e}\delta\omega \qquad (125)\\
&\quad+\nabla\cdot R^{T,(4)}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he}+\nabla\cdot R^{T,(5)}_{\xi,B,h,e}\nabla(\delta\phi^T_{\xi+he}-\delta\phi^T_\xi)+\nabla\cdot R^{T,(6)}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he,B}\\
&\quad+\nabla\cdot R^{T,(7)}_{\xi,B,h,e}\delta\omega+\nabla\cdot R^{T,(8)}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he}+\nabla\cdot R^{T,(9)}_{\xi,B,h,e}\nabla(\delta\phi^T_{\xi+he}-\delta\phi^T_\xi)\\
&\quad+\sum_\Pi\sum_{\pi\in\Pi}\nabla\cdot R^{T,(10),\pi}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he,B'_\pi}+\nabla\cdot R^{T,(11)}_{\xi,B,h,e}\delta\omega+\nabla\cdot R^{T,(12)}_{\xi,B,h,e}\nabla\delta\phi^T_\xi\\
&\quad+\sum_\Pi\sum_{\pi\in\Pi}\nabla\cdot R^{T,(13),\pi}_{\xi,B,h,e}\nabla(\delta\phi^T_{\xi+he,B'_\pi}-\delta\phi^T_{\xi,B'_\pi})\\
&\quad+\sum_\Pi\sum_{\pi\in\Pi}\nabla\cdot R^{T,(14),\pi}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he,B'_\pi}.
\end{aligned}
$$

Resorting to an approximation argument along the same lines as in *Step 1* of the proof of Lemma 12, and employing a duality argument based on the dual operator $(\frac{1}{T} - \nabla\cdot a_\xi^{T,*}\nabla)$, we may deduce from (125) that

$$
\begin{aligned}
\delta F_\phi^{\mathrm{diff}} &= \int g\cdot\nabla(\delta\phi^T_{\xi+he,B} - \delta\phi^T_{\xi,B}) \\
&= \lim_{r\to\infty}\bigg\{\int \mathbb{1}_{B_r}\big(R^{T,(1)}_{\xi,B,h,e}\delta\omega + R^{T,(2)}_{\xi,B,h,e}\nabla\delta\phi^T_\xi\big)\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \int \mathbb{1}_{B_r}\big(R^{T,(3)}_{\xi,B,h,e} + R^{T,(7)}_{\xi,B,h,e} + R^{T,(11)}_{\xi,B,h,e}\big)\delta\omega\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \int \mathbb{1}_{B_r} R^{T,(12)}_{\xi,B,h,e}\nabla\delta\phi^T_\xi\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \int \mathbb{1}_{B_r}\big(R^{T,(4)}_{\xi,B,h,e} + R^{T,(8)}_{\xi,B,h,e}\big)\nabla\delta\phi^T_{\xi+he}\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \int \mathbb{1}_{B_r} R^{T,(6)}_{\xi,B,h,e}\nabla\delta\phi^T_{\xi+he,B}\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \sum_{\Pi}\sum_{\pi\in\Pi}\int \mathbb{1}_{B_r}\big(R^{T,(10),\pi}_{\xi,B,h,e} + R^{T,(14),\pi}_{\xi,B,h,e}\big)\nabla\delta\phi^T_{\xi+he,B'_\pi}\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \int \mathbb{1}_{B_r}\big(R^{T,(5)}_{\xi,B,h,e} + R^{T,(9)}_{\xi,B,h,e}\big)\nabla(\delta\phi^T_{\xi+he} - \delta\phi^T_\xi)\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g) \\
&\quad + \sum_{\Pi}\sum_{\pi\in\Pi}\int \mathbb{1}_{B_r} R^{T,(13),\pi}_{\xi,B,h,e}\nabla(\delta\phi^T_{\xi+he,B'_\pi} - \delta\phi^T_{\xi,B'_\pi})\cdot\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot g)\bigg\}.
\end{aligned}
$$

Thanks to the induction hypothesis (Hdiff2a) resp. (Hdiff2c) and (H2a) resp. (H2c) (the latter already being available to any linearization order $\leq L$) we obtain

$$
\delta F_\phi^{\mathrm{diff}} = \sum_{i=1}^{14}\int G^{T,(i)}_{\xi,B,h,e}\cdot\delta\omega, \tag{126}
$$

for associated random fields $G^{T,(i)}_{\xi,B,h,e}$, $i\in\{1,\ldots,14\}$. We next feed (126) into the spectral gap inequality in form of (48) which entails

$$
\big\langle\big|F_\phi^{\mathrm{diff}}\big|^{2q}\big\rangle^{\frac{1}{q}} \leq C^2q^2\sum_{i=1}^{14}\bigg\langle\bigg|\int\bigg(\fint_{B_1(x)}\big|G^{T,(i)}_{\xi,B,h,e}\big|\bigg)^2\bigg|^q\bigg\rangle^{\frac{1}{q}}. \tag{127}
$$

It remains to estimate the terms on the right hand side of the previous display.

We start with the first two terms on the right hand side of (127), which are precisely those being responsible for the first right hand side term in (120). By duality in $L^q_{\langle\cdot\rangle}$, Hölder's inequality, stationarity of the linearized homogenization correctors, and $(\mathrm{A3})_L$ from Assumption 1 we get

$$
\begin{aligned}
&\bigg\langle\bigg|\int\bigg(\fint_{B_1(x)}\big|G^{T,(1)}_{\xi,B,h,e}\big|\bigg)^2\bigg|^q\bigg\rangle^{\frac{1}{q}} \\
&\leq C^2\big\langle\big\|\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}}\sup_{\langle F^{2q_*}\rangle=1}\int\bigg\langle\bigg|\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}(\nabla\cdot Fg)\bigg|^{2q_*}\bigg\rangle^{\frac{1}{q_*}}.
\end{aligned}
$$

Hence, at least for sufficiently large $q \in [q_0,\infty)$ such that $|(q_0)_*-1|$ is small enough in order to be in the perturbative regime of the annealed Calderón–Zygmund estimate in form of (T8), we deduce from the previous display that

$$\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^{T,(1)}_{\xi,B,h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} \leq C^2\big\langle\big\|\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}}\int|g|^2. \quad (128)$$

For the second term, we instead rely on (H2b) (applied with the choice $\kappa=\tau$) and $(\mathrm{A2})_L$ from Assumption 1 to infer for all $q\in[q_0,\infty)$

$$\begin{aligned}&\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^{T,(2)}_{\xi,B,h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\\ &\leq C^2\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\Big|^{2(\frac{q}{\tau})_*}\Big|\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2(\frac{q}{\tau})_*}\Big\rangle^{\frac{1}{(\frac{q}{\tau})_*}}.\end{aligned}$$

Applying Hölder's inequality with exponents $(\frac{q-\tau}{1-\tau},(\frac{q-\tau}{1-\tau})_*=\frac{q-\tau}{q-1})$ it then follows from $(\frac{q}{\tau})_*=\frac{q}{q-\tau}$ and (T8) that

$$\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^{T,(2)}_{\xi,B,h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} \leq C^2\big\langle\big\|\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big\|^{2\frac{q}{1-\tau}}_{C^\alpha(B_1)}\big\rangle^{\frac{1-\tau}{q}}\int|g|^2,$$

again at least for sufficiently large $q\in[q_0,\infty)$. Combining this with (124) updates the previous display to

$$\begin{aligned}&\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^{T,(2)}_{\xi,B,h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\\ &\leq C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}\int|g|^2+C^2\big\langle\big\|\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big\|^{2\frac{q}{(1-\tau)^2}}_{L^2(B_1)}\big\rangle^{\frac{(1-\tau)^2}{q}}\int|g|^2,\end{aligned} \quad (129)$$

at least for sufficiently large $q\in[q_0,\infty)$.

For the remaining terms on the right hand side of (127), note that all of them incorporate a difference of *lower-order* linearized homogenization correctors. In view of induction hypotheses (Hdiff1), (Hdiff2b) and (Hdiff3), one thus expects them to contribute to the second right hand side term of (120). We verify this by grouping these terms into three categories.

First, we estimate by duality in $L^q_{\langle\cdot\rangle}$, Hölder's inequality, stationarity of the linearized homogenization correctors, and $(\mathrm{A3})_L$ as well as $(\mathrm{A4})_L$ from Assumption 1

$$\begin{aligned}&\sum_{i\in\{3,7,11\}}\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G^{T,(i)}_{\xi,B,h,e}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\\ &\leq C^2\big\langle\big\|he+\nabla\phi^T_{\xi+he}-\nabla\phi^T_\xi\big\|^{4q}_{L^2(B_1)}\big\rangle^{\frac{1}{2q}}\big\langle\big\|\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B}\big\|^{4q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q}}\\ &\qquad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}\\ &\quad+C^2\big\langle\big\|he+\nabla\phi^T_{\xi+he}-\nabla\phi^T_\xi\big\|^{4q}_{L^2(B_1)}\big\rangle^{\frac{1}{2q}}\sum_{\Pi}\prod_{\pi\in\Pi}\big\langle\big\|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}}\\ &\qquad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}\end{aligned}$$

$$
\begin{aligned}
&+ C^2 \sum_{\Pi} \sup_{\pi\in\Pi} \big\langle \big\|\nabla\phi^T_{\xi+he,B'_\pi} - \nabla\phi^T_{\xi,B'_\pi}\big\|^{2q|\Pi|}_{L^2(B_1)}\big\rangle^{\frac{1}{q|\Pi|}} \\
&\qquad\qquad \times \sup_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}} \Big\{1 + \big\langle \big\|\nabla\phi^T_{\xi+he,B'_\pi}\big\|^{2q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q|\Pi|}} + \big\langle \big\|\nabla\phi^T_{\xi,B'_\pi}\big\|^{2q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q|\Pi|}}\Big\} \\
&\qquad\qquad \times \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(\frac{1}{T} - \nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}.
\end{aligned}
$$

Hence, a combination of the induction hypothesis (Hdiff1) with the annealed small-scale Schauder estimate (H3) (which is available to any linearization order $\leq L$) and the perturbative annealed Calderón–Zygmund estimate (T8) entails for sufficiently large $q \in [q_0,\infty)$

$$
\sum_{i\in\{3,7,11\}} \Big\langle \Big| \int \Big(\fint_{B_1(x)} \big|G^{T,(i)}_{\xi,B,h,e}\big| \Big)^2 \Big|^q \Big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'} |h|^{2(1-\frac{\beta}{2})} \int |g|^2. \tag{130}
$$

Second, we estimate by means of (H2b) with $\kappa = \frac{1}{2}$ (which is already available up to any linearization order $\leq L$), stationarity of the linearized homogenization correctors, Hölder's inequality with respect to the exponents $\big(\frac{q-\frac12}{1-\frac12}, (\frac{q-\frac12}{1-\frac12})_* = \frac{q-\frac12}{q-1}\big)$, the fact that $(\frac{q}{\frac12})_* = \frac{q}{q-\frac12}$, and (A2)$_L$ as well as (A4)$_L$ from Assumption 1

$$
\begin{aligned}
&\sum_{i\in\{4,6,8,10,12,14\}} \Big\langle \Big| \int \Big(\fint_{B_1(x)} \big|G^{T,(i)}_{\xi,B,h,e}\big| \Big)^2 \Big|^q \Big\rangle^{\frac{1}{q}} \\
&\leq C^2 \big\langle \big\|he + \nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big\|^{8q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{4q}} \big\langle \big\|\mathbb{1}_{L=1}B + \nabla\phi^T_{\xi+he,B}\big\|^{8q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{4q}} \\
&\qquad\qquad\qquad \times \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(\frac{1}{T} - \nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}} \\
&\quad + C^2 \big\langle \big\|he + \nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big\|^{4q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q}} \\
&\qquad\qquad\qquad \times \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(\frac{1}{T} - \nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}} \\
&\quad + C^2 \big\langle \big\|he + \nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big\|^{8q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{4q}} \sum_{\Pi}\prod_{\pi\in\Pi} \big\langle \big\|\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi+he,B'}\big\|^{8q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{4q|\Pi|}} \\
&\qquad\qquad\qquad \times \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(\frac{1}{T} - \nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}} \\
&\quad + C^2 \sum_{\Pi} \sup_{\pi\in\Pi} \big\langle \big\|\nabla\phi^T_{\xi+he,B'_\pi} - \nabla\phi^T_{\xi,B'_\pi}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}} \\
&\qquad\qquad \times \sup_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}} \Big\{1 + \big\langle \big\|\nabla\phi^T_{\xi+he,B'_\pi}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}} + \big\langle \big\|\nabla\phi^T_{\xi,B'_\pi}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}}\Big\} \\
&\qquad\qquad \times \sup_{\langle F^{2q_*}\rangle=1} \int \Big\langle \Big|\Big(\frac{1}{T} - \nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}.
\end{aligned}
$$

This time, it thus follows from induction hypothesis (Hdiff3) in combination with the annealed small-scale Schauder estimate (H3) (which is already available to any linearization order $\leq L$) and the perturbative annealed Calderón–Zygmund estimate (T8)

$$\sum_{i\in\{4,6,8,10,12,14\}}\Big\langle\Big|\int\Big(\fint_{B_1(x)}\big|G^{T,(i)}_{\xi,B,h,e}\big|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}\int|g|^2,\quad(131)$$

at least for sufficiently large $q\in[q_0,\infty)$.

Third, and last, we estimate based on induction hypothesis (Hdiff2b) with $\kappa=\frac{1}{2}$, stationarity of the linearized homogenization correctors, Hölder's inequality with respect to the exponents $\big(\frac{q-\frac{1}{2}}{1-\frac{1}{2}},(\frac{q-\frac{1}{2}}{1-\frac{1}{2}})_*=\frac{q-\frac{1}{2}}{q-1}\big)$, the fact that $(\frac{q}{\frac{1}{2}})_*=\frac{q}{q-\frac{1}{2}}$, and finally $(\mathrm{A2})_L$ from Assumption 1

$$\begin{aligned}&\sum_{i\in\{5,9,13\}}\Big\langle\Big|\int\Big(\fint_{B_1(x)}\big|G^{T,(i)}_{\xi,B,h,e}\big|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\\&\leq C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}\big\langle\big\|\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B}\big\|^{4q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q}}\\&\qquad\qquad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}\\&\quad+C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}\sum_\Pi\prod_{\pi\in\Pi}\big\langle\big\|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'}\big\|^{4q|\Pi|}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{2q|\Pi|}}\\&\qquad\qquad\times\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(\frac{1}{T}-\nabla\cdot a^{T,*}_\xi\nabla\Big)^{-1}(\nabla\cdot Fg)\Big|^{2q_*}\Big\rangle^{\frac{1}{q_*}}.\end{aligned}$$

We then obtain for sufficiently large $q\in[q_0,\infty)$ the estimate

$$\sum_{i\in\{5,9,13\}}\Big\langle\Big|\int\Big(\fint_{B_1(x)}\big|G^{T,(i)}_{\xi,B,h,e}\big|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}|h|^{2(1-\frac{\beta}{2})}\int|g|^2\quad(132)$$

by means of the same ingredients as for (131).

Collecting the estimates (128)–(132) and feeding them back into (127) eventually entails the asserted estimate (120).

Step 5: (Induction step—Conclusion) As we already argued at the end of *Step 3* of this proof, the estimates from (Hdiff1) and (Hdiff3) now also hold true with B' replaced by B. In order to lift the validity of (Hdiff2a), (Hdiff2b) and (Hdiff2c) to linearization order L, one may in fact follow the principles of the proof of Lemma 14 and adapt them to the arguments from the previous step. This concludes the proof of the estimates (60)–(62) at least in the case of linearized homogenization correctors.

Step 6: (Proof of (60)–(62) for linearized flux correctors) The difference of two linearized flux correctors satisfies the equation

$$\begin{aligned}&\frac{1}{T}(\sigma^T_{\xi+he,B,kl}-\sigma^T_{\xi,B,kl})-\Delta(\sigma^T_{\xi+he,B,kl}-\sigma^T_{\xi,B,kl})\\&=-\nabla\cdot\big((e_l\otimes e_k-e_k\otimes e_l)(q^T_{\xi+he,B}-q^T_{\xi,B})\big).\end{aligned}\quad(133)$$

Following the argument in *Step 2* of the proof of Theorem 5 (see, e.g., (112)), the desired estimate on the difference $\sigma^T_{\xi+he,B,kl}-\sigma^T_{\xi,B,kl}$ of linearized flux correctors boils down to an estimate of linear functionals for the difference $\sigma^T_{\xi+he,B,kl}-\sigma^T_{\xi,B,kl}$ and an estimate on the difference of linearized fluxes $q^T_{\xi+he,B}-q^T_{\xi,B}$.

With respect to the latter, we derive from (49d), (114) and (116) that

$$\begin{aligned}
&q^T_{\xi+he,B}-q^T_{\xi,B}\\
&= (\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big(\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}\big) \qquad (134)\\
&\quad+\big((\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big)\big(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi+he,B}\big)\\
&\quad+\sum_\Pi\big((\partial_\xi^{|\Pi|}A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\big)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big]\\
&\quad-\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})-\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi+he,B'_\pi})\Big].
\end{aligned}$$

Since we already have established the last estimate from (Hdiff1) with B' replaced by B, we may conclude together with the argument leading to (117) that

$$\big\langle\big\|q^T_{\xi+he,B}-q^T_{\xi,B}\big\|^{2q}_{L^2(B_1)}\big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}|h|^{2(1-\beta)}. \qquad (135)$$

For an estimate on linear functionals of the difference $\sigma^T_{\xi+he,B,kl}-\sigma^T_{\xi,B,kl}$, the argument from *Step 1* of the proof of Theorem 5 applied to equation (133) shows that it suffices to have a corresponding estimate on linear functionals of the difference of linearized fluxes $q^T_{\xi+he,B}-q^T_{\xi,B}$ (or more precisely, the analogue of (58) for differences with an additional rate $|h|^{2(1-\beta)}$). However, this in turn is an immediate consequence of the argument in *Step 4* of this proof. Indeed, comparing with the right hand side of (125) the only additional term which has to be dealt with in a sensitivity estimate for (134) is given by

$$(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\nabla\big(\delta\phi^T_{\xi+he,B}-\delta\phi^T_{\xi,B}\big).$$

However, as we already lifted the estimates (Hdiff1) and (Hdiff2b) from B' to B, we immediately obtain the desired sensitivity estimate on differences of linearized fluxes. This in turn implies the estimates

$$\begin{aligned}
\Big\langle\Big|\int g\cdot\big(\nabla\sigma^T_{\xi+he,B}-\nabla\sigma^T_{\xi,B}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}}&\leq C^2q^{2C'}|h|^{2(1-\beta)}\int|g|^2,\\
\Big\langle\Big|\int\frac{1}{T}f\big(\sigma^T_{\xi+he,B}-\sigma^T_{\xi,B}\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}}&\leq C^2q^{2C'}|h|^{2(1-\beta)}\int\Big|\frac{f}{\sqrt{T}}\Big|^2.
\end{aligned} \qquad (136)$$

Feeding back (135) and (136) into the analogue of (112) with respect to equation (133) then yields the asserted estimate (60) for differences of linearized flux correctors. Of course, the first line of (136) is nothing else than the desired estimate (62) in terms of the difference of linearized flux correctors. Finally, the asserted annealed Schauder estimate (61) follows from (133), (134), (T7), (T5), and the fact that (Hdiff3) is already valid up to linearization order L (cf. the previous step of this proof). □

4.9. **Proof of Lemma 17** (Differentiability of massive correctors and the massive version of the homogenized operator)**.** We first consider the case of $q=1$, and argue in favor of (64) by induction over the linearization order. The base case consisting of the correctors of the nonlinear problem is treated in Appendix C by means of Lemma 28. We then establish the estimates (64) and (65) for general $q\in[1,\infty)$—first for the linearized homogenization correctors and then for linearized flux correctors—, and finally conclude with a proof of (63).

Step 1: (Induction hypothesis) Let $L\in\mathbb{N}$, $T\in[1,\infty)$ and $M>0$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_L$, $(\mathrm{A3})_L$ and $(\mathrm{A4})_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place.

For any $0\leq l\leq L{-}1$, any $|\xi|\leq M$, any $|h|\leq 1$, and any collection of unit vectors $v_1',\ldots,v_l',e\in\mathbb{R}^d$, the following first-order Taylor expansion

$$\phi^T_{\xi,B',e,h} := \phi^T_{\xi+he,B'} - \phi^T_{\xi,B'} - \phi^T_{\xi,B'\odot e}h$$

of linearized homogenization correctors in direction $B':=v_1'\odot\cdots\odot v_l'$ is assumed to satisfy—under the above conditions—the following estimate (if $l=0$—and thus B' being an empty symmetric tensor product—$\phi^T_{\xi,B'}$ is understood to denote the localized homogenization corrector ϕ^T_ξ of the nonlinear problem with an additional massive term):

$$\big\langle \big\|\nabla\phi^T_{\xi,B',e,h}\big\|^2_{L^2(B_1)}\big\rangle \leq C^2 h^{4(1-\beta)}. \tag{Hreg}$$

Step 2: (Induction step) We start by writing the equation for the first-order Taylor expansion $\phi^T_{\xi,B,e,h}=\phi^T_{\xi+he,B}-\phi^T_{\xi,B}-\phi^T_{\xi,B\odot e}h$ in a suitable form. To this end, we first derive a suitable representation of the first-order Taylor expansion for the linearized fluxes $q^T_{\xi,B,e,h}=q^T_{\xi+he,B}-q^T_{\xi,B}-q^T_{\xi,B\odot e}h$. Note that these expressions—most importantly $\phi^T_{\xi,B\odot e}$ resp. $q^T_{\xi,B\odot e}$—are indeed well-defined $\mathbf{P}$-almost surely under the assumptions of Lemma 17 thanks to Lemma 8. Furthermore, for a proof of (64) in case of $q=1$ we will not rely on corrector estimates for $\phi^T_{\xi,B\odot e}$ but only on estimates for (differences of) correctors up to linearization order $\leq L$. This is the reason why we can stick for the moment with Assumption 1 realized to linearization order L as claimed in Lemma 17 (cf. also Remark 18).

In view of the definition (49d) of the linearized fluxes, we split this task into two substeps. By adding zero and abbreviating as always $a^T_\xi := (\partial_\xi A)(\omega,\xi{+}\nabla\phi^T_\xi)$, we may rewrite the contribution from the first term on the right hand side of (49d) as follows

$$\begin{aligned}
&a^T_{\xi+he}(\mathbb{1}_{L=1}B{+}\nabla\phi^T_{\xi+he,B}) - a^T_\xi(\mathbb{1}_{L=1}B{+}\nabla\phi^T_{\xi,B}) - a^T_\xi\nabla\phi^T_{\xi,B\odot e}h\\
&= a^T_\xi(\nabla\phi^T_{\xi+he,B}{-}\nabla\phi^T_{\xi,B}{-}\nabla\phi^T_{\xi,B\odot e}h) + (a^T_{\xi+he}{-}a^T_\xi)(\nabla\phi^T_{\xi+he,B}{-}\nabla\phi^T_{\xi,B})\\
&\quad + (a^T_{\xi+he}{-}a^T_\xi)(\mathbb{1}_{L=1}B{+}\nabla\phi^T_{\xi,B}).
\end{aligned} \tag{137}$$

For the contribution from the second right hand side term of (49d), it is useful to split the sum in case of $q^T_{B\odot e}h$ in the following way:

$$\begin{aligned}
&\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L,L+1\}\\ \Pi\neq\{\{1,\ldots,L,L+1\}\}}} (\partial_\xi^{|\Pi|}A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}(B\odot e)'_\pi{+}\nabla\phi^T_{\xi,(B\odot e)'_\pi})\Big]h\\
&= (\partial_\xi^2 A)(\omega,\xi{+}\nabla\phi^T_\xi)\big[(eh{+}\nabla\phi^T_{\xi,e}h)\odot(\mathbb{1}_{L=1}B{+}\nabla\phi^T_{\xi,B})\big]\\
&\quad+\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial_\xi^{|\Pi|+1}A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[(eh{+}\nabla\phi^T_{\xi,e}h)\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi^T_{\xi,B'_\pi})\Big]\\
&\quad+\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}} (\partial_\xi^{|\Pi|}A)(\omega,\xi{+}\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\phi^T_{\xi,B'_\pi\odot e}h\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}{+}\nabla\phi^T_{\xi,B'_{\pi'}})\Big].
\end{aligned} \tag{138}$$

Adding zero several times and combining terms then yields based on (137) and (138) (where we from now on again abbreviate $\sum_{\Pi} := \sum_{\Pi\in\mathrm{Par}\{1,\ldots,L\},\,\Pi\neq\{\{1,\ldots,L\}\}}$)

$$q^T_{\xi+he,B} - q^T_{\xi,B} - q^T_{\xi,B\odot e}h = R_0 + R_1 + R_2 + R_3 + R_4, \tag{139}$$

with the right hand side terms being given by

$$\begin{aligned}
R_0 &:= a^T_\xi(\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} - \nabla\phi^T_{\xi,B\odot e}h),\\
R_1 &:= (a^T_{\xi+he} - a^T_\xi)(\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B}) + \Big\{\big(a^T_{\xi+he} - a^T_\xi\big)(\mathbb{1}_{L=1}B + \nabla\phi^T_{\xi,B})\\
&\qquad -(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[(eh+\nabla\phi^T_{\xi,e}h)\odot(\mathbb{1}_{L=1}B+\nabla\phi^T_{\xi,B})\big]\Big\},\\
R_2 &:= \sum_{\Pi}\Big\{(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi) - (\partial^{|\Pi|}_\xi A)(\omega,\xi+\nabla\phi^T_\xi) - (\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)[eh]\Big\}\\
&\qquad\qquad\qquad\qquad \times\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big],\\
R_3 &:= \sum_{\Pi}\Big\{(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he}) - (\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\Big\}\\
&\qquad\qquad\qquad \times\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big]\\
&\qquad - \sum_{\Pi}(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\nabla\phi^T_{\xi,e}h\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big],
\end{aligned}$$

as well as

$$\begin{aligned}
R_4 &:= \sum_{\Pi}(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})\\
&\qquad \times\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi+he,B'_\pi}) - \bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi + \nabla\phi^T_{\xi,B'_\pi})\Big]\\
&\qquad - \sum_{\Pi}(\partial^{|\Pi|}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big[\sum_{\pi\in\Pi}\nabla\phi^T_{\xi,B'_\pi\odot e}h\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'} + \nabla\phi^T_{\xi,B'_{\pi'}})\Big].
\end{aligned}$$

In particular, we obtain the following equation for the first-order Taylor expansion $\phi^T_{\xi,B,e,h} = \phi^T_{\xi+he,B} - \phi^T_{\xi,B} - \phi^T_{\xi,B\odot e}h$ of linearized homogenization correctors

$$\frac{1}{T}(\phi^T_{\xi+he,B} - \phi^T_{\xi,B} - \phi^T_{\xi,B\odot e}h) - \nabla\cdot a^T_\xi(\nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} - \nabla\phi^T_{\xi,B\odot e}h) = \nabla\cdot\sum_{i=1}^{4}R_i.$$

Applying the weighted energy estimate (T3) to the equation from the previous display then yields the bound

$$\begin{aligned}
&\int \ell_{\gamma,\sqrt{T}}\Big|\Big(\frac{\phi^T_{\xi+he,B} - \phi^T_{\xi,B} - \phi^T_{\xi,B\odot e}h}{\sqrt{T}}, \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} - \nabla\phi^T_{\xi,B\odot e}h\Big)\Big|^2\\
&\leq C^2 \sup_{i=1,\ldots,4}\int \ell_{\gamma,\sqrt{T}}|R_i|^2.
\end{aligned}$$

By stationarity of the linearized homogenization correctors, we may take the expected value in the latter estimate and infer

$$\begin{aligned}&\Big\langle\Big\|\Big(\frac{\phi^T_{\xi+he,B}-\phi^T_{\xi,B}-\phi^T_{\xi,B\odot e}h}{\sqrt{T}},\nabla\phi^T_{\xi+he,B}-\nabla\phi^T_{\xi,B}-\nabla\phi^T_{\xi,B\odot e}h\Big)\Big\|^2_{L^2(B_1)}\Big\rangle\\&\leq C^2\sup_{i=1,\dots,4}\big\langle\|R_i\|^2_{L^2(B_1)}\big\rangle.\end{aligned}\tag{140}$$

It remains to post-process the four right hand side terms of the previous display.

Estimate for R_1: We first rewrite

$$\begin{aligned}a^T_{\xi+he}-a^T_\xi&=(\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\\&\quad+(\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)-(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi).\end{aligned}\tag{141}$$

By means of $(\mathrm{A2})_L$ from Assumption 1, the previous display in particular entails

$$\begin{aligned}&a^T_{\xi+he}-a^T_\xi-(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\big[eh+\nabla\phi^T_{\xi,e}h\big]\\&=\int_0^1\Big\{(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi+she)-(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big\}[eh]\,\mathrm{d}s\\&\quad+\int_0^1(\partial^2_\xi A)\big(\omega,\xi+he+s\nabla\phi^T_{\xi+he}+(1-s)\nabla\phi^T_\xi\big)\big[\nabla\phi^T_{\xi+he}-\nabla\phi^T_\xi-\nabla\phi^T_{\xi,e}h\big]\,\mathrm{d}s\\&\quad+\int_0^1\Big\{(\partial^2_\xi A)\big(\omega,\xi+he+s\nabla\phi^T_{\xi+he}+(1-s)\nabla\phi^T_\xi\big)-(\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big\}\big[\nabla\phi^T_{\xi,e}h\big]\,\mathrm{d}s.\end{aligned}\tag{142}$$

Hence, it follows from (141) and (142), the induction hypothesis (Hreg), the corrector estimates (41) and (42), the corrector estimates for differences (60) and (61), and $(\mathrm{A2})_L$ as well as $(\mathrm{A4})_L$ from Assumption 1 that

$$\big\langle\|R_1\|^2_{L^2(B_1)}\big\rangle\leq C^2h^{4(1-\beta)}.\tag{143}$$

Estimate for R_2: It is a simple consequence of $(\mathrm{A2})_L$ from Assumption 1 that

$$\begin{aligned}&(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)-(\partial^{|\Pi|}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)-(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)[eh]\\&=\int_0^1\Big\{(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi+she)-(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big\}[eh]\,\mathrm{d}s.\end{aligned}\tag{144}$$

It thus follows from (144), Hölder's inequality, the corrector estimates (41) and (42), and $(\mathrm{A2})_L$ as well as $(\mathrm{A4})_L$ from Assumption 1 that

$$\big\langle\|R_2\|^2_{L^2(B_1)}\big\rangle\leq C^2h^4.\tag{145}$$

Estimate for R_3: We first express R_3 in equivalent form as follows:

$$\begin{aligned}R_3=&-\sum_\Pi\Big\{(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi)-(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+\nabla\phi^T_\xi+he)\Big\}\\&\qquad\times\Big[\nabla\phi^T_{\xi,e}h\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})\Big]\\&+\sum_\Pi\Big\{(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_{\xi+he})-(\partial^{|\Pi|}_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\\&\qquad-(\partial^{|\Pi|+1}_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\big[\nabla\phi^T_{\xi,e}h\big]\Big\}\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi^T_{\xi,B'_\pi})\Big].\end{aligned}$$

We also have thanks to $(\mathrm{A2})_L$ from Assumption 1

$$
\begin{aligned}
&(\partial_\xi^{|\Pi|}A)(\omega,\xi{+}he{+}\nabla\phi_{\xi+he}^T)-(\partial_\xi^{|\Pi|}A)(\omega,\xi{+}he{+}\nabla\phi_{\xi}^T)-(\partial_\xi^{|\Pi|+1}A)(\omega,\xi{+}he{+}\nabla\phi_{\xi}^T)\big[\nabla\phi_{\xi,e}^T h\big]\\
&=\int_0^1(\partial_\xi^{|\Pi|+1}A)(\omega,\xi{+}he{+}s\nabla\phi_{\xi+he}^T{+}(1{-}s)\nabla\phi_\xi^T)\big[\nabla\phi_{\xi+he}^T{-}\nabla\phi_\xi^T{-}\nabla\phi_{\xi,e}^T h\big]\,\mathrm{d}s\\
&\quad+\int_0^1\Big\{(\partial_\xi^{|\Pi|+1}A)(\omega,\xi{+}he{+}s\nabla\phi_{\xi+he}^T{+}(1{-}s)\nabla\phi_\xi^T)-(\partial_\xi^{|\Pi|+1}A)(\omega,\xi{+}he{+}\nabla\phi_\xi^T)\Big\}\big[\nabla\phi_{\xi,e}^T h\big]\,\mathrm{d}s.
\end{aligned}
\tag{146}
$$

Hence, it follows from (146), the induction hypothesis (Hreg), an application of Hölder's inequality, the corrector estimates (41) and (42), the corrector estimates for differences (60) and (61), and $(\mathrm{A2})_L$ as well as $(\mathrm{A4})_L$ from Assumption 1 that

$$
\big\langle\|R_3\|_{L^2(B_1)}^2\big\rangle\le C^2h^{4(1-\beta)}. \tag{147}
$$

Estimate for R_4: By adding zero, we may decompose $R_4=R_4'+R_4''$ with

$$
\begin{aligned}
R_4':=&\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi{+}\nabla\phi_\xi^T)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B_\pi'{+}\nabla\phi_{\xi+he,B_\pi'}^T)-\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B_\pi'{+}\nabla\phi_{\xi,B_\pi'}^T)\Big]\\
&-\sum_\Pi(\partial_\xi^{|\Pi|}A)(\omega,\xi{+}\nabla\phi_\xi^T)\Big[\sum_{\pi\in\Pi}\nabla\phi_{\xi,B_\pi'\odot e}^T h\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B_{\pi'}'{+}\nabla\phi_{\xi,B_{\pi'}'}^T)\Big],
\end{aligned}
$$

and where R_4'' can be treated by the arguments from the previous items. Thanks to $(\mathrm{A2})_L$ from Assumption 1 and the Leibniz rule for differences we have the bound

$$
\begin{aligned}
|R_4'|^2\le C^2&\sup_\Pi\sup_{\substack{\pi,\pi'\in\Pi\\ \pi\neq\pi'}}\big|\nabla\phi_{\xi+he,B_\pi'}^T{-}\nabla\phi_{\xi,B_\pi'}^T\big|^2\big|\nabla\phi_{\xi+he,B_{\pi'}'}^T{-}\nabla\phi_{\xi,B_{\pi'}'}^T\big|^2\\
&\times\Big\{1{+}\prod_{\substack{\pi''\in\Pi\\ \pi''\notin\{\pi,\pi'\}}}\Big(\big|\nabla\phi_{\xi,B_{\pi''}'}^T\big|^2{+}\big|\nabla\phi_{\xi+he,B_{\pi''}'}^T\big|^2\Big)\Big\}\\
&+C^2\sup_\Pi\sup_{\pi\in\Pi}\big|\nabla\phi_{\xi+he,B_\pi'}^T{-}\nabla\phi_{\xi,B_\pi'}^T{-}\nabla\phi_{\xi,B_\pi'\odot e}^T h\big|^2\\
&\times\Big\{1{+}\prod_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}\Big(\big|\nabla\phi_{\xi,B_{\pi'}'}^T\big|^2{+}\big|\nabla\phi_{\xi+he,B_{\pi'}'}^T\big|^2\Big)\Big\}.
\end{aligned}
\tag{148}
$$

Hence, it follows as a combination of (148), the induction hypothesis (Hreg), Hölder's inequality, the corrector estimates (41) and (42), as well as the corrector estimates for differences (60) and (61) that

$$
\big\langle\|R_4\|_{L^2(B_1)}^2\big\rangle\le C^2h^{4(1-\beta)}. \tag{149}
$$

Inserting the estimates (143), (145), (147), and (149) back into (140) then finally entails the bound

$$
\big\langle\big\|\nabla\phi_{\xi+he,B}^T{-}\nabla\phi_{\xi,B}^T{-}\nabla\phi_{\xi,B\odot e}^T h\big\|_{L^2(B_1)}^2\big\rangle\le C^2h^{4(1-\beta)}.
$$

This is the asserted estimate (64) on the level of the linearized homogenization corrector in the case of $q=1$. In particular, the map $\xi\mapsto\nabla\phi_{\xi,B}^T$ is Fréchet differentiable with values in the Fréchet space $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Note that as a consequence of (139) we then also get the estimate

$$\big\langle \big\| q^T_{\xi+he,B} - q^T_{\xi,B} - q^T_{\xi,B\odot e} h \big\|^2_{L^2(B_1)} \big\rangle \leq C^2 h^{4(1-\beta)}. \tag{150}$$

In particular, the map $\xi \mapsto q^T_{\xi,B}$ is also Fréchet differentiable with values in the Fréchet space $L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Step 3: (Proof of the estimates (64) *and* (65) *for general $q \in [1,\infty)$)* As we already established qualitative differentiability of the map $\xi \mapsto \nabla\phi^T_{\xi,B}$ in the Fréchet space $L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d)$, we may estimate based on the corrector estimate for differences (60) to linearization order $L{+}1$

$$\begin{aligned}
&\big\langle \big\| \nabla\phi^T_{\xi+he,B} - \nabla\phi^T_{\xi,B} - \nabla\phi^T_{\xi,B\odot e} h \big\|^{2q}_{L^2(B_1)} \big\rangle^{\frac{1}{2q}} \\
&\leq \int_0^1 \big\langle \big\| \nabla\phi^T_{\xi+she,B\odot e} h - \nabla\phi^T_{\xi,B\odot e} h \big\|^{2q}_{L^2(B_1)} \big\rangle^{\frac{1}{2q}} \,\mathrm{d}s \leq C h^{2(1-\beta)},
\end{aligned}$$

which is precisely the asserted bound (64). Based on the corrector estimate for differences (62) to linearization order $L{+}1$, the estimate (65) is derived analogously.

Step 4: (Proof of the estimates (64) *and* (65) *for linearized flux correctors)* The equation for the first-order Taylor expansion $\sigma^T_{\xi+he,B} - \sigma^T_{\xi,B} - \sigma^T_{\xi,B\odot e} h$ of linearized flux correctors is simply given by

$$\begin{aligned}
&\frac{1}{T}(\sigma^T_{\xi+he,B} - \sigma^T_{\xi,B} - \sigma^T_{\xi,B\odot e} h) - \Delta(\sigma^T_{\xi+he,B} - \sigma^T_{\xi,B} - \sigma^T_{\xi,B\odot e} h) \\
&= -\nabla \cdot \big((e_l \otimes e_k - e_k \otimes e_l)(q^T_{\xi+he,B} - q^T_{\xi,B} - q^T_{\xi,B\odot e} h) \big).
\end{aligned}$$

By an application of the weighted energy estimate (T3), the stationarity of linearized flux correctors and linearized fluxes, and the estimate (150) we obtain

$$\big\langle \big\| \nabla\sigma^T_{\xi+he,B} - \nabla\sigma^T_{\xi,B} - \nabla\sigma^T_{\xi,B\odot e} h \big\|^2_{L^2(B_1)} \big\rangle \leq C^2 h^{4(1-\beta)}.$$

In particular, the map $\xi \mapsto \nabla\sigma^T_{\xi,B}$ is Fréchet differentiable with values in the Fréchet space $L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Based on the corrector estimates for differences (60) and (62), we then infer along the same lines as in *Step 3* of this proof that the asserted estimates (64) and (65) indeed hold true for the linearized flux correctors.

Step 5: (Proof of the estimate (63)*)* As this is an immediate consequence of the estimate (150) in combination with the stationarity of the linearized fluxes, we may now conclude the proof of Lemma 17. □

4.10. **Proof of Lemma 19** (Limit passage in the massive approximation)**.** We again argue by induction over the linearization order. For the base case consisting of the correctors of the nonlinear problem we refer to Lemma 29 in Appendix C.

Step 1: (Induction hypothesis) Let $L \in \mathbb{N}$, $T \in [1,\infty)$ and $M > 0$ be fixed. Let the requirements and notation of (A1), $(A2)_L$ and $(A3)_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place.

For any $0 \leq l \leq L{-}1$, any $|\xi| \leq M$, any $|h| \leq 1$, any $T \in [1,\infty)$ and any collection of unit vectors $v'_1,\ldots,v'_l \in \mathbb{R}^d$ the difference $\phi^{2T}_{\xi,B'} - \phi^T_{\xi,B'}$ of linearized homogenization correctors in direction $B' := v'_1 \odot \cdots \odot v'_l$ is assumed to satisfy—under the above conditions—the following estimate (if $l = 0$—and thus B' being an empty symmetric tensor product—$\phi^T_{\xi,B'}$ is understood to denote the localized homogenization corrector ϕ_ξ^T of the nonlinear problem with a massive term):

$$\big\langle \big\|\nabla\phi_{\xi,B'}^{2T} - \nabla\phi_{\xi,B'}^{T}\big\|_{L^2(B_1)}^2\big\rangle \leq C^2\Big(\frac{\mu_*^2(\sqrt{T})}{T}\Big)^{\frac{1}{2^l}}. \tag{Hconv}$$

Step 2: (Induction step) The difference $\phi_{\xi,B}^{2T}-\phi_{\xi,B}^{T}$ of linearized homogenization correctors is subject to the equation

$$\begin{aligned}
&\frac{1}{2T}(\phi_{\xi,B}^{2T}-\phi_{\xi,B}^{T}) - \nabla\cdot a_\xi^{2T}(\nabla\phi_{\xi,B}^{2T}-\nabla\phi_{\xi,B}^{T})\\
&= \frac{1}{2T}\phi_{\xi,B}^{T} - \nabla\cdot\big((\partial_\xi A)(\omega,\xi+\nabla\phi_\xi^T)-(\partial_\xi A)(\omega,\xi+\nabla\phi_\xi^{2T})\big)(\mathbb{1}_{L=1}B+\nabla\phi_{\xi,B}^T)\\
&\quad + \nabla\cdot\sum_\Pi \big((\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi_\xi^{2T})-(\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi_\xi^{T})\big)\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{2T})\Big]\\
&\quad + \nabla\cdot\sum_\Pi (\partial_\xi^{|\Pi|}A)(\omega,\xi+\nabla\phi_\xi^{T})\Big[\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{2T})-\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{T})\Big].
\end{aligned} \tag{151}$$

Here, we made use of the abbreviation $\sum_\Pi := \sum_{\Pi\in\mathrm{Par}\{1,\ldots,L\},\,\Pi\neq\{\{1,\ldots,L\}\}}$. Applying the weighted energy estimate (T3) to equation (151) entails by means of $(\mathrm{A2})_L$ from Assumption 1

$$\begin{aligned}
&\int \ell_{\gamma,\sqrt{T}}\Big|\Big(\frac{\phi_{\xi,B}^{2T}-\phi_{\xi,B}^{T}}{\sqrt{2T}},\nabla\phi_{\xi,B}^{2T}-\nabla\phi_{\xi,B}^{T}\Big)\Big|^2\\
&\leq C^2\int \ell_{\gamma,\sqrt{T}}\frac{1}{2T}\big|\phi_{\xi,B}^T\big|^2 + C^2\int \ell_{\gamma,\sqrt{T}}\big|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\big|^2\big|\mathbb{1}_{L=1}B+\nabla\phi_{\xi,B}^T\big|^2\\
&\quad + C^2\sum_\Pi\int \ell_{\gamma,\sqrt{T}}\Big|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\Big|^2\Big|\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{2T})\Big|^2\\
&\quad + C^2\sum_\Pi\int \ell_{\gamma,\sqrt{T}}\Big|\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{2T})-\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{T})\Big|^2.
\end{aligned}$$

Taking expectation in the previous display, exploiting stationarity of the linearized homogenization correctors, adding zero, and applying Hölder's and Poincaré's inequalities then yields

$$\begin{aligned}
&\big\langle\big\|\nabla\phi_{\xi,B}^{2T}-\nabla\phi_{\xi,B}^{T}\big\|_{L^2(B_1)}^2\big\rangle\\
&\leq C^2\frac{1}{T}\big\langle\big\|\nabla\phi_{\xi,B}^{T}\big\|_{L^2(B_1)}^2\big\rangle + C^2\frac{1}{T}\Big\langle\Big|\fint_{B_1}\phi_{\xi,B}^T\Big|^2\Big\rangle\\
&\quad + C^2\big\langle\big\|\mathbb{1}_{L=1}B+\nabla\phi_{\xi,B}^T\big\|_{C^\alpha(B_1)}^8\big\rangle^{\frac14}\big\langle\big\|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\big\|_{C^\alpha(B_1)}^4\big\rangle^{\frac14}\big\langle\big\|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\big\|_{L^2(B_1)}^2\big\rangle^{\frac12}\\
&\quad + C^2\sum_\Pi\prod_{\pi\in\Pi}\big\langle\big\|\mathbb{1}_{|\pi|=1}B'_\pi+\nabla\phi_{\xi,B'_\pi}^{2T}\big\|_{C^\alpha(B_1)}^{8|\Pi|}\big\rangle^{\frac{1}{4|\Pi|}}\\
&\qquad\qquad \times\big\langle\big\|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\big\|_{C^\alpha(B_1)}^4\big\rangle^{\frac14}\big\langle\big\|\nabla\phi_\xi^{2T}-\nabla\phi_\xi^T\big\|_{L^2(B_1)}^2\big\rangle^{\frac12}\\
&\quad + C^2\sum_\Pi\sup_{\pi\in\Pi}\big\langle\big\|\nabla\phi_{\xi,B'_\pi}^{2T}-\nabla\phi_{\xi,B'_\pi}^{T}\big\|_{C^\alpha(B_1)}^4\big\rangle^{\frac14}\big\langle\big\|\nabla\phi_{\xi,B'_\pi}^{2T}-\nabla\phi_{\xi,B'_\pi}^{T}\big\|_{L^2(B_1)}^2\big\rangle^{\frac12}\\
&\qquad\qquad \times\sup_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}\Big\{1+\big\langle\big\|\nabla\phi_{\xi,B'_{\pi'}}^{2T}\big\|_{C^\alpha(B_1)}^{8|\Pi|}\big\rangle^{\frac{1}{4|\Pi|}}+\big\langle\big\|\nabla\phi_{\xi,B'_{\pi'}}^{T}\big\|_{C^\alpha(B_1)}^{8|\Pi|}\big\rangle^{\frac{1}{4|\Pi|}}\Big\}.
\end{aligned}$$

It is thus a consequence of the induction hypothesis (Hconv) and the corrector estimates (41)–(43) that

$$\big\langle\big\|\nabla\phi^{2T}_{\xi,B}-\nabla\phi^{T}_{\xi,B}\big\|^2_{L^2(B_1)}\big\rangle \leq C^2\Big(\frac{\mu_*^2(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}, \tag{152}$$

which concludes the induction step, and in particular establishes the asserted estimate (66).

Step 3: (Estimates for linearized flux correctors and massive version of homogenized operator) The difference $\sigma^{2T}_{\xi,B}-\sigma^{T}_{\xi,B}$ of linearized flux correctors is subject to the equation

$$\begin{aligned}&\frac{1}{2T}(\sigma^{2T}_{\xi,B,kl}-\sigma^{T}_{\xi,B,kl})-\Delta(\sigma^{2T}_{\xi,B,kl}-\sigma^{T}_{\xi,B,kl})\\&=\frac{1}{2T}\sigma^{T}_{\xi,B,kl}-\nabla\cdot\big((e_l\otimes e_k-e_k\otimes e_l)(q^{2T}_{\xi,B}-q^{T}_{\xi,B})\big).\end{aligned} \tag{153}$$

Applying the weighted energy estimate (T3) to equation (153) thus yields

$$\begin{aligned}&\int \ell_{\gamma,\sqrt{T}}\Big|\Big(\frac{\sigma^{2T}_{\xi,B,kl}-\sigma^{T}_{\xi,B,kl}}{\sqrt{2T}},\nabla\sigma^{2T}_{\xi,B,kl}-\nabla\sigma^{T}_{\xi,B,kl}\Big)\Big|^2\\&\leq C^2\int \ell_{\gamma,\sqrt{T}}\frac{1}{2T}\big|\sigma^{T}_{\xi,B,kl}\big|^2+C^2\int \ell_{\gamma,\sqrt{T}}\big|q^{2T}_{\xi,B}-q^{T}_{\xi,B}\big|^2.\end{aligned}$$

Taking expectation in the previous display, exploiting stationarity of the linearized flux correctors, adding zero, and applying the Poincaré inequality entails the estimate

$$\begin{aligned}&\big\langle\big\|\nabla\sigma^{2T}_{\xi,B,kl}-\nabla\sigma^{T}_{\xi,B,kl}\big\|^2_{L^2(B_1)}\big\rangle\\&\leq C^2\frac{1}{T}\big\langle\big\|\nabla\sigma^{T}_{\xi,B,kl}\big\|^2_{L^2(B_1)}\big\rangle+C^2\frac{1}{T}\Big\langle\Big|\fint_{B_1}\sigma^{T}_{\xi,B,kl}\Big|^2\Big\rangle+C^2\big\langle\big\|q^{2T}_{\xi,B}-q^{T}_{\xi,B}\big\|^2_{L^2(B_1)}\big\rangle.\end{aligned}$$

By means of the corrector estimates (41) and (43) the previous display updates to

$$\big\langle\big\|\nabla\sigma^{2T}_{\xi,B,kl}-\nabla\sigma^{T}_{\xi,B,kl}\big\|^2_{L^2(B_1)}\big\rangle\leq C^2\Big(\frac{\mu_*^2(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}+C^2\big\langle\big\|q^{2T}_{\xi,B}-q^{T}_{\xi,B}\big\|^2_{L^2(B_1)}\big\rangle. \tag{154}$$

It remains to provide an estimate for the difference $q^{2T}_{\xi,B}-q^{T}_{\xi,B}$ of linearized fluxes. The majority of the required work is already done since we already provided an estimate of required type for the divergence terms on the right hand side of (151). By definition (49d) of the linearized flux, it thus suffices to note that by (152) and $(\mathrm{A2})_L$ from Assumption 1 we obtain

$$\begin{aligned}\big\langle\big\|q^{2T}_{\xi,B}-q^{T}_{\xi,B}\big\|^2_{L^2(B_1)}\big\rangle&\leq C^2\big\langle\big\|\nabla\phi^{2T}_{\xi,B}-\nabla\phi^{T}_{\xi,B}\big\|^2_{L^2(B_1)}\big\rangle+C^2\Big(\frac{\mu_*^2(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}\\&\leq C^2\Big(\frac{\mu_*^2(\sqrt{T})}{T}\Big)^{\frac{1}{2^L}}.\end{aligned} \tag{155}$$

Plugging this estimate back into (154) therefore yields the asserted estimate (66) on the level of linearized flux correctors. The estimate (67) is in turn an immediate consequence of (155), the stationarity of the linearized flux, and Jensen's inequality.

Step 4: (Conclusion) As a consequence of (66), there exist stationary gradient fields

$$(\nabla\phi_{\xi,B},\nabla\sigma_{\xi,B})\in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\times L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}}\times\mathbb{R}^d)$$

with vanishing expectation, finite second moments and being subject to the anchoring $\fint_{B_1}\phi_{\xi,B}=0$ resp. $\fint_{B_1}\sigma_{\xi,B}=0$ such that

$$(\nabla\phi^T_{\xi,B},\nabla\sigma^T_{\xi,B,kl})\to(\nabla\phi_{\xi,B},\nabla\sigma_{\xi,B,kl})\quad\text{as }T\to\infty,\tag{156}$$

strongly in $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$.

For any $x_0\in\mathbb{R}^d$, let $f_{x_0}:=\frac{1}{|B_1|}\mathbb{1}_{B_1}-\frac{1}{|B_1(x_0)|}\mathbb{1}_{B_1(x_0)}$. We then take $v^{\mathrm{Neu}}_{x_0}$ to be the solution of the Neumann problem for Poisson's equation $-\Delta v^{\mathrm{Neu}}_{x_0}=f_{x_0}$ in the ball $B_{1+|x_0|}$. We may then estimate by adding zero, exploiting stationarity, and applying Poincaré's inequality and Fatou's lemma

$$\begin{aligned}
&\Big\langle\Big|\fint_{B_1(x_0)}\big|\phi_{\xi,B}\big|^2\Big|^q\Big\rangle^{\frac{1}{2q}}\\
&\leq\Big\langle\Big|\fint_{B_1}\big|\nabla\phi_{\xi,B}\big|^2\Big|^q\Big\rangle^{\frac{1}{2q}}+\Big\langle\Big|\fint_{B_1(x_0)}\phi_{\xi,B}-\fint_{B_1}\phi_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{2q}}\\
&=\Big\langle\Big|\fint_{B_1}\big|\nabla\phi_{\xi,B}\big|^2\Big|^q\Big\rangle^{\frac{1}{2q}}+\Big\langle\Big|\int_{B_{1+|x_0|}}\nabla v^{\mathrm{Neu}}_{x_0}\cdot\nabla\phi_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{2q}}\\
&\leq\liminf_{T\to\infty}\Big\{\Big\langle\Big|\fint_{B_1}\big|\nabla\phi^T_{\xi,B}\big|^2\Big|^q\Big\rangle^{\frac{1}{2q}}+\Big\langle\Big|\fint_{B_1(x_0)}\phi^T_{\xi,B}-\fint_{B_1}\phi^T_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{2q}}\Big\}.
\end{aligned}$$

We now let v_{x_0} be the decaying solution for Poisson's equation $-\Delta v_{x_0}=f_{x_0}$ on $\mathbb{R}^d$, and define $g_{x_0}:=\nabla v_{x_0}$. Note that $\int|g_{x_0}|^2\leq C^2\mu_*^2(1+|x_0|)$ for a constant $C>0$ independent of $x_0\in\mathbb{R}^d$. For every $T\in[1,\infty)$, choose then a compactly supported cutoff $\eta^T_{x_0}\colon\mathbb{R}^d\to[0,1]$ such that $\eta^T_{x_0}\equiv1$ throughout $B_{1+|x_0|}$, and $|\nabla\eta^T_{x_0}|\leq\frac{2}{\sqrt{T}}$ throughout $\mathbb{R}^d$. We may then estimate based on the corrector estimates (39) and (40) as well as the properties of g_{x_0} resp. $\eta^T_{x_0}$

$$\begin{aligned}
&\Big\langle\Big|\fint_{B_1(x_0)}\phi^T_{\xi,B}-\fint_{B_1}\phi^T_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{2q}}\\
&\leq\Big\langle\Big|\int\eta^T_{x_0}g_{x_0}\cdot\nabla\phi_{\xi,B}\Big|^{2q}\Big\rangle^{\frac{1}{2q}}+\Big\langle\Big|\int\frac{\phi^T_{\xi,B}}{T}(g_{x_0}\cdot\nabla)(T\eta^T_{x_0})\Big|^{2q}\Big\rangle^{\frac{1}{2q}}\\
&\leq Cq^{C'}\mu_*(1+|x_0|),
\end{aligned}$$

uniformly over all $T\in[1,\infty)$. By the corrector estimates (39), the previous two displays, and an analogous argument for $\sigma_{\xi,B}$, it follows in summary that

$$\Big\langle\Big|\fint_{B_1(x_0)}\big|\big(\phi_{\xi,B},\sigma_{\xi,B}\big)\big|^2\Big|^q\Big\rangle^{\frac{1}{2q}}\leq Cq^{C'}\mu_*(1+|x_0|).\tag{157}$$

In particular, on one side we may infer from (157) that

$$(\phi_{\xi,B},\sigma_{\xi,B})\in H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\times H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d}_{\mathrm{skew}}\times\mathbb{R}^d)\quad\text{almost surely.}$$

On the other side, we also learn from (157) (by a covering argument and definition (20) of the scaling function) that the pair $(\phi_{\xi,B},\sigma_{\xi,B})$ features almost surely sublinear growth at infinity in the precise sense of Definition 4.

It remains to verify the validity of the associated PDE (30a) for the linearized homogenization correctors resp. the associated PDEs for the linearized flux correctors (30c) and (30d) (almost surely in a distributional sense). To this end, we first note that as a consequence of the corrector estimates (41) and (43) and stationarity of the linearized correctors $(\phi^T_{\xi,B}, \sigma^T_{\xi,B}, \psi^T_{\xi,B})$ that

$$\begin{aligned}
&\Big\langle \fint_{B_1(x_0)} \Big|\Big(\phi^T_{\xi,B}, \sigma^T_{\xi,B}, \frac{\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big|^2 \Big\rangle^{\frac{1}{2}} \\
&\leq \Big\langle \fint_{B_1} \Big|\Big(\nabla\phi^T_{\xi,B}, \nabla\sigma^T_{\xi,B}, \frac{\nabla\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big|^2 \Big\rangle^{\frac{1}{2}} + \Big\langle \Big| \fint_{B_1} \Big(\phi^T_{\xi,B}, \sigma^T_{\xi,B}, \frac{\psi^T_{\xi,B}}{\sqrt{T}}\Big)\Big|^2 \Big\rangle^{\frac{1}{2}} \\
&\leq C\mu_*(\sqrt{T}).
\end{aligned}$$

In particular,

$$\Big(\frac{\phi^T_{\xi,B}}{\sqrt{T}}, \frac{\sigma^T_{\xi,B}}{\sqrt{T}}, \frac{\psi^T_{\xi,B}}{T}\Big) \to 0 \quad \text{strongly in } L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d). \tag{158}$$

Moreover, due to the strong convergence (156) in $L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d)$ of the gradients, the gradients also converge $\mathbf{P} \otimes \mathcal{L}^d$ almost everywhere in the product space $\Omega \times K$, for every compact $K \subset \mathbb{R}^d$. In particular, by a straightforward inductive argument we deduce convergence of the linearized fluxes from (49d) resp. (30b) in the sense of

$$q^T_{\xi,B} \to q_{\xi,B} \quad \mathbf{P} \otimes \mathcal{L}^d \text{ almost everywhere in } \Omega \times K, \tag{159}$$

for every compact $K \subset \mathbb{R}^d$. Uniform boundedness of $(q^T_{\xi,B})_{T\geq 1}$ in $L^q_{\langle\cdot\rangle} L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for every pair of exponents $q \in [1,\infty)$ and $p \in [2,\infty)$ (which is a consequence of the corrector estimates (42), the definition (49d) of the linearized flux, and $(\mathrm{A2})_L$ from Assumption 1) upgrades (159) to strong convergence

$$q^T_{\xi,B} \to q_{\xi,B} \quad \text{strongly in } L^2_{\langle\cdot\rangle} L^2_{\mathrm{loc}}(\mathbb{R}^d), \tag{160}$$

which by stationarity of the linearized fluxes $q^T_{\xi,B}$ and $q_{\xi,B}$ in particular entails (68). Validity of the PDEs (30a), (30c) and (30d) (almost surely in a distributional sense) thus follows from taking the limit in the corresponding massive versions (49a), (49b) and (49e). This concludes the proof of Lemma 19. □

4.11. **Proof of Theorem 1** (Corrector estimates for higher-order linearizations)**.** We proceed in three steps.

Step 1: (Proof of the corrector estimates (16)–(19)*)* The estimates (16) and (17) are immediate consequences of the corresponding estimates (39) and (41) from Theorem 5, an application of Fatou's inequality together with Lemma 19, and multilinearity of the map $B \mapsto \phi_{\xi,B}$. A similar argument based in addition on the first step of the proof of Lemma 23, the Schauder estimate (42), and the representation of a Hölder seminorm in terms of a Campanato seminorm yields the annealed Schauder estimate (18). For the estimate (19), we simply refer to (157).

Step 2: (Qualitative differentiability of linearized correctors) We argue, under the assumptions of $(A2)_{L+1}$ and $(A3)_{L+1}$ of Assumption 1, that the maps $\xi \mapsto \phi_{\xi,B}$ and $\xi \mapsto \sigma_{\xi,B}$ are Fréchet differentiable with values in the Fréchet space $L^q_{\langle\cdot\rangle} H^1_{\mathrm{loc}}(\mathbb{R}^d)$, and that for every unit vector $e \in \mathbb{R}^d$ we have the following representation of the directional derivative: $\partial_\xi(\phi_{\xi,B}, \sigma_{\xi,B})[e] = (\phi_{\xi,B\odot e}, \sigma_{\xi,B\odot e})$.

On the level of the (stationary) corrector gradients, it follows from (64) by an application of Fatou's inequality together with Lemma 19 that

$$\begin{aligned}&\Big\langle\big\|\big(\nabla\phi_{\xi+he,B}-\nabla\phi_{\xi,B}-\nabla\phi_{\xi,B\odot e}h,\nabla\sigma_{\xi+he,B}-\nabla\sigma_{\xi,B}-\nabla\sigma_{\xi,B\odot e}h\big)\big\|_{L^2(B_1)}^{2q}\Big\rangle^{\frac{1}{q}}\\&\leq C^2q^{2C'}|B|^2|h|^{4(1-\beta)}\end{aligned}$$

for all unit vectors $e\in\mathbb{R}^d$ and all $|h|\leq 1$. On the level of the correctors themselves, note that the argument for (19) in *Step 1* of this proof is linear in the variable $(\phi_{\xi,B},\sigma_{\xi,B})$. Hence, we may run it based on the first-order Taylor expansion $(\phi_{\xi+he,B}-\phi_{\xi,B}-\phi_{\xi,B\odot e}h,\sigma_{\xi+he,B}-\sigma_{\xi,B}-\sigma_{\xi,B\odot e}h)$ which in light of (65) entails the estimate

$$\begin{aligned}&\Big\langle\big\|\big(\phi_{\xi+he,B}-\phi_{\xi,B}-\phi_{\xi,B\odot e}h,\sigma_{\xi+he,B}-\sigma_{\xi,B}-\sigma_{\xi,B\odot e}h\big)\big\|_{L^2(B_1(x_0))}^{2q}\Big\rangle^{\frac{1}{q}}\\&\leq C^2q^{2C'}|B|^2|h|^{4(1-\beta)}\mu_*^2(1+|x_0|)\end{aligned}$$

for all unit vectors $e\in\mathbb{R}^d$, all $|h|\leq 1$ and all $x_0\in\mathbb{R}^d$. The two previous displays immediately imply the claim.

Step 3: (Proof of the estimates (23)–(25)*)* As a consequence of the previous step and Taylor's formula, we may represent the (well-defined) Taylor expansions from (21) resp. (22) in terms of the one-parameter family of linearized correctors $(\phi_{s\xi+(1-s)\xi_0,B\odot(\xi-\xi_0)^{\odot(K+1)}},\sigma_{s\xi+(1-s)\xi_0,B\odot(\xi-\xi_0)^{\odot(K+1)}})$, $s\in[0,1]$. The estimates (23)–(25) thus follow from the corrector estimates (17)–(19).

This concludes the proof of Theorem 1. □

4.12. **Proof of Theorem 2** (Higher-order regularity of the homogenized operator)**.** Differentiability of the homogenized operator, the representation of the directional derivatives (28), and the corresponding bound (27) are in the case of first-order differentiability immediate consequences of the estimates (209), (68) and (212), and in case of higher-order differentiability of the estimates (63) and (68). □

Appendix A. A toolbox from elliptic regularity theory

The aim of this appendix is to list (and in parts to prove) several results from (deterministic resp. random) elliptic regularity theory. We start with the probably most basic result concerning the Caccioppoli and hole filling estimates.

Lemma 20 (Caccioppoli inequality and hole filling estimate)**.** *Let $a\colon\mathbb{R}^d\to\mathbb{R}^{d\times d}$ be a uniformly elliptic and bounded coefficient field with respect to constants (λ,Λ). For a given $T\in[1,\infty)$ and $f,g\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, let $u\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a solution of*

$$\frac{1}{T}u-\nabla\cdot a\nabla u=\frac{1}{T}f+\nabla\cdot g\quad\text{in }\mathbb{R}^d.$$

Then, we have for all $x_0\in\mathbb{R}^d$ and all $R>0$ the Caccioppoli estimate

$$\begin{aligned}&\Big\|\Big(\frac{u}{\sqrt{T}},\nabla u\Big)\Big\|_{L^2(B_R(x_0))}^2\\&\lesssim_{d,\lambda,\Lambda}\inf_{b\in\mathbb{R}}\Big\{\frac{1}{R^2}\big\|u-b\big\|_{L^2(B_{2R}(x_0))}^2+\frac{1}{T}|b|^2\Big\}+\Big\|\Big(\frac{f}{\sqrt{T}},g\Big)\Big\|_{L^2(B_{2R}(x_0))}^2.\end{aligned}\tag{T1}$$

Moreover, there exists $\delta = \delta(d, \lambda, \Lambda) \in (0, 1)$ such that for all $r < R$ and all $x_0 \in \mathbb{R}^d$ we have the hole filling estimate

$$\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|^2_{L^2(B_r(x_0))} \lesssim_{d,\lambda,\Lambda} \Big(\frac{R}{r}\Big)^{-\delta}\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|^2_{L^2(B_R(x_0))} + \int_{B_R(x_0)} \frac{r^\delta}{|x-x_0|^\delta}\Big|\Big(\frac{f}{\sqrt{T}}, g\Big)\Big|^2 \,\mathrm{d}x. \tag{T2}$$

Proof. The standard proofs carry over immediately to the setting with an additional massive term. □

Perturbing a uniformly elliptic PDE by a massive term has the very convenient consequence of entailing a weighted energy estimate in terms of a suitable exponential weight. More precisely, we have the following standard result.

Lemma 21 (Exponential localization). *Let $a\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be a uniformly elliptic and bounded coefficient field with respect to constants (λ, Λ). For given $T \in [1, \infty)$ and $f, g \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a solution of*

$$\frac{1}{T}u - \nabla\cdot a\nabla u = \frac{1}{T}f + \nabla\cdot g \quad \text{in } \mathbb{R}^d.$$

Assume that $\limsup_{R\to\infty} R^{-2k-d}\|(u, \nabla u, f, g)\|^2_{L^2(B_R)} = 0$ for some $k \in \mathbb{N}$. For each $\gamma > 0$, $R > 0$ and $x_0 \in \mathbb{R}^d$ define the weight $\ell^{x_0}_{\gamma,R}(x) := \frac{1}{R^d}\exp(-\frac{\gamma|x-x_0|}{R})$. Then, there exists $\gamma = \gamma(d, \lambda, \Lambda)$ such that for all $R \geq \sqrt{T}$ we have the weighted energy estimate

$$\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|_{L^2(\mathbb{R}^d;\ell^{x_0}_{\gamma,R}\,\mathrm{d}x)} \lesssim_{d,\lambda,\Lambda} \Big\|\Big(\frac{f}{\sqrt{T}}, g\Big)\Big\|_{L^2(\mathbb{R}^d;\ell^{x_0}_{\gamma,R}\,\mathrm{d}x)}. \tag{T3}$$

Proof. The idea is to test the equation with $u\ell^{x_0}_{\gamma,R}$, and to run an absorption argument which gets facilitated by an appropriate choice of $\gamma \in (0, 1)$. Details of this (standard) argument are provided in, e.g., the proof of [18, Lemma 45]. □

In case of Hölder continuous coefficients, classical elliptic regularity provides local Schauder estimates and local Calderón–Zygmund estimates. The corresponding version for the massive approximation with an additional explicit dependence of the constant on the Hölder norm of the coefficient field reads as follows.

Lemma 22 (Local regularity estimates for Hölder continuous coefficients, cf. [29]). *Consider $\alpha \in (0, 1)$ and $T \geq 1$, and let $a \in C^\alpha(B_2)$ be a uniformly elliptic and bounded coefficient field with respect to constants (λ, Λ). Let $p \in [2, \infty)$, and for given $f, g \in L^p(B_2)$ let $u \in H^1(B_2)$ be a solution of*

$$\frac{1}{T}u - \nabla\cdot a\nabla u = \frac{1}{T}f + \nabla\cdot g \quad \text{in } B_2.$$

Then, the following local Calderón–Zygmund estimate holds true

$$\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|_{L^p(B_1)} \lesssim_{d,\lambda,\Lambda,\alpha,p} \|a\|^{\frac{d}{\alpha}(\frac{1}{2}-\frac{1}{p})}_{C^\alpha(B_2)}\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|_{L^2(B_2)} + \Big\|\Big(\frac{f}{\sqrt{T}}, g\Big)\Big\|_{L^p(B_2)}. \tag{T4}$$

Furthermore, in case of $f = 0$ and $g \in C^\alpha(B_2)$ the following local Schauder estimate is satisfied

$$\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|_{C^\alpha(B_1)} \lesssim_{d,\lambda,\Lambda,\alpha} \|a\|_{C^\alpha(B_2)}^{\frac{d}{\alpha}(\frac{1}{2}+\frac{1}{d})}\Big\|\Big(\frac{u}{\sqrt{T}}, \nabla u\Big)\Big\|_{L^2(B_2)} + \|a\|_{C^\alpha(B_2)}^{\frac{1}{\alpha}-1}\|g\|_{C^\alpha(B_2)}. \tag{T5}$$

Proof. The claims are standard except for the explicitly spelled-out dependence of the estimates on the Hölder regularity of the coefficient field and the uniformity with respect to the parameter $T \in [1,\infty)$. A proof can be found in [29, Lemma A.3]. □

The previous result is typically applied to the random setting on the level of the linearized coefficient field $a_\xi^T := (\partial_\xi A)(\omega, \xi+\nabla\phi_\xi^T)$. However, one first needs to verify that the linearized coefficient field is actually Hölder regular in order to apply the above local regularity theory. Moreover, the arguments in the main text require stretched exponential moments for the corresponding Hölder norm. The key step towards these goals is a proof on the level of the correctors.

Lemma 23 (Annealed Hölder regularity for the corrector of the nonlinear problem)**.** *Let the requirements and notation of (A1), (A2)$_0$, (A3)$_0$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Given $\xi \in \mathbb{R}^d$ and $T \in [1,\infty)$, denote by $\phi_\xi^T \in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ the unique solution of the corrector equation (44a). Let finally $M > 0$ be fixed.*

There exist $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M) > 0$, $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta) > 0$ and $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $q \in [1,\infty)$ and all $|\xi| \le M$

$$\big\langle \|\nabla\phi_\xi^T\|_{C^\alpha(B_1)}^{2q}\big\rangle^{\frac{1}{q}} \le C^2 q^{2C'}. \tag{T6}$$

Proof. Note first that by smuggling in a spatial average over the unit ball, and subsequently applying the triangle inequality and Jensen's inequality, we obtain for all $\alpha \in (0,1)$ and all $p \ge 1$

$$\bigg(\fint_{B_1} \big|\nabla\phi_\xi^T\big|^p\bigg)^{\frac{1}{p}} \le \big[\nabla\phi_\xi^T\big]_{C^\alpha(B_1)} + \bigg(\fint_{B_1} \big|\nabla\phi_\xi^T\big|^2\bigg)^{\frac{1}{2}}.$$

The left hand side term of the previous display converges to $\|\nabla\phi_\xi^T\|_{L^\infty(B_1)}$ as we let $p \to \infty$. Hence, in light of the corrector bounds (45) it suffices to derive an annealed estimate for the Hölder seminorm. More precisely, we have to prove that there exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)$, $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}})$ and $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $q \in [1,\infty)$ and all $|\xi| \le M$ it holds

$$\big\langle \big[\nabla\phi_\xi^T\big]_{C^\alpha(B_1)}^{2q}\big\rangle^{\frac{1}{q}} \le C^2 q^{2C'}. \tag{161}$$

For a proof of (161), we start by recalling the equivalence of Hölder seminorms and Campanato seminorms. In other words, we face the task of finding constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)$, $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}})$ and $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $q \in [1,\infty)$ and all $|\xi| \le M$ it holds

$$\bigg\langle \bigg| \sup_{x_0\in B_1} \sup_{0<\kappa\le 1} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \bigg|^q \bigg\rangle^{\frac{1}{q}} \le C^2 q^{2C'}. \tag{162}$$

To this end, let us introduce some auxiliary quantities. First, define the random variable

$$\mathcal{X}_\eta := \sup_{x,y\in B_4,\, x\neq y} \frac{|\omega(x)-\omega(y)|}{|x-y|^\eta}. \tag{163}$$

The non-negative random variable $\mathcal{X}_\eta$ has stretched exponential moments because of condition (R) from Assumption 3. In addition, for every $m\in\mathbb{N}_0$ we define

$$\text{a scale } r_m := 2^{-m} \text{ and the event } \mathcal{A}_m^\eta := \{r_m^{-\eta}-1 \leq \mathcal{X}_\eta < r_{m+1}^{-\eta}-1\}. \tag{164}$$

Consider also $\theta=\theta(d,\lambda,\Lambda,\eta)\in(0,1)$. (The precise choice of θ will be determined further below.) Finally, for every $x_0\in B_1$ we define a field

$$u_{\xi,x_0}^T := \xi\cdot(x-x_0) + \phi_\xi^T. \tag{165}$$

Note that $\nabla u_{\xi,x_0}^T$ is stationary and that in the sequel we may freely switch between $\nabla\phi_\xi^T$ and $\nabla u_{\xi,x_0}^T$ in (162) for fixed $x_0\in B_1$.

Decomposing for every $m\in\mathbb{N}_0$, every $x_0\in B_1$, and every $\alpha\in(0,1)$

$$\begin{aligned}
&\sup_{0<\kappa\leq 1} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \\
&\leq \sup_{0<\kappa\leq\theta r_m} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 + 2C(d,\lambda,\Lambda,\alpha,\eta)\frac{1}{r_m^{d+2\alpha}} \fint_{B_2(0)} \big|\nabla\phi_\xi^T\big|^2
\end{aligned}$$

exploiting that $\mathbb{1}_\Omega = \sum_{m=0}^\infty \mathbb{1}_{\mathcal{A}_m^\eta}$ by definition (164) of the events $\mathcal{A}_m^\eta$, and relying on the stationarity of $\nabla\phi_\xi^T$, we obtain for all $\alpha\in(0,1)$ the estimate

$$\begin{aligned}
&\bigg\langle\bigg| \sup_{x_0\in B_1}\sup_{0<\kappa\leq 1} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \bigg|^q\bigg\rangle^{\frac{1}{q}} \\
&\lesssim_{d,\lambda,\Lambda,\alpha,\eta} \sum_{m=0}^\infty \big\langle\big\|\nabla\phi_\xi^T\big\|_{L^2(B_1)}^{4q}\big\rangle^{\frac{1}{2q}} r_m^{-d-2\alpha}\mathbf{P}\big[\mathcal{A}_m^\eta\big]^{\frac{1}{2q}} \\
&\quad + \sum_{m=0}^\infty \bigg\langle\bigg| \sup_{x_0\in B_1}\sup_{0<\kappa\leq\theta r_m} \mathbb{1}_{\mathcal{A}_m^\eta} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \bigg|^q\bigg\rangle^{\frac{1}{q}}.
\end{aligned}$$

As a consequence of the corrector estimates from Proposition 6, the definition (164) of the scales r_m and the events $\mathcal{A}_m^\eta$, and the random variable $\mathcal{X}_\eta$ from (163) admitting stretched exponential moments, the previous display updates to

$$\begin{aligned}
&\bigg\langle\bigg| \sup_{x_0\in B_1}\sup_{0<\kappa\leq 1} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \bigg|^q\bigg\rangle^{\frac{1}{q}} \lesssim_{d,\lambda,\Lambda,\alpha,\eta} C^2 q^{2C'}|\xi|^2 \\
&\quad + \sum_{m=0}^\infty \bigg\langle\bigg| \sup_{x_0\in B_1}\sup_{0<\kappa\leq\theta r_m} \mathbb{1}_{\mathcal{A}_m^\eta} \frac{1}{\kappa^{2\alpha}} \fint_{B_\kappa(x_0)} \Big|\nabla\phi_\xi^T - \fint_{B_\kappa(x_0)} \nabla\phi_\xi^T\Big|^2 \bigg|^q\bigg\rangle^{\frac{1}{q}}.
\end{aligned} \tag{166}$$

To estimate the second right hand side term in (166), we proceed (not surprisingly) by harmonic approximation and split the task into two parts.

Claim 1: For all $\mu \in (0,d)$ there exists $\theta = \theta(d,\lambda,\Lambda,\eta,\mu) \in (0,1)$ such that for all $x_0 \in B_1$, all $m \in \mathbb{N}_0$ and all $\kappa \in (0,\theta r_m]$ it holds

$$\begin{aligned}&\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_\kappa(x_0)} |\nabla u_{\xi,x_0}^T|^2 \\ &\lesssim_{d,\lambda,\Lambda,\eta,\mu} \mathbb{1}_{\mathcal{A}_m^\eta} \Big(\frac{\kappa}{r_m}\Big)^\mu \Big\{ r_m^{2+d}|\xi|^2 + \int_{B_{r_m}(x_0)} |\nabla u_{\xi,x_0}^T|^2 \Big\}.\end{aligned} \tag{167}$$

For a proof of (167), fix $x_0 \in B_1$, $R \in (0,\theta r_m)$ and $\kappa \in (0,\frac{R}{2})$. Note that we may write the equation of u_{ξ,x_0}^T defined in (165) in form of

$$\begin{aligned}&\frac{1}{T} u_{\xi,x_0}^T - \nabla \cdot A(\omega(x_0), \nabla u_{\xi,x_0}^T) \\ &= \frac{1}{T}\xi \cdot (x - x_0) - \nabla \cdot \big\{ A(\omega(x_0), \nabla u_{\xi,x_0}^T) - A(\omega, \nabla u_{\xi,x_0}^T)\big\}.\end{aligned} \tag{168}$$

We then consider the harmonic approximation with massive term

$$\begin{aligned}\frac{1}{T} v_R - \nabla \cdot A(\omega(x_0), \nabla v_R) = 0 \qquad &\text{in } B_R(x_0), \\ v_R = u_{\xi,x_0}^T \quad &\text{on } \partial B_R(x_0).\end{aligned} \tag{169}$$

Moser iteration applied to the equation

$$\frac{1}{T}\partial_i v_R - \nabla \cdot (\partial_\xi A)(\omega(x_0), \nabla v_R) \nabla \partial_i v_R = 0 \tag{170}$$

entails that

$$\int_{B_\kappa(x_0)} |\nabla v_R|^2 \lesssim_d \kappa^d \sup_{x \in B_{\frac{R}{2}}(x_0)} |\nabla v_R|^2 \lesssim_{d,\lambda,\Lambda} \Big(\frac{\kappa}{R}\Big)^d \int_{B_R(x_0)} |\nabla v_R|^2. \tag{171}$$

Moreover, by a simple energy estimate and $(A3)_0$ from Assumption 1 we have

$$\begin{aligned}\int_{B_\kappa(x_0)} |\nabla u_{\xi,x_0}^T - \nabla v_R|^2 &\le \int_{B_R(x_0)} |\nabla u_{\xi,x_0}^T - \nabla v_R|^2 \\ &\lesssim_{d,\lambda,\Lambda} \frac{1}{T} R^{2+d}|\xi|^2 + R^{2\eta}\mathcal{X}_\eta^2 \int_{B_R(x_0)} |\nabla u_{\xi,x_0}^T|^2.\end{aligned} \tag{172}$$

As $R \le \theta r_m$ and $T \ge 1$, we obtain from definition (163) of the random variable $\mathcal{X}_\eta$ and definition (164) of the event $\mathcal{A}_m^\eta$ that

$$\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_\kappa(x_0)} |\nabla u_{\xi,x_0}^T - \nabla v_R|^2 \lesssim_{d,\lambda,\Lambda} \mathbb{1}_{\mathcal{A}_m^\eta} \Big\{ R^{2+d}|\xi|^2 + \theta^{2\eta} \int_{B_R(x_0)} |\nabla u_{\xi,x_0}^T|^2 \Big\}. \tag{173}$$

Combining the estimates (171) and (173) thus shows that

$$\begin{aligned}&\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_\kappa(x_0)} |\nabla u_{\xi,x_0}^T|^2 \\ &\lesssim_{d,\lambda,\Lambda} \mathbb{1}_{\mathcal{A}_m^\eta} \Big\{ R^\mu r_m^{2+d-\mu}|\xi|^2 + \Big(\theta^{2\eta} + \Big(\frac{\kappa}{R}\Big)^d\Big) \int_{B_R(x_0)} |\nabla u_{\xi,x_0}^T|^2 \Big\}.\end{aligned} \tag{174}$$

Note that the estimate from the previous display is trivially fulfilled in the regime $R \in (0,\theta r_m)$ and $\kappa \in (\frac{R}{2}, R)$. Choosing $\theta = \theta(d,\lambda,\Lambda,\eta,\mu)$ sufficiently small so that one can iterate (174) (see, e.g., [19, Lemma 5.13]), we obtain for all $R \in (0, \theta r_m]$ and all $\kappa \in (0, R]$

$$\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_\kappa(x_0)} |\nabla u_{\xi,x_0}^T|^2 \lesssim_{d,\lambda,\Lambda,\eta,\mu} \mathbb{1}_{\mathcal{A}_m^\eta} \Big(\frac{\kappa}{R}\Big)^\mu \Big\{ R^\mu r_m^{2+d-\mu} |\xi|^2 + \int_{B_R(x_0)} |\nabla u_{\xi,x_0}^T|^2 \Big\}.$$

This in turn immediately implies the claim (167).

Claim 2: There exist $\theta = \theta(d,\lambda,\Lambda,\eta) \in (0,1)$ and $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$ such that for all $x_0 \in B_1$, all $m \in \mathbb{N}_0$, all $R \in (0,\theta r_m]$ and all $\kappa \in (0,R]$

$$\begin{aligned} &\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_\kappa(x_0)} \Big| \nabla u_{\xi,x_0}^T - \fint_{B_\kappa(x_0)} \nabla u_{\xi,x_0}^T \Big|^2 \\ &\lesssim_{d,\lambda,\Lambda,\eta} \mathbb{1}_{\mathcal{A}_m^\eta} \Big(\frac{\kappa}{R}\Big)^{2\alpha+d} \Big\{ \int_{B_R(x_0)} \Big| \nabla u_{\xi,x_0}^T - \fint_{B_R(x_0)} \nabla u_{\xi,x_0}^T \Big|^2 \\ &\qquad + R^{2\alpha+d} (1+\mathcal{X}_\eta^2) |\xi|^2 \\ &\qquad + R^{2\alpha+d} (1+\mathcal{X}_\eta^2) \fint_{B_{r_m}(x_0)} |\nabla u_{\xi,x_0}^T|^2 \Big\}. \end{aligned} \tag{175}$$

For a proof, fix $x_0 \in B_1$, $R \in (0,\theta r_m)$ and $\kappa \in (0, \frac{R}{8})$. We first introduce a suitable decomposition for the gradient of the solution v_R of (169) because of the massive term appearing in the equation. More precisely, consider for all $i \in \{1,\ldots,d\}$ the auxiliary Dirichlet problem

$$\begin{aligned} -\nabla \cdot A(\omega(x_0), \nabla v_R) \nabla w_{i,R} &= -\frac{1}{T} \partial_i v_R && \text{in } B_{\frac{R}{2}}(x_0), \\ w_{i,R} &= 0 && \text{on } \partial B_{\frac{R}{2}}(x_0). \end{aligned} \tag{176}$$

By the De Giorgi–Nash–Moser estimate in combination with Moser iteration applied to the equation

$$-\nabla \cdot A(\omega(x_0), \nabla v_R)(\nabla w_{i,R} - \nabla \partial_i v_R) = 0 \qquad \text{in } B_{\frac{R}{2}}(x_0),$$

we then find $\gamma = \gamma(d,\lambda,\Lambda) \in (0,1)$ such that for all $i \in \{1,\ldots,d\}$ we have excess decay in form of

$$\begin{aligned} &\int_{B_\kappa(x_0)} \Big| (w_{i,R} - \partial_i v_R) - \fint_{B_\kappa(x_0)} (w_{i,R} - \partial_i v_R) \Big|^2 \\ &\lesssim_{d,\lambda,\Lambda} \Big(\frac{\kappa}{R}\Big)^{2\gamma+d} \int_{B_{\frac{R}{4}}(x_0)} \Big| (w_{i,R} - \partial_i v_R) - \fint_{B_{\frac{R}{4}}(x_0)} (w_{i,R} - \partial_i v_R) \Big|^2. \end{aligned} \tag{177}$$

Moreover, by a simple energy estimate for (176) (rewriting to this end the right hand side of (176) in form of $-\frac{1}{T}\partial_i v_R = -\frac{1}{T}\nabla \cdot e_i(v_R - \fint_{B_{\frac{R}{2}}(x_0)} v_R))$ in combination with Poincaré's inequality and $T \geq 1$ we obtain for all $i \in \{1, \ldots, d\}$

$$\begin{aligned}\int_{B_\kappa(x_0)} \Big|w_{i,R} - \fint_{B_\kappa(x_0)} w_{i,R}\Big|^2 &\leq \int_{B_{\frac{R}{2}}(x_0)} \Big|w_{i,R} - \fint_{B_{\frac{R}{2}}(x_0)} w_{i,R}\Big|^2 \\ &\lesssim_d R^2 \int_{B_{\frac{R}{2}}(x_0)} |\nabla w_{i,R}|^2 \\ &\lesssim_{d,\lambda,\Lambda} R^2 \int_{B_{\frac{R}{2}}(x_0)} \Big|v_R - \fint_{B_{\frac{R}{2}}(x_0)} v_R\Big|^2 \\ &\lesssim_{d,\lambda,\Lambda} R^4 \int_{B_R(x_0)} |\nabla v_R - \nabla u^T_{\xi,x_0}|^2 + R^4 \int_{B_R(x_0)} |\nabla u^T_{\xi,x_0}|^2.\end{aligned} \tag{178}$$

Finally, because of (172) and $T \geq 1$ we get

$$\begin{aligned}\int_{B_\kappa(x_0)} |\nabla u^T_{\xi,x_0} - \nabla v_R|^2 &\leq \int_{B_R(x_0)} |\nabla u^T_{\xi,x_0} - \nabla v_R|^2 \\ &\lesssim_{d,\lambda,\Lambda} R^{2+d}|\xi|^2 + R^{2\eta}\mathcal{X}_\eta^2 \int_{B_R(x_0)} |\nabla u^T_{\xi,x_0}|^2.\end{aligned} \tag{179}$$

In total, the decomposition $\partial_i u^T_{\xi,x_0} = (\partial_i u^T_{\xi,x_0} - \partial_i v_R) + (\partial_i v_R - w_{i,R}) + w_{i,R}$ together with the estimates (177)–(179) and $R \leq 1$ implies that for all $i \in \{1, \ldots, d\}$ we have the estimate

$$\begin{aligned}&\int_{B_\kappa(x_0)} \Big|\partial_i u^T_{\xi,x_0} - \fint_{B_\kappa(x_0)} \partial_i u^T_{\xi,x_0}\Big|^2 \\ &\lesssim_{d,\lambda,\Lambda} \Big(\frac{\kappa}{R}\Big)^{2\gamma+d} \int_{B_R(x_0)} \Big|\partial_i u^T_{\xi,x_0} - \fint_{B_R(x_0)} \partial_i u^T_{\xi,x_0}\Big|^2 \\ &\quad + R^{2+d}|\xi|^2 + R^{2\eta}\big(1+\mathcal{X}_\eta^2\big) \int_{B_R(x_0)} |\nabla u^T_{\xi,x_0}|^2.\end{aligned}$$

Now, the estimate (167) (with κ replaced by R) entails

$$\mathbb{1}_{\mathcal{A}_m^\eta} \int_{B_R(x_0)} |\nabla u^T_{\xi,x_0}|^2 \lesssim_{d,\lambda,\Lambda,\eta,\mu} \mathbb{1}_{\mathcal{A}_m^\eta} R^\mu \Big\{ r_m^{2+d-\mu}|\xi|^2 + \fint_{B_{r_m}(x_0)} |\nabla u^T_{\xi,x_0}|^2 \Big\}.$$

The previous two displays in turn show that after choosing $\mu = \mu(d,\lambda,\Lambda,\eta) \in (0,d)$ appropriately there exists $\alpha = \alpha(d,\lambda,\Lambda) \in (0, \eta \wedge \gamma)$ such that

$$\begin{aligned}&\mathbb{1}_{\mathcal{A}_m} \int_{B_\kappa(x_0)} \Big|\partial_i u^T_{\xi,x_0} - \fint_{B_\kappa(x_0)} \partial_i u^T_{\xi,x_0}\Big|^2 \\ &\lesssim_{d,\lambda,\Lambda,\eta} \mathbb{1}_{\mathcal{A}_m} \Big(\frac{\kappa}{R}\Big)^{2\gamma+d} \int_{B_R(x_0)} \Big|\partial_i u^T_{\xi,x_0} - \fint_{B_R(x_0)} \partial_i u^T_{\xi,x_0}\Big|^2 \\ &\quad + \mathbb{1}_{\mathcal{A}_m} R^{2\alpha+d} \Big\{ \big(1+\mathcal{X}_\eta^2\big)|\xi|^2 + \big(1+\mathcal{X}_\eta^2\big) \fint_{B_{r_m}(x_0)} |\nabla u^T_{\xi,x_0}|^2 \Big\}.\end{aligned}$$

Iterating the previous display then establishes the asserted estimate (175).

Conclusion: We make use of (175) in order to estimate the second right hand side term of (166). More precisely, the estimate (175) applied with $R = \theta r_m$ entails that it holds

$$\sup_{x_0\in B_1}\sup_{0<\kappa\leq\theta r_m}\mathbb{1}_{\mathcal{A}_m^\eta}\frac{1}{\kappa^{2\alpha}}\fint_{B_\kappa(x_0)}\Big|\nabla u^T_{\xi,x_0}-\fint_{B_\kappa(x_0)}\nabla u^T_{\xi,x_0}\Big|^2$$
$$\lesssim_{d,\lambda,\Lambda,\eta}\mathbb{1}_{\mathcal{A}_m^\eta}(1+\mathcal{X}_\eta^2)r_m^{-d-2\alpha}\int_{B_2}|\nabla u^T_{\xi,x_0}|^2+\mathbb{1}_{\mathcal{A}_m^\eta}(1+\mathcal{X}_\eta^2)|\xi|^2.$$

Plugging this back into the second right hand side term of (166), and then making use of Hölder's inequality, the corrector estimates from Proposition 6, the definition (164) of the scales r_m and the events $\mathcal{A}_m^\eta$, and that the random variable $\mathcal{X}_\eta$ from (163) admits stretched exponential moments, therefore upgrades (166) to

$$\Big\langle\Big|\sup_{x_0\in B_1}\sup_{0<\kappa\leq 1}\frac{1}{\kappa^{2\alpha}}\fint_{B_\kappa(x_0)}\Big|\nabla\phi^T_\xi-\fint_{B_\kappa(x_0)}\nabla\phi^T_\xi\Big|^2\Big|^q\Big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}|\xi|^2.$$

This concludes the proof of (162), which in turn implies (161). □

An immediate consequence of Lemma 23 is now the following regularity result for the linearized coefficient field $a^T_\xi:=(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$.

Lemma 24 (Annealed Hölder regularity for the linearized coefficient)**.** *Let the requirements and notation of (A1), (A2)$_1$, (A3)$_1$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Given $\xi\in\mathbb{R}^d$ and $T\in[1,\infty)$, denote by $\phi^T_\xi\in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ the unique solution of the corrector equation (44a). Let finally $M>0$ be fixed.*

Then, there exist $C=C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)$, $C'=C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta)$ and $\alpha=\alpha(d,\lambda,\Lambda)\in(0,\eta)$ such that for all $q\in[1,\infty)$ and all $|\xi|\leq M$

$$\big\langle\|(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\|^{2q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q}}\leq C^2q^{2C'}. \tag{T7}$$

Proof. Because of (A2)$_1$ resp. (A3)$_1$ from Assumption 1 as well as (R) from Assumption 3, this is an immediate consequence of Lemma 23 and the chain rule. □

In terms of regularity theory in the random setting, the last missing ingredient is given by an annealed Calderón–Zygmund estimate. We emphasize that for the purpose of the present work, it suffices to consider a perturbative regime.

Lemma 25 (The annealed Calderón–Zygmund estimate: the perturbative regime, cf. [29])**.** *Consider an ensemble of uniformly elliptic and bounded coefficient fields $a\colon\mathbb{R}^d\to\mathbb{R}^{d\times d}$ with respect to constants (λ,Λ). There exists $c=c(d,\lambda,\Lambda)>0$ such that for all $q\in[2,2{+}c)$ and all random fields $f,g\in L^2(\mathbb{R}^d;L^q_{\langle\cdot\rangle})$, the solution of*

$$\frac{1}{T}u-\nabla\cdot a\nabla u=\frac{1}{T}f+\nabla\cdot g\quad\textit{in }\mathbb{R}^d$$

is subject to the estimate

$$\Big\|\Big\langle\Big|\Big(\frac{u}{\sqrt{T}},\nabla u\Big)\Big|^q\Big\rangle^{\frac{1}{q}}\Big\|^2_{L^2(\mathbb{R}^d)}\lesssim_{d,\lambda,\Lambda}\Big\|\Big\langle\Big|\Big(\frac{f}{\sqrt{T}},g\Big)\Big|^q\Big\rangle^{\frac{1}{q}}\Big\|^2_{L^2(\mathbb{R}^d)}. \tag{T8}$$

Proof. We refer the reader to the proof of [29, Proposition 7.1(i)]. Strictly speaking, the result is stated and proved for the massless version ($T=\infty$) only. The argument, however, carries over easily to the case of the massive approximation. Moreover, in [29] it is assumed (for simplicity only) that $|a\xi|^2\leq a^{-1}\xi\cdot\xi$. The reader may consult [35, Proof of Theorem 2.51, Step 3] for the general setting of a λ-uniformly elliptic and Λ-bounded coefficient field. □

Appendix B. Existence of localized, higher-order linearized homogenization correctors and flux correctors

Proof of Lemma 8 (Existence of localized correctors). We proceed by an induction over the linearization order $L \in \mathbb{N}$. For the sake of completeness, we refer the reader to [18, Lemma 18] for the existence of localized correctors ϕ_ξ^T of the nonlinear corrector problem (44a).

Step 1: (Base case) Let the requirements and notation of (A1), $(A2)_0$ and $(A3)_0$ of Assumption 1 be in place, and consider an *arbitrary* measurable parameter field $\tilde\omega\colon \mathbb{R}^d \to \mathbb{R}^n$, as well as a unit vector $v \in \mathbb{R}^d$. Then, there exists a unique solution

$$\Big(\frac{\phi_{\xi,v}^T(\cdot,\tilde\omega)}{\sqrt{T}}, \nabla\phi_{\xi,v}^T(\cdot,\tilde\omega)\Big) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d) \tag{180}$$

of the first-order linearized corrector problem with massive term given by

$$\frac{1}{T}\phi_{\xi,v}^T - \nabla\cdot(\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi_\xi^T)\nabla\phi_{\xi,v}^T = \nabla\cdot(\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi_\xi^T)v. \tag{181}$$

Under the stronger set of conditions (A1), $(A2)_1$ and $(A3)_1$ of Assumption 1, and under the stronger requirement (50) on the parameter field $\tilde\omega\colon \mathbb{R}^d \to \mathbb{R}^n$, we in addition claim that for all $p \geq 2$

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\Big|\Big(\frac{\phi_{\xi,v}^T(\cdot,\tilde\omega)}{\sqrt{T}}, \nabla\phi_{\xi,v}^T(\cdot,\tilde\omega)\Big)\Big|^p < \infty. \tag{182}$$

Proof of first claim (Existence of solutions to (181) in the function space (180)): For any $R \geq 1$, there exists a unique Lax-Milgram solution $\phi_{\xi,v}^{T,R} \in H^1(\mathbb{R}^d)$ of

$$\frac{1}{T}\phi_{\xi,v}^{T,R} - \nabla\cdot(\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi_\xi^T)\nabla\phi_{\xi,v}^{T,R} = \nabla\cdot\mathbb{1}_{B_R}(\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi_\xi^T)v. \tag{183}$$

The sequence $\big(\frac{\phi_{\xi,v}^{T,R}(\cdot,\tilde\omega)}{\sqrt{T}}, \nabla\phi_{\xi,v}^{T,R}(\cdot,\tilde\omega)\big)_{R\geq 1}$ is Cauchy in $L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$ as a consequence of the weighted energy estimate (T3) applied to differences of solutions to (183), and the limit is easily identified as a distributional solution to (181). Uniqueness follows again by an application of the weighted energy estimate (T3), this time with respect to two solutions of (181) in the function space (180).

Proof of second claim (Improved regularity (182) under stronger assumptions): Thanks to the assumption (50) on the parameter field $\tilde\omega$, the proof of Lemma 23 (in particular the proof of the annealed estimate (161) for the Hölder seminorm) carries over essentially verbatim to the present setting. Indeed, one simply needs to replace stochastic moments $\langle|\cdot|^q\rangle$ by $\limsup_{R\to\infty}\fint_{B_R(x_0)}|\cdot|^p$, the random variable $\mathcal{X}_\eta$ defined in (163) by the field $\mathcal{X}_\eta(x,\tilde\omega) := \sup_{y,z\in B_1(x),\,y\neq z}\frac{|\tilde\omega(y)-\tilde\omega(z)|}{|y-z|^\eta}$, the condition (R) of Assumption 3 by the assumption (50), and the usage of stationarity by the additional supremum over $x_0 \in \mathbb{R}^d$ in the condition (182). In particular, we obtain for all $p \geq 2$ an "annealed" Schauder estimate

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\big\|\nabla\phi_\xi^T(\cdot,\tilde\omega)\big\|^p_{C^\alpha(B_1(x))} < \infty. \tag{184}$$

The upshot of this is that we obtain an "annealed" Hölder estimate on the linearized coefficient $(\partial_\xi A)(\tilde{\omega},\xi{+}\nabla\phi_\xi^T(\cdot,\tilde{\omega}))$ in form of

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\big\|(\partial_\xi A)(\tilde{\omega},\xi{+}\nabla\phi_\xi^T(\cdot,\tilde{\omega}))\big\|^p_{C^\alpha(B_1(x))}<\infty \tag{185}$$

for some suitable $\alpha=\alpha(d,\lambda,\Lambda)\in(0,\eta)$ and all $p\geq 2$.

The information provided by (185) is now leveraged as follows. By means of the local Calderón–Zygmund estimate (T4) applied to the equation (181), the estimate (185) and the regularity $\big(\frac{\phi_{\xi,v}^T(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,v}^T(\cdot,\tilde{\omega})\big)\in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d)$ we infer that for all $p\geq 2$ it holds

$$\begin{aligned}
&\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\Big|\Big(\frac{\phi_{\xi,v}^T(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,v}^T(\cdot,\tilde{\omega})\Big)\Big|^p\\
&\lesssim\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\Big\|\Big(\frac{\phi_{\xi,v}^T(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,v}^T(\cdot,\tilde{\omega})\Big)\Big\|^p_{L^p(B_1(x))}\,\mathrm{d}x<\infty.
\end{aligned}$$

This concludes the proof of (182).

Step 2: (Formulation of the induction hypotheses) Let $L\geq 2$ and $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_{L-1}$ and $(\mathrm{A3})_{L-1}$ of Assumption 1 be in place. Fix also a parameter field $\tilde{\omega}\colon\mathbb{R}^d\to\mathbb{R}^n$ subject to the condition (50). For any linearization order $1\leq l\leq L{-}1$, and any collection of unit vectors $v_1',\ldots,v_l'\in\mathbb{R}^d$ we assume that—under the above conditions—the associated localized lth-order linearized homogenization corrector in direction $B':=v_1'\odot\cdots\odot v_l'$

$$\Big(\frac{\phi_{\xi,B'}^T(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,B'}^T(\cdot,\tilde{\omega})\Big)\in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d) \tag{186}$$

exists, and is subject to the estimate

$$\sup_{x_0\in\mathbb{R}^d}\limsup_{R\to\infty}\fint_{B_R(x_0)}\Big|\Big(\frac{\phi_{\xi,B'}^T(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,B'}^T(\cdot,\tilde{\omega})\Big)\Big|^p<\infty \tag{187}$$

for all $p\geq 2$.

Step 3: (Induction step) Let $L\geq 2$ and $T\in[1,\infty)$ be fixed. Let the requirements and notation of (A1), $(\mathrm{A2})_{L-1}$ and $(\mathrm{A3})_{L-1}$ of Assumption 1 be in place. Let $\tilde{\omega}\colon\mathbb{R}^d\to\mathbb{R}^n$ be a parameter field subject to the condition (50). We finally fix a set of unit vectors $v_1,\ldots,v_L\in\mathbb{R}^d$ and define $B:=v_1\odot\cdots\odot v_L$.

As a consequence of the induction hypothesis (187), there exists for any $R\geq 1$ a unique Lax-Milgram solution $\phi_{\xi,B}^{T,R}(\cdot,\tilde{\omega})\in H^1(\mathbb{R}^d)$ of

$$\begin{aligned}
&\frac{1}{T}\phi_{\xi,B}^{T,R}(\cdot,\tilde{\omega})-\nabla\cdot(\partial_\xi A)\big(\tilde{\omega},\xi{+}\nabla\phi_\xi^T(\cdot,\tilde{\omega})\big)\nabla\phi_{\xi,B}^{T,R}(\cdot,\tilde{\omega})\\
&=\nabla\cdot\mathbb{1}_{B_R}\sum_{\substack{\Pi\in\mathrm{Par}\{1,\ldots,L\}\\ \Pi\neq\{\{1,\ldots,L\}\}}}(\partial_\xi^{|\Pi|}A)\big(\tilde{\omega},\xi{+}\nabla\phi_\xi^T(\cdot,\tilde{\omega})\big)\Big[\bigodot_{\pi\in\Pi}\big(\mathbb{1}_{|\pi|=1}B_\pi'{+}\nabla\phi_{\xi,B_\pi'}^T(\cdot,\tilde{\omega})\big)\Big].
\end{aligned}$$

An application of the weighted energy estimate (T3) to differences of solutions with respect to the equation from the previous display, and making use of the induction hypothesis (187) shows that the sequence $\big(\frac{\phi_{\xi,B}^{T,R}(\cdot,\tilde{\omega})}{\sqrt{T}},\nabla\phi_{\xi,B}^{T,R}(\cdot,\tilde{\omega})\big)_{R\geq 1}$ is Cauchy in the desired function space $L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}{\times}\mathbb{R}^d)$. Details are left to the reader. Moreover, the limit constitutes the unique distributional solution of the linearized corrector problem (49a) in the required function space. The proof of (51) follows along the same lines as the argument in favor of (182). This in turn concludes the proof of the induction step.

Step 4: (Existence of linearized flux correctors) This is a straightforward consequence of standard arguments relying on the form of the flux corrector equations (49b) resp. (49c), the already established existence and regularity results for linearized homogenization correctors $\phi^T_{\xi,B}$, and the definition (49d) of linearized fluxes $q^T_{\xi,B}$.

Step 5: (Almost sure existence for random parameter fields) As a consequence of the small-scale regularity condition (R) of Assumption 3 and Birkhoff's ergodic theorem (recall to this end Assumption 2), there exists a subset $\Omega' \subset \Omega$ of full $\mathbf{P}$-measure such that all random fields $\omega \in \Omega'$ satisfy the condition (50). Hence, the claim on almost sure existence of linearized correctors for random parameter fields follows immediately from the previous four steps of this proof. This in turn concludes the proof of Lemma 8. □

Lemma 26 (Gâteaux differentiability of localized correctors with respect to parameter fields)**.** *Let $L \in \mathbb{N}$ and $T \in [1,\infty)$ be fixed. Let the requirements and notation of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1 be in place. We also fix a parameter field $\tilde{\omega} \colon \mathbb{R}^d \to \mathbb{R}^n$ subject to the condition (50). Consider in addition a a perturbation $\delta\tilde{\omega} \in C^{\eta}_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\tilde{\omega}]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$.*

Then, for every $\xi \in \mathbb{R}^d$ and every $B := v_1 \odot \cdots \odot v_L$ formed by unit vectors $v_1, \ldots, v_L \in \mathbb{R}^d$, the associated linearized homogenization corrector $\phi^T_{\xi,B}(\cdot,\tilde{\omega})$ from Lemma 8 is Gâteaux differentiable at $\tilde{\omega}$ in direction of $\delta\tilde{\omega}$ in the Fréchet space $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. The corresponding Gâteaux derivative $\delta\phi^T_{\xi,B}(\cdot,\tilde{\omega})$ satisfies

$$\Big(\frac{\delta\phi^T_{\xi,B}(\cdot,\tilde{\omega})}{\sqrt{T}}, \nabla\delta\phi^T_{\xi,B}(\cdot,\tilde{\omega})\Big) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d), \tag{188}$$

and it is the unique solution of the equation (198) below. Moreover, for all $p \geq 2$

$$\sup_{x_0\in\mathbb{R}^d} \limsup_{R\to\infty} \fint_{B_R(x_0)} \Big|\Big(\frac{\delta\phi^T_{\xi,B}(\cdot,\tilde{\omega})}{\sqrt{T}}, \nabla\delta\phi^T_{\xi,B}(\cdot,\tilde{\omega})\Big)\Big|^p < \infty. \tag{189}$$

Analogous statements hold true for the linearized flux correctors $\sigma^T_{\xi,B}(\cdot,\tilde{\omega})$ resp. $\psi^T_{\xi,B}(\cdot,\tilde{\omega})$ from Lemma 8.

In particular, under the requirements of (A1), (A2)$_L$ and (A3)$_L$ of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3, there exists a set $\Omega' \subset \Omega$ of full $\mathbf{P}$-measure on which the existence of Gâteaux derivatives for (higher-order) linearized correctors is guaranteed in the above sense for all random parameter fields $\omega \in \Omega'$, with directions given by all $\delta\omega \in C^{\eta}_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$. We in fact have for all $q \geq 1$ as $h \to 0$

$$\Big\langle\Big\|\frac{1}{\sqrt{T}}\Big(\frac{\phi^T_{\xi,B}(\cdot,\omega{+}h\delta\omega) - \phi^T_{\xi,B}(\cdot,\omega)}{h} - \delta\phi^T_{\xi,B}(\cdot,\omega)\Big)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} \to 0, \tag{190}$$

$$\Big\langle\Big\|\frac{\nabla\phi^T_{\xi,B}(\cdot,\omega{+}h\delta\omega) - \nabla\phi^T_{\xi,B}(\cdot,\omega)}{h} - \nabla\delta\phi^T_{\xi,B}(\cdot,\omega)\Big\|^{2q}_{L^2(B_1)}\Big\rangle^{\frac{1}{q}} \to 0, \tag{191}$$

and analogously for the linearized flux correctors $\sigma^T_{\xi,B}$ resp. $\psi^T_{\xi,B}$.

Proof. Step 1: Let the requirements and notation of (A1), $(A2)_0$ and $(A3)_0$ of Assumption 1 be in place, fix a parameter field $\tilde\omega\colon \mathbb{R}^d\to\mathbb{R}^n$ subject to the condition (50), and consider a perturbation $\delta\tilde\omega\in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\tilde\omega]_\infty\|_{L^2(\mathbb{R}^d)}<\infty$. In this setting, consider the unique $H^1_{\mathrm{uloc}}(\mathbb{R}^d)$-solutions of

$$\frac{1}{T}\phi^T_\xi(\cdot,\tilde\omega)-\nabla\cdot A\big(\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega)\big)=0,$$

$$\frac{1}{T}\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\nabla\cdot A\big(\tilde\omega+h\delta\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)\big)=0,$$

$$\frac{1}{T}\delta\phi^T_\xi(\cdot,\tilde\omega)-\nabla\cdot(\partial_\xi A)\big(\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega)\big)\nabla\delta\phi^T_\xi(\cdot,\tilde\omega)=\nabla\cdot(\partial_\omega A)\big(\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega)\big)\delta\omega.$$

Expressing now the equation satisfied by $\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)$ in form of

$$\begin{aligned}
&\frac{1}{T}\big(\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)\big)\\
&-\nabla\cdot\bigg(\int_0^1(\partial_\xi A)\big(\tilde\omega,\xi+s\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)+(1-s)\nabla\phi^T_\xi(\cdot,\tilde\omega)\big)\,\mathrm{d}s\bigg)\\
&\qquad\qquad\qquad\qquad\times\big(\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\nabla\phi^T_\xi(\cdot,\tilde\omega)\big)\\
&=\nabla\cdot A\big(\tilde\omega+h\delta\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)\big)-\nabla\cdot A\big(\tilde\omega,\xi+\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)\big),
\end{aligned}\tag{192}$$

it follows due to $(A2)_0$ and $(A3)_0$ from Assumption 1 and the weighted energy estimate (T3) applied to the equation from the previous display that

$$\sup_{x_0\in\mathbb{R}^d}\bigg\|\bigg(\frac{\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)}{\sqrt{T}},\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\nabla\phi^T_\xi(\cdot,\tilde\omega)\bigg)\bigg\|_{L^2(B_1(x_0))}\lesssim|h|.$$

Computing next the equation satisfied by $\frac{1}{h}\big(\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)\big)-\delta\phi^T_\xi(\cdot,\tilde\omega)$, applying to it the weighted energy estimate (T3), inserting the estimates from the previous display, employing the conditions $(A2)_0$ and $(A3)_0$ from Assumption 1, and finally making use of the dominated convergence theorem, one obtains as $h\to0$

$$\bigg\|\frac{1}{\sqrt{T}}\bigg(\frac{\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)}{h}-\delta\phi^T_\xi(\cdot,\tilde\omega)\bigg)\bigg\|_{L^2(B_1(x_0))}\to0,\tag{193}$$

$$\bigg\|\frac{\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\nabla\phi^T_\xi(\cdot,\tilde\omega)}{h}-\nabla\delta\phi^T_\xi(\cdot,\tilde\omega)\bigg\|_{L^2(B_1(x_0))}\to0\tag{194}$$

for all $x_0\in\mathbb{R}^d$. Under the stronger assumptions of $(A2)_1$ and $(A3)_1$ from Assumption 1, we register for later purposes that as a consequence of (184)–(185), $\tilde\omega+h\delta\tilde\omega$ satisfying (50) uniformly in $|h|\le1$, and small-scale elliptic regularity in form of (T4), we have in addition to (51)

$$\bigg\|\bigg(\frac{\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)}{\sqrt{T}},\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)\bigg)\bigg\|_{L^p_{\mathrm{erg}}(\mathbb{R}^d)}<\infty,\tag{195}$$

$$\bigg\|\bigg(\frac{\delta\phi^T_\xi(\cdot,\tilde\omega)}{\sqrt{T}},\nabla\delta\phi^T_\xi(\cdot,\tilde\omega)\bigg)\bigg\|_{L^p_{\mathrm{erg}}(\mathbb{R}^d)}<\infty,\tag{196}$$

$$\bigg\|\bigg(\frac{\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\phi^T_\xi(\cdot,\tilde\omega)}{\sqrt{T}},\nabla\phi^T_\xi(\cdot,\tilde\omega+h\delta\tilde\omega)-\nabla\phi^T_\xi(\cdot,\tilde\omega)\bigg)\bigg\|_{L^p_{\mathrm{erg}}(\mathbb{R}^d)}\lesssim_p|h|\tag{197}$$

for all $p\ge2$ (with constants independent of $|h|\le1$).

Step 2: By an induction over the linearization order $1 \leq l \leq L$ starting with (196), one may provide solutions with the regularity (188) and (189) to the equations obtained from the linearized corrector problem (49a) by formally differentiating with respect to the parameter field. Indeed, by the same arguments as in the proof of Lemma 8, there exists a unique solution $(\frac{\delta\phi^T_{\xi,B}(\cdot,\tilde\omega)}{\sqrt{T}}, \nabla\delta\phi^T_{\xi,B}(\cdot,\tilde\omega)) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$ with the additional regularity (189) to the equation

$$\begin{aligned}
&\frac{1}{T}\delta\phi^T_{\xi,B} - \nabla\cdot(\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\nabla\delta\phi^T_{\xi,B} \\
&= \nabla\cdot(\partial_\omega\partial_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\big[\delta\tilde\omega\odot\nabla\phi^T_{\xi,B}\big] \\
&\quad + \nabla\cdot(\partial^2_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\big[\nabla\delta\phi^T_\xi\odot\nabla\phi^T_{\xi,B}\big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial_\omega\partial^{|\Pi|}_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\Big[\delta\tilde\omega\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi^T_{\xi,B'_\pi})\Big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial^{1+|\Pi|}_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\Big[\nabla\delta\phi^T_\xi\odot\bigodot_{\pi\in\Pi}(\mathbb{1}_{|\pi|=1}B'_\pi{+}\nabla\phi^T_{\xi,B'_\pi})\Big] \\
&\quad + \nabla\cdot\sum_\Pi(\partial^{|\Pi|}_\xi A)(\tilde\omega,\xi{+}\nabla\phi^T_\xi(\cdot,\tilde\omega))\Big[\sum_{\pi\in\Pi}\nabla\delta\phi^T_{\xi,B'_\pi}\odot\bigodot_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}}(\mathbb{1}_{|\pi'|=1}B'_{\pi'}{+}\nabla\phi^T_{\xi,B'_{\pi'}})\Big].
\end{aligned} \tag{198}$$

In order to identify $\delta\phi^T_{\xi,B}(\cdot,\tilde\omega)$ as the Gâteaux derivative of $\phi^T_{\xi,B}(\cdot,\tilde\omega)$ in direction of some $\delta\tilde\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\tilde\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$, one proceeds as follows. For any $|h| \leq 1$, note that $\tilde\omega + h\delta\tilde\omega$ also satisfies (50). In particular, for any $|h| \leq 1$ one may construct a linearized homogenization corrector $\phi^T_{\xi,B}(\cdot,\tilde\omega+h\delta\tilde\omega)$ in the precise sense of Lemma 8 satisfying in particular (51). Based on that observation, the next step consists of studying the equation satisfied by

$$\frac{1}{h}\big(\phi^T_{\xi,B}(\cdot,\tilde\omega{+}h\delta\tilde\omega){-}\phi^T_{\xi,B}(\cdot,\tilde\omega)\big) - \delta\phi^T_{\xi,B}(\cdot,\tilde\omega). \tag{199}$$

As a consequence of $(A2)_L$ and $(A3)_L$ from Assumption 1, the weighted energy estimate (T3) applied to the equation satisfied by the expansion from the previous display, and an induction over the linearization order $1 \leq l \leq L$ (first providing an estimate on differences $\frac{1}{h}\big(\phi^T_{\xi,B}(\cdot,\tilde\omega+h\delta\tilde\omega) - \phi^T_{\xi,B}(\cdot,\tilde\omega)\big)$, and then proceeding on with (199) as in the first step of the proof, inductively relying on the estimates (195)–(197) and their analogs in terms of linearized correctors), we obtain

$$\bigg\|\frac{1}{\sqrt{T}}\bigg(\frac{\phi^T_{\xi,B}(\cdot,\tilde\omega{+}h\delta\tilde\omega) - \phi^T_{\xi,B}(\cdot,\tilde\omega)}{h}{-}\delta\phi^T_{\xi,B}(\cdot,\tilde\omega)\bigg)\bigg\|_{L^2(B_1(x_0))} \to 0, \tag{200}$$

$$\bigg\|\frac{\nabla\phi^T_{\xi,B}(\cdot,\tilde\omega{+}h\delta\tilde\omega) - \nabla\phi^T_{\xi,B}(\cdot,\tilde\omega)}{h}{-}\nabla\delta\phi^T_{\xi,B}(\cdot,\tilde\omega)\bigg\|_{L^2(B_1(x_0))} \to 0 \tag{201}$$

for all $x_0 \in \mathbb{R}^d$. This entails the asserted qualitative directional differentiability for the linearized homogenization corrector $\phi^T_{\xi,B}(\cdot,\tilde\omega)$.

Finally, the claims from the statement of Lemma 26 concerning linearized flux correctors and random parameter fields now follow from the previous considerations as in the proof of Lemma 8. For a proof of the bounds (190)–(191), we first verify the claim on the level of the corrector gradient $\nabla\phi^T_\xi$, assuming in the course only (A1), $(A2)_0$ and $(A3)_0$ of Assumption 1.

Applying the weighted energy estimate (T3) to the equation satisfied by ϕ_ξ^T in form of (recall that $A(\omega,0)=0$)

$$\frac{1}{T}\big(\xi\cdot x+\phi_\xi^T\big)-\nabla\cdot\bigg(\int_0^1(\partial_\xi A)\big(\omega,s\xi+s\nabla\phi_\xi^T\big)\,\mathrm{d}s\bigg)\big(\xi+\nabla\phi_\xi^T\big)=\frac{1}{T}\xi\cdot x,$$

we obtain for all $q\geq 1$

$$\big\langle\|\xi+\nabla\phi_\xi^T\|_{L^2(B_1)}^{2q}\big\rangle^{\frac{1}{q}}\lesssim\sqrt{T}^d\int\ell^0_{\gamma,\sqrt{T}}\Big(\frac{|x|}{\sqrt{T}}\Big)^2\lesssim\sqrt{T}^d.$$

Since the same bound holds true in terms of $\phi_\xi^T(\cdot,\omega+h\delta\omega)$ uniformly over all $|h|\leq 1$, it follows from applying (T3) with respect to the equation (192) for all $q\geq 1$

$$\Big\langle\Big\|\Big(\frac{1}{\sqrt{T}}\big(\phi_\xi^T(\cdot,\omega)-\phi_\xi^T(\cdot,\omega+h\delta\omega)\big),\nabla\phi_\xi^T(\cdot,\omega)-\nabla\phi_\xi^T(\cdot,\omega+h\delta\omega)\Big)\Big\|_{L^2(B_1)}^{2q}\Big\rangle^{\frac{1}{q}}$$
$$\lesssim\sqrt{T}^d|h|^2\|\delta\omega\|^2_{C^\alpha_{\mathrm{uloc}}}\bigg\langle\bigg|\int\ell^0_{\gamma,\sqrt{T}}\big|\xi+\nabla\phi_\xi^T(\cdot,\omega+h\delta\omega)\big|^2\bigg|^q\bigg\rangle^{\frac{1}{q}}\lesssim\sqrt{T}^d|h|^2.$$

By a similar reasoning, one obtains for all $q\geq 1$ the bound

$$\Big\langle\Big\|\Big(\frac{1}{\sqrt{T}}\delta\phi_\xi^T,\nabla\delta\phi_\xi^T\Big)\Big\|_{L^2(B_1)}^{2q}\Big\rangle^{\frac{1}{q}}\lesssim\sqrt{T}^d.$$

Hence, in view of the previous three displays and the almost sure convergence from (193)–(194), we deduce from Vitali's theorem that for all $q\geq 1$ as $h\to 0$

$$\bigg\langle\bigg\|\frac{1}{\sqrt{T}}\Big(\frac{\phi_\xi^T(\cdot,\omega+h\delta\omega)-\phi_\xi^T(\cdot,\omega)}{h}-\delta\phi_\xi^T(\cdot,\omega)\Big)\bigg\|_{L^2(B_1)}^{2q}\bigg\rangle^{\frac{1}{q}}\to 0, \tag{202}$$

$$\bigg\langle\bigg\|\frac{\nabla\phi_\xi^T(\cdot,\omega+h\delta\omega)-\nabla\phi_\xi^T(\cdot,\omega)}{h}-\nabla\delta\phi_\xi^T(\cdot,\omega)\bigg\|_{L^2(B_1)}^{2q}\bigg\rangle^{\frac{1}{q}}\to 0. \tag{203}$$

Arguing inductively over the linearization order finally ensures (190) and (191). ☐

Appendix C. Corrector bounds for higher-order linearizations: Proofs for the base cases

C.1. **Proof of Proposition 6** (Estimates for localized homogenization correctors of the nonlinear problem)**.** The corrector estimates (45) hold true by a combination of [18, Proof of Lemma 24, Lemma 26, Estimate (99)]. The small-scale annealed Schauder estimate (47) is in turn already established by Lemma 23.

For a proof of (46a) and (46b), consider a perturbation $\delta\omega\in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)}<\infty$. Differentiating the defining equation (44a) for the localized homogenization corrector with respect to the parameter field in the direction of $\delta\omega$ yields (cf. the first step in the proof of Lemma 26)

$$\frac{1}{T}\delta\phi_\xi^T-\nabla\cdot(\partial_\xi A)(\omega,\xi+\nabla\phi_\xi^T)\nabla\delta\phi_\xi^T=\nabla\cdot(\partial_\omega A)(\omega,\xi+\nabla\phi_\xi^T)\delta\omega,$$

with $(\frac{\delta\phi_\xi^T}{\sqrt{T}},\nabla\delta\phi_\xi^T)\in L^2(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$. Due to $\xi+\nabla\phi_\xi^T\in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^d)$ as well as $\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big)\in L^2(\mathbb{R}^d;\mathbb{R}^d)$ (the latter being a consequence of the energy estimate), the random field defined by

$$G_\xi^T:=\big((\partial_\omega A)(\omega,\xi+\nabla\phi_\xi^T)\big)^*\nabla\Big(\frac{1}{T}-\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f+\nabla\cdot g\Big) \tag{204}$$

satisfies $\big[G_\xi^T\big]_1 \in L^2(\mathbb{R}^d;\mathbb{R}^n)$. Hence, by the usual duality argument we may compute (with $a_\xi^{T,*}$ denoting the transpose of the uniformly elliptic and bounded coefficient field $(\partial_\xi A)(\omega,\xi{+}\nabla\phi_\xi^T)$)

$$\begin{aligned}
&\int g\cdot\nabla\delta\phi_\xi^T - \int f\frac{1}{T}\delta\phi_\xi^T \\
&= \int (\partial_\omega A)(\omega,\xi{+}\nabla\phi_\xi^T)\delta\omega\cdot\nabla\Big(\frac{1}{T}{-}\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}f{+}\nabla\cdot g\Big).
\end{aligned}$$

This proves (46a) in terms of the random field G_ξ^T defined by (204). The asserted integrability of G_ξ^T will be a consequence of the subsequent argument.

From the previous display, it follows by duality in $L^q_{\langle\cdot\rangle}$, the stationarity of the localized homogenization corrector ϕ_ξ^T, and Hölder's inequality that

$$\begin{aligned}
\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G_\xi^T|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} &\leq C^2\big\langle\big\|\xi{+}\nabla\phi_\xi^T\big\|_{L^2(B_1)}^{2\frac{q\vee q_0}{\kappa}}\big\rangle^{\frac{1}{\frac{q\vee q_0}{\kappa}}} \\
&\times \sup_{\langle F^{2q_*}=1\rangle}\int\Big\langle\Big|\nabla\Big(\frac{1}{T}{-}\nabla\cdot a_\xi^{T,*}\nabla\Big)^{-1}\Big(\frac{1}{T}Ff{+}\nabla\cdot Fg\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.
\end{aligned}$$

For large enough $q_0\in[1,\infty)$, we may then apply the annealed Calderón–Zygmund estimate from (T8) to infer from the previous display that

$$\begin{aligned}
&\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G_\xi^T|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} \\
&\leq C^2\big\langle\big\|\xi{+}\nabla\phi_\xi^T\big\|_{L^2(B_1)}^{2\frac{q\vee q_0}{\kappa}}\big\rangle^{\frac{1}{\frac{q\vee q_0}{\kappa}}}\sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(Fg,\frac{Ff}{\sqrt{T}}\Big)\Big|^{2(\frac{q\vee q_0}{\kappa})_*}\Big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}.
\end{aligned}$$

The sensitivity estimate (46b) now follows from the corrector estimates (45). Note that the asserted integrability of G_ξ^T is also a consequence of the previous display.

Finally, for a proof of (46c) denote by $G_\xi^{T,r}$ the random fields defined by (204), however with (g,f) replaced by (g_r,f_r). Note also that the definition (204) even makes sense for merely square integrable pairs (g,f). Hence, since the right hand side of the definition (204) depends linearly on the data (g,f), applying the estimate from the previous display in terms of the difference $G_\xi^{T,r}{-}G_\xi^T$ thus yields together with an application of Hölder's inequality with respect to $(\frac{2q-1}{2(q-1)},2q{-}1)$

$$\begin{aligned}
\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G_\xi^T{-}G_\xi^{T,r}|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} &\lesssim_q \sup_{\langle F^{2q_*}\rangle=1}\int\Big\langle\Big|\Big(F(g{-}g_r),\frac{F(f{-}f_r)}{\sqrt{T}}\Big)\Big|^{2(2q)_*}\Big\rangle^{\frac{1}{(2q)_*}} \\
&\lesssim_q \int\Big\langle\Big|(g{-}g_r),\frac{(f{-}f_r)}{\sqrt{T}}\Big)\Big|^{2(2q)}\Big\rangle^{\frac{1}{(2q)}}.
\end{aligned}$$

This, of course, shows (46c) and thus concludes the proof of Proposition 6. □

C.2. **Estimates for differences of localized homogenization correctors of the nonlinear problem.** We next turn to a result which provides a proof of the base case for the induction in the proof of Lemma 16.

Lemma 27 (Estimates for differences of localized homogenization correctors of the nonlinear problem)**.** *Let the requirements and notation of (A1), (A2)$_1$ and (A3)$_1$*

of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Let $T \in [1,\infty)$ and $M>0$ be fixed. For any vector $\xi \in \mathbb{R}^d$ let

$$\Big(\frac{\phi_\xi^T}{\sqrt{T}}, \nabla\phi_\xi^T\Big) \in L^2_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}\times\mathbb{R}^d)$$

denote the unique solution of the localized corrector problem (44a). For any unit vector $e \in \mathbb{R}^d$ and any $|h| \leq 1$, the difference of localized homogenization correctors $\phi_{\xi+he}^T - \phi_\xi^T$ then satisfies the following estimates:

- *There exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta)$ such that for all $|\xi| \leq M$, all $q \in [1,\infty)$, and all compactly supported and square-integrable f,g we have* corrector estimates for differences

$$\begin{aligned}\Big\langle\Big|\int g\cdot\big(\nabla\phi_{\xi+he}^T-\nabla\phi_\xi^T\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'}|h|^2\int|g|^2,\\ \Big\langle\Big|\int \frac{1}{T}f\big(\phi_{\xi+he}^T-\phi_\xi^T\big)\Big|^{2q}\Big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'}|h|^2\int\Big|\frac{f}{\sqrt{T}}\Big|^2,\\ \Big\langle\Big\|\Big(\frac{\phi_{\xi+he}^T-\phi_\xi^T}{\sqrt{T}},\nabla\phi_{\xi+he}^T-\nabla\phi_\xi^T\Big)\Big\|_{L^2(B_1)}^{2q}\Big\rangle^{\frac{1}{q}} &\leq C^2 q^{2C'}|h|^2.\end{aligned} \tag{205}$$

- *Fix $p \in (2,\infty)$, and let $g \in \bigcap_{q\geq1} L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$ be a compactly supported and $L^p(\mathbb{R}^d)$-valued random field. Then there exists a random field $G_{\xi,h,e}^T$ satisfying $\big[G_{\xi,h,e}^T\big]_1 \in \bigcap_{q\geq1} L^{2q}_{\langle\cdot\rangle}L^2(\mathbb{R}^d;\mathbb{R}^n)$, and which is related to g via $\phi_{\xi+he}^T-\phi_\xi^T$ in the sense that, $\mathbf{P}$-almost surely, it holds for all $\delta\omega \in C^\eta_{\mathrm{uloc}}(\mathbb{R}^d;\mathbb{R}^n)$ with $\|[\delta\omega]_\infty\|_{L^2(\mathbb{R}^d)} < \infty$*

$$\int g\cdot\nabla\big(\delta\phi_{\xi+he}^T-\delta\phi_\xi^T\big) = \int G_{\xi,h,e}^T\cdot\delta\omega. \tag{206a}$$

There exists $q_0 = q_0(d,\lambda,\Lambda) \in (1,\infty)$ such that for any $\kappa \in (0,1]$, there then exist constants $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},M,\eta,\kappa)$ and $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta)$ such that for all $|\xi| \leq M$ and all $q \in [1,\infty)$ the random field $G_{\xi,h,e}^T$ gives rise to a sensitivity estimate for differences of correctors

$$\Big\langle\Big|\int\Big(\fint_{B_1(x)}|G_{\xi,h,e}^T|\Big)^2\Big|^q\Big\rangle^{\frac{1}{q}} \leq C^2q^{2C'}|h|^2\sup_{\langle F^{2q_*}\rangle=1}\int\big\langle|Fg|^{2(\frac{q\vee q_0}{\kappa})_*}\big\rangle^{\frac{1}{(\frac{q\vee q_0}{\kappa})_*}}. \tag{206b}$$

If $(g_r)_{r\geq1}$ denotes a sequence in $\bigcap_{q\geq1} L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$ of compactly supported and $L^p(\mathbb{R}^d)$-valued random fields, denote by $G_{\xi,h,e}^{T,r}$, $r\geq1$, the random field associated to g_r, $r\geq1$, in the sense of the condition (206a). Let g be an $L^p_{\mathrm{loc}}(\mathbb{R}^d)$-valued random field such that $g_r \to g$ as $r\to\infty$ in $\bigcap_{q\geq1} L^2(\mathbb{R}^d;L^{2q}_{\langle\cdot\rangle})$. Then there exists a random field $G_{\xi,h,e}^T$ with

$$\big[G_{\xi,h,e}^{T,r}-G_{\xi,h,e}^T\big]_1 \to 0 \text{ as } r\to\infty \text{ in } \bigcap_{q\geq1} L^{2q}_{\langle\cdot\rangle}L^2(\mathbb{R}^d;\mathbb{R}^n), \tag{206c}$$

and the limit random field is subject to the sensitivity estimate (206b).

- *There exists* $\alpha = \alpha(d,\lambda,\Lambda) \in (0,\eta)$, $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},\eta,M)$ *as well as* $C' = C'(d,\lambda,\Lambda,C'_{\mathrm{reg}},\eta)$ *such that for all* $|\xi| \leq M$, *and all* $q \in [1,\infty)$ *we have a* small-scale annealed Schauder estimate *of the form*

$$\big\langle \big\|\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi\big\|^{2q}_{C^\alpha(B_1)}\big\rangle^{\frac{1}{q}} \leq C^2 q^{2C'}|h|^2. \tag{207}$$

Proof. The estimates (205) follow from a combination of the qualitative differentiability result [18, Lemma 27] with [18, Proof of Lemma 28, Lemma 30, Estimate (100)]. Note that these results are even available under the weaker small-scale regularity condition (R) from Assumption 3 as we still have annealed Hölder regularity of the linearized coefficient fields $(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$ at our disposal, see Lemma 24. For a proof of the remaining assertions, note that the equation for the difference of localized correctors is given by

$$\begin{aligned}
&\frac{1}{T}(\phi^T_{\xi+he} - \phi^T_\xi) - \nabla\cdot\big\{A(\omega,\xi{+}\nabla\phi^T_{\xi+he}) - A(\omega,\xi{+}\nabla\phi^T_\xi)\big\}\\
&= -\nabla\cdot\big\{A(\omega,\xi{+}\nabla\phi^T_{\xi+he}) - A(\omega,\xi{+}\nabla\phi^T_{\xi+he}{+}he)\big\}.
\end{aligned}$$

By means of $(\mathrm{A2})_0$ from Assumption 1, we may express the equation for the difference in equivalent form as follows:

$$\begin{aligned}
&\frac{1}{T}(\phi^T_{\xi+he} - \phi^T_\xi) - \nabla\cdot\bigg(\int_0^1 (\partial_\xi A)(\omega,\xi{+}s\nabla\phi^T_{\xi+he}{+}(1{-}s)\nabla\phi^T_\xi)\,\mathrm{d}s\bigg)(\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi)\\
&= -\nabla\cdot\int_0^1 (\partial_\xi A)(\omega,\xi{+}\nabla\phi^T_{\xi+he}{+}she)\,\mathrm{d}s\,he.
\end{aligned}$$

The coefficient in the equation of the previous display is uniformly elliptic and bounded with respect to the constants (λ,Λ) from Assumption 1, and by means of Lemma 24 Hölder continuous with an annealed estimate for the associated Hölder norm of the form (T7). In particular, applying the local Schauder estimate (T5) to the equation from the previous display in combination with the corrector estimates (47) and (205) implies the small-scale annealed Schauder estimate (207) for differences of localized correctors.

The remaining assertions (206a)–(206c) follow by an argument similar to the proof of (46a)–(46c), which in view of the right hand side term in the equation for the difference $\phi^T_{\xi+he}{-}\phi^T_\xi$ in particular relies on the already established ingredients (46a)–(46c). Details are left to the interested reader. ☐

C.3. **Differentiability of localized homogenization correctors of the nonlinear problem.** The base case of the induction in the proof of Lemma 17 is covered by the following result.

Lemma 28 (Differentiability of localized homogenization correctors of the nonlinear problem)**.** *Let the requirements and notation of (A1), (A2)*$_1$ *and (A3)*$_1$ *of Assumption 1, (P1) and (P2) of Assumption 2, and (R) of Assumption 3 be in place. Let* $T \in [1,\infty)$ *and* $M > 0$ *be fixed. For any* $\xi \in \mathbb{R}^d$ *let* $\phi^T_\xi \in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ *resp.* $\phi^T_{\xi,e} \in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ *denote the unique solutions of the problems* (44a) *resp.* (49a)*, and let* q^T_ξ *resp.* $q^T_{\xi,e}$ *denote the associated fluxes from* (44b) *resp.* (49d)*.*

For any unit vector $e \in \mathbb{R}^d$ *and any* $|h| \leq 1$*, the first-order Taylor expansion of localized homogenization correctors* $\phi^T_{\xi+he}{-}\phi^T_\xi{-}\phi^T_{\xi,e}h$ *then satisfies the following* *estimate: there exists a constant* $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},C'_{\mathrm{reg}},\eta,M)$ *such that for all* $|\xi| \leq M$ *it holds*

$$\big\langle \big\|\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi - \nabla\phi^T_{\xi,e}h\big\|^2_{L^2(B_1)}\big\rangle \leq C^2h^4. \tag{208}$$

In particular, the map $\xi \mapsto \nabla\phi^T_\xi$ *is Fréchet differentiable with values in the Fréchet space* $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$*. Finally, we also have the estimate*

$$\big|\big\langle q^T_{\xi+he}\big\rangle - \big\langle q^T_\xi\big\rangle - \big\langle q^T_{\xi,e}h\big\rangle\big|^2 \leq C^2h^4. \tag{209}$$

Proof. For a proof of (208), we start by computing the equation for the first-order Taylor expansion of localized homogenization correctors $\phi^T_{\xi+he} - \phi^T_\xi - \phi^T_{\xi,e}h$. Abbreviating $a^T_\xi := (\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)$ and adding zero yields

$$\begin{aligned}
&\frac{1}{T}(\phi^T_{\xi+he} - \phi^T_\xi - \phi^T_{\xi,e}h) - \nabla\cdot a^T_\xi(\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi - \nabla\phi^T_{\xi,e}h)\\
&= -\nabla\cdot\Big\{(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\nabla\phi^T_{\xi+he} - A(\omega,\xi+he+\nabla\phi^T_{\xi+he})\Big\}\\
&\quad + \nabla\cdot\Big\{(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\nabla\phi^T_\xi - A(\omega,\xi+\nabla\phi^T_\xi)\Big\}\\
&\quad - \nabla\cdot(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)eh.
\end{aligned}$$

Adding zero again, we may rewrite the equation from the previous display in the following equivalent form

$$\frac{1}{T}(\phi^T_{\xi+he} - \phi^T_\xi - \phi^T_{\xi,e}h) - \nabla\cdot a^T_\xi(\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi - \nabla\phi^T_{\xi,e}h) = -\nabla\cdot\sum_{i=1}^3 R_i \tag{210}$$

with the divergence form right hand side terms given by

$$\begin{aligned}
R_1 &:= -\Big\{A(\omega,\xi+he+\nabla\phi^T_\xi) - A(\omega,\xi+\nabla\phi^T_\xi) - (\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)eh\Big\}\\
R_2 &:= A(\omega,\xi+he+\nabla\phi^T_\xi) - A(\omega,\xi+he+\nabla\phi^T_{\xi+he})\\
&\quad - (\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)(\nabla\phi^T_\xi - \nabla\phi^T_{\xi+he})\\
R_3 &:= -\Big\{\big((\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi) - (\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\big)(\nabla\phi^T_\xi - \nabla\phi^T_{\xi+he})\Big\}.
\end{aligned}$$

As a consequence of the weighted energy estimate (T3) applied to equation (210) as well as stationarity we then obtain

$$\big\langle \big\|\nabla\phi^T_{\xi+he} - \nabla\phi^T_\xi - \nabla\phi^T_{\xi,e}h\big\|^2_{L^2(B_1)}\big\rangle \leq C^2\sum_{i=1}^3\big\langle\big\|R_i\big\|^2_{L^2(B_1)}\big\rangle.$$

Observe that by means of $(\mathrm{A2})_0$ from Assumption 1 we may express the right hand side terms of (210) as follows:

$$\begin{aligned}
R_1 &= -\int_0^1 \Big\{(\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi+she) - (\partial_\xi A)(\omega,\xi+\nabla\phi^T_\xi)\Big\}eh\,\mathrm{d}s,\\
R_2 &= \int_0^1 \Big\{(\partial_\xi A)(\omega,\xi+he+s\nabla\phi^T_\xi+(1-s)\nabla\phi^T_{\xi+he}) - (\partial_\xi A)(\omega,\xi+he+\nabla\phi^T_\xi)\Big\}\\
&\qquad\qquad\times(\nabla\phi^T_\xi - \nabla\phi^T_{\xi+he})\,\mathrm{d}s,\\
R_3 &= \int_0^1 (\partial^2_\xi A)(\omega,\xi+\nabla\phi^T_\xi+she)\big[eh\odot(\nabla\phi^T_\xi - \nabla\phi^T_{\xi+he})\big].
\end{aligned}$$

The previous two displays in combination with $(\mathrm{A2})_1$ from Assumption 1, the corrector estimates for differences (205), and Hölder's inequality then imply the asserted estimate (208).

For a proof of (209), observe first that because of stationarity and Jensen's inequality we obtain

$$\big|\big\langle q_{\xi+he}^T\big\rangle - \big\langle q_\xi^T\big\rangle - \big\langle q_{\xi,e}^T h\big\rangle\big|^2 \leq \big\langle\big\|q_{\xi+he}^T - q_\xi^T - q_{\xi,e}^T h\big\|_{L^2(B_1)}^2\big\rangle.$$

Moreover, by definition (44b) resp. (49d) of the fluxes we have

$$q_{\xi+he}^T - q_\xi^T - q_{\xi,e}^T h = a_\xi^T(\nabla\phi_{\xi+he}^T - \nabla\phi_\xi^T - \nabla\phi_{\xi,e}^T h) - \sum_{i=1}^3 R_i.$$

Hence, the above reasoning for the proof of (208) together with the estimate (208) itself then entails the estimate (209). This concludes the proof of Lemma 28. □

C.4. **Limit passage in massive approximation of the nonlinear corrector problem.** We finally formulate the result covering the base case of the induction in the proof of Lemma 19.

Lemma 29 (Limit passage in massive approximation of the nonlinear corrector problem)**.** *Let the requirements and notation of (A1), $(A2)_0$ and $(A3)_0$ of Assumption 1, as well as (P1) and (P2) of Assumption 2 be in place. Let $T \in [1,\infty)$ be fixed. For any given vector $\xi \in \mathbb{R}^d$ let $\phi_\xi^T \in H^1_{\mathrm{uloc}}(\mathbb{R}^d)$ denote the unique solution of (44a), and let q_ξ^T denote the associated flux from (44b).*

There exists $C = C(d,\lambda,\Lambda,C_{\mathrm{sg}},C_{\mathrm{reg}},C'_{\mathrm{reg}},M)$ such that for all $|\xi| \leq M$ it holds

$$\big\langle\big\|\nabla\phi_\xi^{2T} - \nabla\phi_\xi^T\big\|_{L^2(B_1)}^2\big\rangle \leq C^2\frac{\mu_*^2(\sqrt{T})}{T}. \tag{211}$$

In particular, the sequence $(\nabla\phi_\xi^T)_{T\in[1,\infty)}$ is Cauchy in $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$ (with respect to the strong topology). The limit gives rise to the unique homogenization corrector of the nonlinear PDE in the sense of Definition 3. Finally, it holds

$$\big\langle\big|q_\xi^T - q_\xi\big|^2\big\rangle \to 0 \tag{212}$$

with the limiting flux q_ξ defined in (29b).

Proof. For a proof of the error estimate (211), we first note that

$$\frac{1}{2T}(\phi_\xi^{2T} - \phi_\xi^T) - \nabla\cdot\Big(\int_0^1(\partial_\xi A)(\omega,\xi+s\nabla\phi_\xi^{2T}+(1-s)\nabla\phi_\xi^T)\,\mathrm{d}s\Big)(\nabla\phi_\xi^{2T} - \nabla\phi_\xi^T) = \frac{1}{2T}\phi_\xi^T.$$

As a consequence of the weighted energy estimate (T3) applied to equation of the previous display, stationarity, smuggling in a spatial average of ϕ_ξ^T over B_1, and Poincaré's inequality, we thus get

$$\begin{aligned}\big\langle\|\nabla\phi_\xi^{2T} - \nabla\phi_\xi^T\|_{L^2(B_1)}^2\big\rangle &\leq C^2\frac{1}{T}\big\langle\|\phi_\xi^T\|_{L^2(B_1)}^2\big\rangle \\ &\leq C^2\frac{1}{T}\big\langle\|\nabla\phi_\xi^T\|_{L^2(B_1)}^2\big\rangle + C^2\frac{1}{T}\Big\langle\Big|\fint_{B_1}\phi_\xi^T\Big|^2\Big\rangle.\end{aligned}$$

By the same argument as the one for (43), we obtain $\langle|\fint_{B_1}\phi_\xi^T|^2\rangle \leq C^2\mu_*^2(\sqrt{T})$. Hence, exploiting in addition the corrector estimate (45) we may deduce from the previous display the desired bound (211). It then follows from [18, Lemma 33] that the corresponding limit yields the unique homogenization corrector of the nonlinear

PDE in the sense of Definition 3. Finally, the estimate (212) is a consequence of stationarity, Jensen's inequality, and the strong convergence of $\nabla\phi_\xi^T$ to $\nabla\phi_\xi$ in the space $L^2_{\langle\cdot\rangle}L^2_{\mathrm{loc}}(\mathbb{R}^d)$. □

References

[1] S. Armstrong, S. J. Ferguson, and T. Kuusi. Higher-order linearization and regularity in nonlinear homogenization. *Archive for Rational Mechanics and Analysis*, 237(2):631–741, 2020. doi:10.1007/s00205-020-01519-1.

[2] S. Armstrong, S. J. Ferguson, and T. Kuusi. Homogenization, linearization, and large-scale regularity for nonlinear elliptic equations. *Communications on Pure and Applied Mathematics*, 2020. doi:10.1002/cpa.21902.

[3] S. Armstrong, T. Kuusi, and J.-C. Mourrat. The additive structure of elliptic homogenization. *Inventiones mathematicae*, 208(3):999–1154, 2016. doi:10.1007/s00222-016-0702-4.

[4] S. Armstrong and J.-C. Mourrat. Lipschitz regularity for elliptic equations with random coefficients. *Archive for Rational Mechanics and Analysis*, 219(1):255–348, 2015. doi:10.1007/s00205-015-0908-4.

[5] S. Armstrong and C. K. Smart. Stochastic homogenization of fully nonlinear uniformly elliptic equations revisited. *Calculus of Variations and Partial Differential Equations*, 50(3-4):967–980, 2013. doi:10.1007/s00526-013-0663-z.

[6] S. Armstrong and C. K. Smart. Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity. *The Annals of Probability*, 42(6):2558–2594, 2014. doi:10.1214/13-aop833.

[7] S. Armstrong and C. K. Smart. Quantitative stochastic homogenization of convex integral functionals. *Annales scientifiques de l'École normale supérieure*, 49(2):423–481, 2016. doi:10.24033/asens.2287.

[8] L. A. Caffarelli and P. E. Souganidis. Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media. *Inventiones mathematicae*, 180(2):301–360, 2010. doi:10.1007/s00222-009-0230-6.

[9] L. A. Caffarelli, P. E. Souganidis, and L. Wang. Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media. *Communications on Pure and Applied Mathematics*, 58(3):319–361, 2004. doi:10.1002/cpa.20069.

[10] J. Conlon and A. Naddaf. On homogenization of elliptic equations with random coefficients. *Electronic Journal of Probability*, 5:paper no. 9, 58 pp., 2000. doi:10.1214/ejp.v5-65.

[11] J. G. Conlon and A. Fahim. Strong convergence to the homogenized limit of parabolic equations with random coefficients. *Transactions of the American Mathematical Society*, 367(5):3041–3093, 2014. doi:10.1090/s0002-9947-2014-06005-4.

[12] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization. *Annali di Matematica Pura ed Applicata*, 144(1):347–389, 1986. doi:10.1007/bf01760826.

[13] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, (368):28–42, 1986. doi:10.1515/crll.1986.368.28.

[14] M. Duerinckx, J. Fischer, and A. Gloria. Scaling limit of the homogenization commutator for gaussian coefficient fields. *Preprint*, 2019. arXiv:1910.04088.

[15] M. Duerinckx, A. Gloria, and F. Otto. Robustness of the pathwise structure of fluctuations in stochastic homogenization. *Probability Theory and Related Fields*, 178(1-2):531–566, 2020. doi:10.1007/s00440-020-00983-w.

[16] M. Duerinckx, A. Gloria, and F. Otto. The structure of fluctuations in stochastic homogenization. *Commun. Math. Phys.*, 377(1):259–306, 2020. doi:10.1007/s00220-020-03722-3.

[17] M. Duerinckx and F. Otto. Higher-order pathwise theory of fluctuations in stochastic homogenization. *Stoch PDE: Anal Comp*, 8(3):625–692, 2019. doi:10.1007/s40072-019-00156-4.

[18] J. Fischer and S. Neukamm. Optimal homogenization rates in stochastic homogenization of nonlinear uniformly elliptic equations and systems. *Preprint*, 2020. arXiv:1908.02273v2.

[19] M. Giaquinta and L. Martinazzi. *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*. Scuola Normale Superiore, 2012.

[20] A. Gloria, S. Neukamm, and F. Otto. An optimal quantitative two-scale expansion in stochastic homogenization of discrete elliptic equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(2):325–346, 2014. doi:10.1051/m2an/2013110.

[21] A. Gloria, S. Neukamm, and F. Otto. Quantitative estimates in stochastic homogenization for correlated coefficient fields. *Preprint*, 2019. arXiv:1910.05530.
[22] A. Gloria, S. Neukamm, and F. Otto. A regularity theory for random elliptic operators. *Milan Journal of Mathematics*, 88(1):99–170, 2020. doi:10.1007/s00032-020-00309-4.
[23] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3):779–856, 2011. doi:10.1214/10-aop571.
[24] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. App. Probab.*, 22(1):1–28, 2012. doi:10.1214/10-aap745.
[25] A. Gloria and F. Otto. The corrector in stochastic homogenization: optimal rates, stochastic integrability, and fluctuations. *Preprint*, 2015. arXiv:1510.08290.
[26] A. Gloria and F. Otto. Quantitative results on the corrector equation in stochastic homogenization. *Journal of the European Mathematical Society*, 19(11):3489–3548, 2017. doi:10.4171/jems/745.
[27] Y. Gu and J.-C. Mourrat. Scaling limit of fluctuations in stochastic homogenization. *Multiscale Modeling & Simulation*, 14(1):452–481, 2016. doi:10.1137/15m1010683.
[28] S. Hensel. Stochastic homogenization of nonlinear uniformly elliptic equations: Fluctuations of the corrector and its (higher-order) linearizations. *In preparation*.
[29] M. Josien and F. Otto. The annealed Calderon-Zygmund estimate as convenient tool in quantitative stochastic homogenization. *Preprint*, 2020. arXiv:2005.08811.
[30] S. M. Kozlov. Averaging of random operators. *Mathematics of the USSR-Sbornik*, 37(2):167–180, 1980. doi:10.1070/sm1980v037n02abeh001948.
[31] S. M. Kozlov. Averaging of difference schemes. *Mathematics of the USSR-Sbornik*, 57(2):351–369, 1987. doi:10.1070/sm1987v057n02abeh003072.
[32] R. Künnemann. The diffusion limit for reversible jump processes on $\mathbb{Z}^d$ with ergodic random bond conductivities. *Communications in Mathematical Physics*, 90(1):27–68, 1983. doi:10.1007/bf01209386.
[33] P.-L. Lions and P. E. Souganidis. Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Communications on Pure and Applied Mathematics*, 56(10):1501–1524, 2003. doi:10.1002/cpa.10101.
[34] P. Malliavin. *Stochastic Analysis*. Springer Berlin Heidelberg, 1997.
[35] S. Müller. Nonlinear partial differential equations I. Lectures notes. https://www.iam.uni-bonn.de/users/lienstromberg/teaching/npde1/.
[36] A. Naddaf and T. Spencer. Estimates on the variance of some homogenization problems, 1998.
[37] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam-New York, 1981.
[38] R. W. Schwab. Stochastic homogenization for some nonlinear integro-differential equations. *Communications in Partial Differential Equations*, 38(2):171–198, 2012. doi:10.1080/03605302.2012.741176.
[39] V. V. Yurinskii. Averaging of symmetric diffusion in random medium. *Siberian Mathematical Journal*, 27(4):603–613, 1987. doi:10.1007/bf00969174.

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