

# THE SHARP INTERFACE LIMIT OF A NAVIER–STOKES/ALLEN–CAHN SYSTEM WITH CONSTANT MOBILITY: CONVERGENCE RATES BY A RELATIVE ENERGY APPROACH

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ABSTRACT. We investigate the sharp interface limit of a diffuse interface system that couples the Allen–Cahn equation with the instationary Navier–Stokes system in a bounded domain in  $\mathbb{R}^d$  with  $d \in \{2, 3\}$ . This model is used to describe a propagating front in a viscous incompressible flow with the width of the transition layer being characterized by a small parameter  $\varepsilon > 0$ . We show that the solutions converge to a limit two-phase fluid system with surface tension that couples the mean curvature flow and the Navier–Stokes system. The main assumptions are that the evolution of the limit system is sufficiently regular and that the associated evolving interface does not intersect the boundary of the container. For quantitatively well-prepared initial data, we even establish an optimal convergence rate. This is the first rigorous result of this kind which is valid in all physically relevant ambient dimensions.

**Keywords:** Allen–Cahn equation, complex fluid, relative entropy, mean curvature flow, sharp interface limit.

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## 1. INTRODUCTION

1.1. **Context.** Curvature driven interface evolution is a challenging topic in PDE theory due to the inherent emergence of topology changes. Indeed, classical descriptions (i.e., via parametrization of the evolving interface) cease to work once the solution approaches the first time of such a topology change. One popular tool to describe the dynamics even after these consists of phase-field models, where instead of a sharp interface one considers a diffuse interface layer of finite width (which is typically small related to a given scaling parameter  $\varepsilon$ ). In the specific example of the evolution of two (macroscopically) immiscible fluids in the presence of surface tension, an associated fundamental phase-field model is the so-called *model H* coupling Navier–Stokes dynamics with the fourth-order Cahn–Hilliard dynamics. This model was in fact introduced in [27, 33].

Next to obvious PDE questions concerning existence and uniqueness of solutions to such a phase-field model, another highly relevant and natural question is the consistency with a sharp interface model. Indeed, by formally taking the limit  $\varepsilon \rightarrow 0$  one hopes to identify the underlying dynamics in terms of the sharp interface model under consideration, and therefore to justify the transition to the phase-field model. Needless to say, for numerical applications apart from qualitative convergence results also quantitative results in the form of convergence rates in certain norms are of interest.

In the context of model H (and even for more general models allowing, e.g., a non-constant density), the formal derivation of the associated sharp interface limit model by means of asymptotic expansion techniques is performed in [3]. Rigorous justifications of these arguments are scarce though. For global-in-time convergence of solutions to model H (or a related model) to a rather weak solution to the sharp interface limit model (i.e., a concept of varifold solutions), one may consult [4] (or [11], respectively). Short-time (quantitative) convergence to strong solutions of the sharp interface limit model is in turn shown in the recent works [6] and [7], where, however, only the setting of the stationary Stokes operator is treated. At the time of this writing, the rigorous derivation of convergence rates incorporating the full Navier–Stokes dynamics in model H remains an important open problem in the field.

In the present work, we study instead of model H (or related models) a phase-field model with fluid mechanical coupling in ambient dimension  $d \in \{2, 3\}$  which is based on the second-order Allen–Cahn operator. We further consider the scaling regime of constant mobility, and with respect to the fluid mechanical modeling we restrict ourselves to the case of Navier–Stokes dynamics with constant density and constant viscosity (we refer to Subsection 1.2 for a mathematical formulation of the model). Phase-field models of such Navier–Stokes/Allen–Cahn type were first introduced in [36, 46]. Formal asymptotic expansions (cf. [1]) suggest that the associated sharp-interface limit model is given by a Navier–Stokes two-phase flow with surface tension, where, as a consequence of the constant mobility assumption, the interface separating the two fluid phases is not merely transported by the fluids but is also subject to mean curvature flow (we again refer to Subsection 1.2 for a mathematical formulation of the corresponding model). In particular, the mass of each individual fluid phase is not preserved and the resulting model may be interpreted as a simplified model for two-phase fluid flow incorporating phase transitions and

surface tension. For global-in-time existence of weak solutions to the phase-field model, we refer to [45] (see also [26] for longtime behavior of solutions).

In this setting, the main result of the present work rigorously justifies the formally derived sharp interface limit model in ambient dimension  $d \in \{2, 3\}$ . Our convergence result holds true on the time horizon of existence of a (sufficiently regular) strong solution of the sharp interface limit. We even derive sharp convergence rates in strong norms under the assumption of correspondingly well-prepared initial data. For precise mathematical statements of these two main results, we refer to Theorem 1 and Corollary 2 below. We mention that a similar convergence result was recently established in [2] (cf. also [37]) by a completely different approach (we provide further comments on the methods later). The results of [2] are, however, restricted to ambient dimension  $d = 2$ , so that our work, to the best of our knowledge, is the first to rigorously justify the above sharp interface limit in all physically relevant dimensions. It has to be said, though, that the authors of [2] are in addition able to treat the regime of different viscosities. Even though we expect this to be manageable also in the framework of our strategy (without restrictions on the ambient dimension), this may very well lead to a doubling of the length of the present paper (cf. the corresponding challenges in [22]). For this reason, we restrict ourselves to the most basic setting. We finally mention that [5] contains a preliminary convergence result preceding ours and the one of [2], where the authors replace the full Navier–Stokes dynamics by the stationary Stokes operator (and again only in ambient dimension  $d = 2$ ).

Without coupling to a fluid mechanical system, the rigorous convergence of phase-field models to sharp interface evolutions is of course an already extensively studied subject in the literature. For (qualitative and/or quantitative) convergence of solutions of the Allen–Cahn equation towards various weak or strong notions of solutions for mean curvature flow, one may consult, e.g., the works [20, 21, 24, 29, 34, 49, 52]. For the inclusion of constant contact angles, corresponding results can be found in [9, 10, 17, 18, 28, 32, 38, 39, 48, 50], whereas (qualitative and/or quantitative) convergence of solutions of the vectorial Allen–Cahn equation towards evolution by multiphase mean curvature flow is the subject of [25, 43]. The results on the connection of the Allen–Cahn equation with mean curvature flow have a fourth-order analogue, namely, the convergence of solutions of the Cahn–Hilliard equation towards evolution by Mullins–Sekerka flow as, e.g., shown in [14, 19, 44] (cf. also the discussion in [52]). For a corresponding result with disparate mobilities, we refer to the recent work [41]. Finally, we mention [42] for a scaling limit result modeling nematic-isotropic phase transitions in the context of Landau–De Gennes theory of liquid crystals.

In order to establish the above convergence results, a variety of techniques and frameworks is used depending on the precise goals (i.e., qualitative long-time convergence towards weak solutions of the sharp interface limit model vs. quantitative convergence towards sufficiently smooth solutions of the sharp interface limit model until the latter run into their first topology change). In fact, most of the above works are either based on notions from geometric measure theory (e.g., varifolds and their first variations), gradient flow techniques in the spirit of [51] (cf. also [52]), or combining rigorous asymptotic expansions with a stability analysis of the linearized operator associated with the phase-field model. Especially in the context of the previously mentioned quantitative convergence results incorporating a fluid

mechanical coupling, the latter strategy seems to be the only one used so far to the best of our knowledge.

However, in the recent inspiring work [24], a new approach for the derivation of convergence rates was introduced and implemented for the simplest setting of the Allen–Cahn equation posed on the full space  $\mathbb{R}^d$ . This strategy is closest to the gradient flow perspective in the sense that it first generates a distance measure for the difference of the phase-field and sharp interface solutions based on the phase-field energy, and then estimates its time evolution by a Gronwall-type argument based, amongst other things, on the dissipation structure of the phase-field model. The method developed in [24] is therefore reminiscent of a well-established technique to establish (weak-strong) uniqueness of solutions in the context of a variety of classical continuum mechanics models (e.g., incompressible and compressible Navier–Stokes flow, or conservation laws): the so-called relative entropy method. In fact, the work [24] draws motivation from recent results extending the relative entropy method (and thus weak-strong uniqueness) to problems incorporating geometric evolution. This was first implemented for binormal curvature flow of curves in  $\mathbb{R}^3$  in [35], or for Navier–Stokes two-phase flow with surface tension in [22]. Subsequent extensions are able to deal with multiphase mean curvature flow [23] (cf. also [30]), contact angle problems [28, 31], or a novel notion of varifold solutions for mean curvature flow in the spirit of ideas from De Giorgi [29]. In view of these developments, it is not surprising that also [24] already led to several follow-up works in the context of scaling limits for phase-field models, see [25, 32, 42, 47].

In the present work, we continue this story and extend the approach from [24] to the case of a Navier–Stokes/Allen–Cahn model with constant mobility. We refer the reader to Section 2 and especially Section 3 for a mathematical account on our strategy.

**1.2. The phase-field model and its sharp interface limit.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be a bounded domain with orientable and  $C^2$ -boundary. We then consider the following most basic Navier–Stokes/Allen–Cahn problem in  $\Omega$

$$\partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon = \Delta v_\varepsilon - \nabla \pi_\varepsilon - \nabla \cdot (\varepsilon \nabla \varphi_\varepsilon \otimes \nabla \varphi_\varepsilon) \quad \text{in } \Omega \times (0, T_0), \quad (1a)$$

$$\nabla \cdot v_\varepsilon = 0 \quad \text{in } \Omega \times (0, T_0), \quad (1b)$$

$$\partial_t \varphi_\varepsilon + (v_\varepsilon \cdot \nabla) \varphi_\varepsilon = \Delta \varphi_\varepsilon - \frac{1}{\varepsilon^2} W'(\varphi_\varepsilon) \quad \text{in } \Omega \times (0, T_0), \quad (1c)$$

$$v_\varepsilon(\cdot, 0) = v_{\varepsilon,0} \quad \text{in } \Omega, \quad (1d)$$

$$\varphi_\varepsilon(\cdot, 0) = \varphi_{\varepsilon,0} \in [-1, 1] \quad \text{in } \Omega, \quad (1e)$$

$$(\varphi_\varepsilon, v_\varepsilon) = (-1, 0) \quad \text{on } \partial\Omega \times (0, T_0). \quad (1f)$$

Here,  $W$  denotes a double-well potential satisfying standard assumptions (see, e.g., [5]). We fix the associated surface tension constant by

$$c_0 := \int_{-1}^1 \sqrt{2W(r)} \, dr.$$

In the sharp interface limit  $\varepsilon \rightarrow 0$  one expects to obtain a simplified model for a two-phase fluid flow with phase transitions, which in our case reads

$$\partial_t v + (v \cdot \nabla) v = \Delta v - \nabla \pi \quad \text{in } (\Omega \times (0, T_*) \setminus \mathcal{I}), \quad (2a)$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega \times (0, T_*), \quad (2b)$$

$$\partial_t \chi + (v \cdot \nabla) \chi = -H_{\mathcal{I}} |\nabla \chi| \quad \text{in } \Omega \times (0, T_*), \quad (2c)$$

$$[[v]] = 0, \quad \text{on } \mathcal{I}, \quad (2d)$$

$$[[2\nabla^{\text{sym}}v - p\text{Id}]] \frac{\nabla\chi}{|\nabla\chi|} = c_0 H_{\mathcal{I}} \frac{\nabla\chi}{|\nabla\chi|} \quad \text{on } \mathcal{I}, \quad (2e)$$

$$v(\cdot, 0) = v_0 \quad \text{in } \Omega, \quad (2f)$$

$$\chi(\cdot, 0) = \chi_0 \quad \text{in } \Omega, \quad (2g)$$

$$(\chi, v) = (0, 0) \quad \text{on } \partial\Omega \times (0, T_*). \quad (2h)$$

In the above system,  $\chi$  denotes a time-dependent characteristic function. From now on, we assume that we are provided with a sufficiently regular solution  $(\chi, v)$  for the sharp interface limit model (for precise assumptions, we refer to Definition 4 below). In particular,  $\text{supp} |\nabla\chi| \cap (\Omega \times [0, T_*])$  shall model a sufficiently regular interface

$$\mathcal{I} = \bigcup_{t \in [0, T_*]} \mathcal{I}(t) \times \{t\} \quad (3)$$

in space-time associated with an open and sufficiently regular set

$$\Omega_+ = \bigcup_{t \in [0, T_*]} \Omega_+(t) \times \{t\} \subset \Omega \times (0, T_*).$$

The map  $\chi$  is assumed to be  $\equiv 1$  in  $\Omega_+$  as well as  $\equiv 0$  in  $(\Omega \times [0, T_*]) \setminus \overline{\Omega_+}$ . Denoting for all  $t \in (0, T_*)$  by  $\mathbf{n}_{\mathcal{I}}(\cdot, t)$  the associated unit normal along  $\mathcal{I}(t)$  pointing inside  $\Omega_+(t)$ , the evolution equation for the interface then translates into (all scalar geometric quantities are oriented with respect to the normal vector field  $\mathbf{n}_{\mathcal{I}}$ )

$$V_{\mathcal{I}} = \mathbf{n}_{\mathcal{I}} \cdot v + H_{\mathcal{I}} \quad \text{on } \mathcal{I}. \quad (4)$$

Here,  $V_{\mathcal{I}}(\cdot, t)$  and  $H_{\mathcal{I}}(\cdot, t)$  denote the normal speed and the scalar mean curvature of the interface  $\mathcal{I}(t)$ ,  $t \in [0, T_*)$ . Finally, in order to avoid issues originating from contact point dynamics, we assume throughout the rest of this work that

$$\overline{\Omega_+(t)} \subset \Omega \quad \text{for all } t \in [0, T_*]. \quad (5)$$

## 2. MAIN RESULT

The main result of the present work is concerned with a rigorous convergence result for solutions of the phase-field model (1a)–(1f) (even quantitatively with sharp convergence rates when starting from well-prepared initial data) towards strong solutions of the sharp interface limit model (2a)–(2h). As already mentioned in the introduction, our approach is not based on the strategy of combining rigorous asymptotic expansions with stability estimates for the underlying linearized operator. In contrast, our results are facilitated by the introduction of two error functionals which aim to encode the difference between a solution of the phase-field model and a solution for the expected sharp interface limit. More precisely, directly inspired by the recent work of Fischer, Laux and Simon [24] on the (quantitative) convergence of solutions of the Allen–Cahn equation to classical solutions of mean curvature flow, we will work on  $[0, T_*)$  with a relative energy

$$\begin{aligned} E[\varphi_\varepsilon, v_\varepsilon | \chi, v] &:= \int_{\Omega} \frac{1}{2} |v_\varepsilon - v|^2 dx \\ &+ \int_{\Omega} \frac{\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon) - \xi \cdot \nabla \left( \int_{-1}^{\varphi_\varepsilon} \sqrt{2W(r)} dr \right) dx \end{aligned} \quad (6)$$

as well as with a suitable measure for the difference in the phase indicators

$$E_{\text{vol}}[\varphi_\varepsilon|\chi] := \int_{\Omega} \left| c_0\chi - \int_{-1}^{\varphi_\varepsilon} \sqrt{2W(r)} dr \right| |\vartheta| dx. \quad (7)$$

In the above definitions, the vector field  $\xi$  will be a suitable extension of the unit normal vector field of the smoothly evolving interface  $\mathcal{I}$  whereas  $\vartheta$  will be a suitable truncation of the associated signed distance function (for precise assumptions on  $\xi$  and  $\vartheta$ , we refer to Subsection 3.1 and Subsection 3.3, respectively). Both qualitative and quantitative convergence then follows from a Gronwall-type stability estimate for the two error functionals (6) and (7). The precise statement reads as follows.

**Theorem 1** (Error estimates for general initial data). *Let  $d \in \{2, 3\}$ , let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with orientable and  $C^2$ -boundary  $\partial\Omega$ , and consider two finite time horizons  $0 < T_* \leq T_0 < \infty$ . Let  $\varepsilon \in (0, 1)$  and let  $(\varphi_\varepsilon, v_\varepsilon)$  be a weak solution of the Navier–Stokes/Allen–Cahn system (1a)–(1f) with time horizon  $T_0$  and initial data  $(\varphi_{\varepsilon,0}, v_{\varepsilon,0})$  in the sense of Definition 3. Finally, let  $(\chi, v)$  be a strong solution for the sharp interface limit model (2a)–(2h) with time horizon  $T_*$  and initial data  $(\chi_0, v_0)$  in the sense of Definition 4.*

*Then, for a.e.  $T \in (0, T_*)$  there exists a constant  $C = C(\chi, v, T) \in (0, \infty)$ , a continuous vector field  $\xi = \xi(\chi, v, T): \Omega \times [0, T] \rightarrow \mathbb{R}^d$ , and a continuous weight  $\vartheta = \vartheta(\chi, v, T): \Omega \times [0, T] \rightarrow \mathbb{R}$  such that the relative energy  $E[\varphi_\varepsilon, v_\varepsilon|\chi, v]$  and the error in the phase indicators  $E_{\text{vol}}[\varphi_\varepsilon|\chi]$  defined by (6) and (7), respectively, satisfy the estimates*

$$E[\varphi_\varepsilon, v_\varepsilon|\chi, v](t) \leq e^{Ct} (E[\varphi_\varepsilon, v_\varepsilon|\chi, v](0) + E_{\text{vol}}[\varphi_\varepsilon|\chi](0)), \quad \text{for a.e. } t \in [0, T], \quad (8)$$

$$E_{\text{vol}}[\varphi_\varepsilon|\chi](t) \leq e^{Ct} (E[\varphi_\varepsilon, v_\varepsilon|\chi, v](0) + E_{\text{vol}}[\varphi_\varepsilon|\chi](0)), \quad \text{for a.e. } t \in [0, T]. \quad (9)$$

Based on the straightforward construction of well-prepared initial data for the phase indicator (see, e.g., [24]), the above theorem leads to a sharp  $L^1$ -convergence rate for the phase-field and to a sharp  $L^2$ -convergence rate for the fluid velocity.

**Corollary 2** (Sharp convergence rates for well-prepared initial data). *Let the assumptions and notation of Theorem 1 be in place. For well-prepared initial data in the sense of*

$$E[\varphi_\varepsilon, v_\varepsilon|\chi, v](0) + E_{\text{vol}}[\varphi_\varepsilon|\chi](0) \leq C(\chi_0)\varepsilon^2, \quad (10)$$

*where  $C(\chi_0) \in (0, \infty)$  denotes a constant depending only on the initial geometry of the strong sharp interface limit solution, it follows that for a.e.  $T \in (0, T_*)$  there exists a constant  $C = C(\chi, v, T) \in (0, \infty)$  such that*

$$\| (v_\varepsilon - v)(\cdot, t) \|_{L^2(\Omega)} + \left\| \left( c_0\chi(\cdot, t) - \int_{-1}^{\varphi_\varepsilon(\cdot, t)} \sqrt{2W(r)} dr \right) \right\|_{L^1(\Omega)} \leq C e^{Ct} \varepsilon \quad (11)$$

*for a.e.  $t \in [0, T]$ .*

The proof of Theorem 1 is given in Section 6 and is based on the following key ingredients. As a preliminary, we provide throughout the remainder of this section precise definitions for the solution concepts associated with the system of PDEs (1a)–(1f) and (2a)–(2h), respectively. Section 3 introduces the definition of the two error functionals for which Theorem 1 claims the stability estimates (8) and (9), respectively. We also collect several important coercivity properties of these error functionals. These in turn are needed to appropriately estimate preliminary

bounds for the time evolution of the two error functionals as derived in Section 4 and Section 5, respectively.

Starting point of our journey is the following precise concept of weak solutions for the phase-field model.

**Definition 3** (Weak solutions of the Navier–Stokes/Allen–Cahn system (1a)–(1f)). Let  $d \in \{2, 3\}$ , let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with orientable and  $C^2$ -boundary, and consider a finite time horizon  $0 < T_0 < \infty$ . Let  $\varepsilon \in (0, 1)$ , and consider an initial velocity field  $v_{\varepsilon,0} \in H^1(\Omega)$  in combination with an initial order parameter  $\varphi_{\varepsilon,0} \in H^1(\Omega)$  such that  $\varphi_{\varepsilon,0} \in [-1, 1]$  holds true  $\mathcal{L}^d$ -a.e. in  $\Omega$ ,  $\nabla \cdot v_{\varepsilon,0} = 0$  in the distributional sense in  $\Omega$ , as well as  $(\varphi_{\varepsilon,0} - (-1), v_{\varepsilon,0}) \in H_0^1(\Omega, \mathbb{R} \times \mathbb{R}^d)$ .

A pair of measurable maps  $(\varphi_\varepsilon, v_\varepsilon): \Omega \times (0, T_0) \rightarrow \mathbb{R} \times \mathbb{R}^d$  is called a *weak solution of the Navier–Stokes/Allen–Cahn system (1a)–(1f) with time horizon  $T_0$  and initial data  $(\varphi_{\varepsilon,0}, v_{\varepsilon,0})$*  if the following conditions are satisfied:

i) In terms of regularity, it has to hold

$$v_\varepsilon \in L^2(0, T_0; H^1(\Omega; \mathbb{R}^d)) \cap L^\infty(0, T_0; L^2(\Omega; \mathbb{R}^d)), \quad (12a)$$

$$\varphi_\varepsilon \in H^1(0, T_0; L^2(\Omega; [-1, 1])) \cap L^2(0, T_0; H^2(\Omega)), \quad (12b)$$

such that in addition (1f) holds in form of

$$(\varphi_\varepsilon(\cdot, t) - (-1), v_\varepsilon(\cdot, t)) \in H_0^1(\Omega, \mathbb{R} \times \mathbb{R}^d) \quad \text{for a.e. } t \in (0, T_0). \quad (12c)$$

ii) The velocity field  $v_\varepsilon$  is a solution of (1a)–(1b) and (1d) in the sense that

$$\begin{aligned} & \int_\Omega v_\varepsilon(\cdot, T) \cdot \eta(\cdot, T) dx - \int_\Omega v_{\varepsilon,0} \cdot \eta(\cdot, 0) dx \\ &= \int_0^T \int_\Omega v_\varepsilon \cdot \partial_t \eta dx dt + \int_0^T \int_\Omega v_\varepsilon \otimes v_\varepsilon : \nabla \eta dx dt \\ & \quad - \int_0^T \int_\Omega \nabla v_\varepsilon : \nabla \eta dx dt + \int_0^T \int_\Omega \varepsilon \nabla \varphi_\varepsilon \otimes \nabla \varphi_\varepsilon : \nabla \eta dx dt \end{aligned} \quad (12d)$$

holds true for a.e.  $T \in (0, T_0)$  and all  $\eta \in C_c^\infty([0, T_0]; C_c^\infty(\Omega; \mathbb{R}^d))$  with  $\nabla \cdot \eta = 0$ , as well as

$$(\nabla \cdot v_\varepsilon)(\cdot, t) = 0 \quad (12e)$$

for a.e.  $t \in (0, T_0)$  in the distributional sense in  $\Omega$ .

iii) The order parameter  $\varphi_\varepsilon$  satisfies (1c) and (1e) in form of

$$\partial_t \varphi_\varepsilon + (v_\varepsilon \cdot \nabla) \varphi_\varepsilon = \Delta \varphi_\varepsilon - \frac{1}{\varepsilon^2} W'(\varphi_\varepsilon) \quad \text{a.e. in } \Omega \times (0, T_0), \quad (12f)$$

$$\varphi_\varepsilon(\cdot, t) \rightarrow \varphi_{\varepsilon,0} \quad \text{strongly in } L^2(\Omega) \text{ as } t \downarrow 0. \quad (12g)$$

iv) Defining for every admissible pair  $(\varphi, v)$  the total energy functional by

$$E[\varphi, v] := \int_\Omega \frac{1}{2} |v|^2 dx + \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} W(\varphi) dx, \quad (12h)$$

sharp energy dissipation is required in form of

$$E[\varphi_\varepsilon(\cdot, T), v_\varepsilon(\cdot, T)] + \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 + \varepsilon |(\partial_t + (v_\varepsilon \cdot \nabla)) \varphi_\varepsilon|^2 dx dt \quad (12i)$$

$$\leq E[\varphi_{\varepsilon,0}, v_{\varepsilon,0}]$$

for a.e.  $T \in (0, T_0)$ .

We conclude this section with a precise concept of strong solutions for the sharp interface limit model.

**Definition 4** (Strong solutions of the sharp interface limit system (2a)–(2h)). Let  $d \in \{2, 3\}$ , let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with orientable and  $C^2$ -boundary, and consider a finite time horizon  $0 < T_* < \infty$ . Consider an initial velocity field  $v_0 \in (H_0^1 \cap C^1)(\Omega; \mathbb{R}^d)$  in combination with an initial phase indicator  $\chi_0 \in BV(\Omega; \{0, 1\})$  such that  $\nabla \cdot v_0 = 0$  pointwise in  $\Omega$ , the set  $\Omega_+(0) := \{\chi_0=1\}$  yields an open subset of  $\Omega$  consisting of finitely many connected components with  $\overline{\Omega_+(0)} \subset \Omega$ , the associated boundary  $\partial\Omega_+(0)$  is orientable and uniformly of class  $C^3$ , and finally the following compatibility conditions hold true:

$$\begin{aligned} \llbracket v_0 \rrbracket &= 0 \text{ on } \mathcal{I}(0), \\ (\text{Id} - \mathbf{n}_{\mathcal{I}(0)} \otimes \mathbf{n}_{\mathcal{I}(0)}) \llbracket \nabla^{\text{sym}} v_0 \cdot \mathbf{n}_{\mathcal{I}(0)} \rrbracket &= 0 \text{ on } \mathcal{I}(0). \end{aligned}$$

A pair of measurable maps  $(\chi, v) : \Omega \times (0, T_*) \rightarrow \{0, 1\} \times \mathbb{R}^d$  is called a *strong solution of the sharp interface limit system (2a)–(2h) with time horizon  $T_*$  and initial data  $(\chi_0, v_0)$*  if the following requirements hold true:

- i) The fluid velocity  $v$  is subject to the regularity

$$v \in H^1(0, T_*; L^2(\Omega; \mathbb{R}^d)) \cap L^2(0, T_*; H^1(\Omega; \mathbb{R}^d)) \quad (13a)$$

such that the corresponding boundary condition (2h) as well as the corresponding initial condition (2f) are satisfied in form of

$$v(\cdot, t) \in H_0^1(\Omega; \mathbb{R}^d) \quad \text{for a.e. } t \in (0, T_*), \quad (13b)$$

$$v(\cdot, t) \rightarrow v_0 \quad \text{strongly in } L^2(\Omega) \text{ as } t \downarrow 0. \quad (13c)$$

Furthermore, for each  $T \in [0, T_*)$  there exists  $C = C(T) \in (0, \infty)$  such that

$$v \in L^\infty(0, T; C^1(\Omega; \mathbb{R}^d)), \quad (13d)$$

with a corresponding bound on the highest-order derivative of the form

$$\|\nabla v\|_{L^\infty(\Omega \times (0, T))} \leq C. \quad (13e)$$

- ii) The velocity field  $v$  solves (2a)–(2b) and (2d)–(2e) in the sense that

$$\begin{aligned} & \int_{\Omega} v(\cdot, T) \cdot \eta(\cdot, T) dx - \int_{\Omega} v_0 \cdot \eta(\cdot, 0) dx \\ &= \int_0^T \int_{\Omega} v \cdot \partial_t \eta dx dt - \int_0^T \int_{\Omega} \eta \cdot (v \cdot \nabla) v dx dt \\ & \quad - \int_0^T \int_{\Omega} \nabla v : \nabla \eta dx dt + c_0 \int_0^T \int_{\Omega} \frac{\nabla \chi}{|\nabla \chi|} \otimes \frac{\nabla \chi}{|\nabla \chi|} : \nabla \eta d|\nabla \chi| dt \end{aligned} \quad (13f)$$

holds true for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^\infty([0, T_*]; C_c^\infty(\Omega; \mathbb{R}^d))$  with  $\nabla \cdot \eta = 0$ , as well as

$$(\nabla \cdot v)(\cdot, t) = 0 \quad (13g)$$

for a.e.  $t \in (0, T_*)$  in the distributional sense in  $\Omega$ .

- iii) Define for every  $t \in (0, T_*)$  the set  $\Omega_+(t) := \{\chi(\cdot, t)=1\}$ . There then exists a map  $\Psi : \overline{\Omega} \times [0, T_*) \rightarrow \overline{\Omega}$  such that  $\Psi(\cdot, 0) = \text{Id}$ ,  $\Psi(\cdot, t) \in C^3\text{-Diffeo}(\overline{\Omega}, \overline{\Omega})$  for



all  $t \in (0, T_*)$ , and  $\Omega_+(t) = \Psi(\Omega_+(0), t)$  for all  $t \in (0, T_*)$ . With respect to regularity in time, we require that for all  $T \in (0, T_*)$  it holds

$$\Psi \in C^1([0, T]; C^1(\overline{\Omega}; \mathbb{R}^d)) \cap C([0, T]; C^3(\overline{\Omega}; \mathbb{R}^d)). \quad (13h)$$

Note that the corresponding initial condition (2g) is thus a consequence of (13h). Finally, the corresponding boundary condition (2h) is satisfied in form of

$$\overline{\Omega_+(t)} \subset \Omega \quad \text{for all } t \in (0, T_*). \quad (13i)$$

- iv) The geometric evolution equation (2c) is required to be satisfied in its strong form given by (4), where the space-time interface  $\mathcal{I} = \bigcup_{t \in [0, T_*]} \mathcal{I}(t) \times \{t\}$  is defined by  $\mathcal{I}(t) := \partial\Omega_+(t)$ ,  $t \in [0, T_*]$ .

In the appendix, we provide some detailed comments on how to obtain such strong solutions for the sharp interface limit model starting from the work of Abels and Moser [8].

### 3. THE RELATIVE ENERGY FUNCTIONAL

**3.1. Definition of the relative energy functional.** In this subsection, we follow ideas first developed in [24]. We start by defining the appropriate *BV*-pendant of  $c_0\chi$  at the level of the phase-field model via

$$\psi_\varepsilon(x, t) := \int_{-1}^{\varphi_\varepsilon(x, t)} \sqrt{2W}(r) dr, \quad (x, t) \in \Omega \times (0, T_0). \quad (14)$$

We then introduce a “unit-normal vector field”  $\mathbf{n}_\varepsilon \in \mathbb{S}^{d-1}$  by means of

$$\mathbf{n}_\varepsilon(\cdot, t) := \begin{cases} \frac{\nabla\varphi_\varepsilon(\cdot, t)}{|\nabla\varphi_\varepsilon(\cdot, t)|} & \text{if } \nabla\varphi_\varepsilon(\cdot, t) \neq 0, \\ s & \text{else,} \end{cases} \quad t \in (0, T_0), \quad (15)$$

with  $s \in \mathbb{S}^{d-1}$  being a fixed but otherwise arbitrary unit vector. Note that because of the previous two definitions we always have the relations

$$\mathbf{n}_\varepsilon |\nabla\varphi_\varepsilon| = \nabla\varphi_\varepsilon \quad \text{and} \quad \mathbf{n}_\varepsilon |\nabla\psi_\varepsilon| = \nabla\psi_\varepsilon \quad \text{throughout } \Omega \times (0, T_0). \quad (16)$$

We now define a measure for the difference between the phase-field approximation  $(\varphi_\varepsilon, v_\varepsilon)$  and the classical solution  $(\chi, v)$  for the sharp interface limit model as follows: for  $t \in (0, T_*) \subset (0, T_0)$  let

$$\begin{aligned} E[\varphi_\varepsilon, v_\varepsilon | \chi, v](t) & \\ := \int_\Omega \frac{1}{2} |(v_\varepsilon - v)(\cdot, t)|^2 dx &+ \int_\Omega \frac{\varepsilon}{2} |\nabla\varphi_\varepsilon(\cdot, t)|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon(\cdot, t)) - (\xi \cdot \nabla\psi_\varepsilon)(\cdot, t) dx. \end{aligned} \quad (17)$$

The vector field  $\xi$  associated with the strong sharp interface limit solution  $(\chi, v)$  is yet to be determined. Note already that the above functional has a form reminiscent of that of a relative entropy. Indeed,

$$E[\varphi_\varepsilon, v_\varepsilon | \chi, v] = E[\varphi_\varepsilon, v_\varepsilon] - \int_\Omega v_\varepsilon \cdot v + \int_\Omega \frac{1}{2} |v|^2 dx - \int_\Omega \xi \cdot \nabla\psi_\varepsilon dx, \quad (18)$$

where we dropped for notational convenience the dependence on the time variable. Continuing in this fashion, a straightforward computation moreover shows

$$\begin{aligned} E[\varphi_\varepsilon, v_\varepsilon | \chi, v] & \\ = \int_\Omega \frac{1}{2} |v_\varepsilon - v|^2 + \int_\Omega \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla\varphi_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_\varepsilon)} \right)^2 &+ \int_\Omega (1 - \xi \cdot \mathbf{n}_\varepsilon) |\nabla\psi_\varepsilon|. \end{aligned} \quad (19)$$

Ensuring non-negativity of  $E[\varphi_\varepsilon, v_\varepsilon | \chi, v]$  thus motivates  $|\xi| \leq 1$  as an absolute minimal assumption on the vector field  $\xi$ .

The main goal of our approach is to derive a stability estimate on  $E[\varphi_\varepsilon, v_\varepsilon | \chi, v]$  in terms of the initial distance  $E_0 := E[\varphi_{\varepsilon,0}, v_{\varepsilon,0} | \chi_0, v_0]$  by a Gronwall argument, cf. the stability estimate (8) in Theorem 1. This in turn necessitates control on the time evolution of the relative energy. In particular, we will need an appropriate control on the time evolution of the vector field  $\xi$ . To this end, it turns out to be beneficial (mostly for clarity of exposition and efficient organization of terms in the relative energy inequality) to introduce a second vector field  $B$ . One should keep in mind the following interpretation of the pair  $(\xi, B)$ : the vector field  $\xi$  will represent a suitable extension of the unit normal vector field  $\mathbf{n}_{\mathcal{I}}$  for the smoothly evolving sharp interface  $\mathcal{I}$ , whereas  $B$  shall denote a suitable extension of the associated normal velocity vector of  $\mathcal{I}$ . More precisely, we impose the following conditions on the pair  $(\xi, B)$  which will turn out to be sufficient.

For every  $T \in [0, T_*)$  there are  $c = c(\chi, T) \in (0, 1)$  and  $C = C(\chi, v, T) \in (0, \infty)$  such that it holds (employing the convenient notation  $\text{dist}(\cdot, \mathcal{I})$  representing the space-time map  $\overline{\Omega} \times [0, T_*) \ni (x, t) \mapsto \text{dist}(x, \mathcal{I}(t))$ ):

- (Regularity estimates) In terms of qualitative regularity, it has to hold

$$\xi \in C^{0,1}([0, T]; L^\infty(\Omega; \mathbb{R}^d)) \cap L^\infty([0, T]; C_c^{1,1}(\Omega; \mathbb{R}^d)), \quad (20a)$$

$$B \in L^\infty([0, T]; C_c^{0,1}(\overline{\Omega}; \mathbb{R}^d)), \quad (20b)$$

with a bound on the corresponding highest-order derivatives in form of

$$\|(\partial_t \xi, \nabla^2 \xi, \nabla B)\|_{L^\infty(\Omega \times (0, T))} \leq C. \quad (20c)$$

- (Coercivity and consistency) It holds

$$|\xi| \leq 1 - c \min\{\text{dist}^2(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T] \quad (20d)$$

$$\xi = \mathbf{n}_{\mathcal{I}} \text{ and } \nabla \cdot \xi = -H_{\mathcal{I}} \quad \text{on } \mathcal{I}. \quad (20e)$$

- (Approximate transport of  $\xi$  by  $B$ ) We have

$$|\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi| \leq C \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T], \quad (20f)$$

$$|\xi \cdot (\partial_t + B \cdot \nabla) \xi| \leq C \min\{\text{dist}^2(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T]. \quad (20g)$$

- (Interpretation of  $B$  as a normal velocity) It holds

$$|(B - v) \cdot \xi + \nabla \cdot \xi| \leq C \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T], \quad (20h)$$

$$|\xi \cdot (\xi \cdot \nabla) B| \leq C \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T]. \quad (20i)$$

**3.2. Coercivity properties of the relative energy functional.** We collect several useful coercivity estimates for  $E[\varphi_\varepsilon, v_\varepsilon | \chi, v]$ , dropping in the process again for notational convenience the dependence on the time variable. The proof of these estimates is provided afterwards.

**Lemma 5.** *Let the assumptions and notation of Subsection 3.1 be in place. First, the relative energy  $E[\varphi_\varepsilon, v_\varepsilon | \chi, v]$  provides control on the error in the fluid velocity by means of*

$$\int_{\Omega} \frac{1}{2} |v_\varepsilon - v|^2 dx \leq E[\varphi_\varepsilon, v_\varepsilon | \chi, v]. \quad (21)$$

Second, it entails a “tilt-excess” type control on the geometry in the form of

$$\int_{\Omega} (1 - \mathbf{n}_{\varepsilon} \cdot \xi) |\nabla \psi_{\varepsilon}| dx \leq E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v]. \quad (22)$$

Third, one obtains control on the lack of equipartition of energy through

$$\int_{\Omega} \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right)^2 dx \leq E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v]. \quad (23)$$

Fourth, for every  $T \in [0, T_*)$  there exists a constant  $C = C(\chi, v, T) \in (0, \infty)$  such that for all times in  $[0, T]$  it holds

$$\int_{\Omega} |\mathbf{n}_{\varepsilon} - \xi|^2 |\nabla \psi_{\varepsilon}| dx + \int_{\Omega} \min\{\text{dist}^2(\cdot, \mathcal{I}), 1\} |\nabla \psi_{\varepsilon}| dx \leq CE[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v], \quad (24)$$

$$\int_{\Omega} |\mathbf{n}_{\varepsilon} - \xi|^2 \varepsilon |\nabla \varphi_{\varepsilon}|^2 dx + \int_{\Omega} \min\{\text{dist}^2(\cdot, \mathcal{I}), 1\} \varepsilon |\nabla \varphi_{\varepsilon}|^2 dx \leq CE[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v], \quad (25)$$

as well as finally

$$\int_{\Omega} (\min\{\text{dist}(\cdot, \mathcal{I}), 1\} + \sqrt{1 - \mathbf{n}_{\varepsilon} \cdot \xi}) |\varepsilon |\nabla \varphi_{\varepsilon}|^2 - |\nabla \psi_{\varepsilon}| dx \leq CE[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v]. \quad (26)$$

*Proof.* The assertions (22)–(26) are essentially already contained and proved in [24, Subsection 2.3]. For the sake of completeness, we re-produce the straightforward argument here.

By (14) and (15), we can write (17) as the sum of three integrals with each featuring a non-negative integrand:

$$\begin{aligned} E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](t) &= \int_{\Omega} \frac{1}{2} |(v_{\varepsilon} - v)(\cdot, t)|^2 dx \\ &\quad + \int_{\Omega} |\nabla \psi_{\varepsilon}(\cdot, t)| - (\xi \cdot \nabla \psi_{\varepsilon})(\cdot, t) dx \\ &\quad + \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon}(\cdot, t)|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(\cdot, t)) - |\nabla \psi_{\varepsilon}(\cdot, t)| dx. \end{aligned}$$

Using (14) and the chain rule, we can complete the square in the last integral above, and this proves (21), (22) and (23). The estimate (24) follows from (20d) and (22).

Next, adding zero and employing Young’s inequality in the form of

$$\begin{aligned} \varepsilon |\nabla \varphi_{\varepsilon}|^2 &= |\nabla \psi_{\varepsilon}| + \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right) \\ &\leq |\nabla \psi_{\varepsilon}| + \frac{1}{2} \varepsilon |\nabla \varphi_{\varepsilon}|^2 + \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right)^2 \end{aligned}$$

we infer

$$\begin{aligned} &\int_{\Omega} (1 - \mathbf{n}_{\varepsilon} \cdot \xi) \varepsilon |\nabla \varphi_{\varepsilon}|^2 dx \\ &\leq 2 \int_{\Omega} (1 - \mathbf{n}_{\varepsilon} \cdot \xi) |\nabla \psi_{\varepsilon}| dx + 4 \int_{\Omega} \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right)^2 dx \\ &\stackrel{(22), (23)}{\leq} CE[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v], \end{aligned} \quad (27)$$

which then together with  $2(1 - \mathbf{n}_{\varepsilon} \cdot \xi) \geq |\mathbf{n}_{\varepsilon} - \xi|^2$  and (20d) leads to (25). Finally

$$\int_{\Omega} \min\{\text{dist}(\cdot, \mathcal{I}), 1\} |\varepsilon |\nabla \varphi_{\varepsilon}|^2 - |\nabla \psi_{\varepsilon}| dx$$

$$\stackrel{(20d)}{\leq} C \int_{\Omega} \sqrt{1 - \mathbf{n}_{\varepsilon} \cdot \xi} \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \left| \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right| dx.$$

So (26) is a consequence of (27) and the Cauchy–Schwarz inequality.  $\square$

**3.3. Error in the phase indicators.** The second step in the proof consists of deriving an estimate for the error in the phase indicators  $\varphi_{\varepsilon}$  and  $\chi$  based on the stability estimate for the relative energy  $E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v]$ . This in turn is done by means of a stability estimate à la Gronwall (cf. the estimate (9) in Theorem 1) for an appropriate error functional defined for all  $t \in [0, T_*)$  as follows:

$$E_{\text{vol}}[\varphi_{\varepsilon} | \chi](t) := \int_{\Omega} |\psi_{\varepsilon}(\cdot, t) - c_0 \chi(\cdot, t)| |\vartheta(\cdot, t)| dx. \quad (28)$$

Here,  $\vartheta: \bar{\Omega} \times [0, T_*) \rightarrow [-1, 1]$  denotes a (by the velocity field  $B$  transported) time-dependent weight function satisfying the following conditions, which in turn will be sufficient to derive a stability estimate for  $E_{\text{vol}}[\varphi_{\varepsilon} | \chi]$ .

For every  $T \in [0, T_*)$  there are  $c = c(\chi, T) \in (0, 1)$  and  $C = C(\chi, v, T) \in (1, \infty)$  such that it holds (employing again the convenient notation  $\text{dist}(\cdot, \mathcal{I})$  representing the space-time map  $\bar{\Omega} \times [0, T_*) \ni (x, t) \mapsto \text{dist}(x, \mathcal{I}(t))$ ):

- (Regularity estimates) We have

$$\vartheta \in C^{0,1}([0, T]; L^{\infty}(\bar{\Omega})) \cap L^{\infty}([0, T]; C^{0,1}(\bar{\Omega})) \quad (29a)$$

with a corresponding estimate for the highest-order derivatives in form of

$$\|(\partial_t \vartheta, \nabla \vartheta)\|_{L^{\infty}(\Omega \times (0, T))} \leq C. \quad (29b)$$

- (Coercivity and consistency) It holds

$$c \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \leq |\vartheta| \leq C \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T], \quad (29c)$$

as well as for all  $t \in [0, T_*)$

$$\vartheta(\cdot, t) < 0 \quad \text{a.e. in the interior of } \{\chi(\cdot, t) = 1\} \cap \Omega, \quad (29d)$$

$$\vartheta(\cdot, t) > 0 \quad \text{a.e. in the interior of } \{\chi(\cdot, t) = 0\} \cap \Omega. \quad (29e)$$

- (Transport equation) Finally, it is required that

$$|\partial_t \vartheta + (B \cdot \nabla) \vartheta| \leq C \min\{\text{dist}(\cdot, \mathcal{I}), 1\} \quad \text{a.e. on } \Omega \times [0, T]. \quad (29f)$$

As in the case of the relative energy functional, we conclude the discussion of this subsection by recording useful coercivity properties of the error functional  $E_{\text{vol}}[\varphi_{\varepsilon} | \chi]$  (dropping yet again the dependence on the time variable). More precisely, for every  $T \in [0, T_*)$  there exists a constant  $C = C(\chi, v, T) \in (0, \infty)$  such that for all times in  $[0, T]$  it obviously holds that

$$\int_{\Omega} \min\{\text{dist}(\cdot, \mathcal{I}), 1\} |\psi_{\varepsilon} - c_0 \chi| dx \leq C E_{\text{vol}}[\varphi_{\varepsilon} | \chi]. \quad (30)$$

More importantly, it holds for all  $\lambda \in (0, 1)$

$$\int_{\Omega} |c_0 \chi - \psi_{\varepsilon}| |v_{\varepsilon} - v| dx \leq \frac{C}{\lambda} (E_{\text{vol}}[\varphi_{\varepsilon} | \chi] + E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v]) + \lambda \int_{\Omega} |\nabla v_{\varepsilon} - \nabla v|^2 dx. \quad (31)$$

*Proof of the coercivity estimate (31).* Let  $\delta = \delta(\chi, T) \in (0, \frac{1}{2}]$  be the thickness of the tubular neighborhood  $B_\delta(\mathcal{I}(t)) := \{x \in \mathbb{R}^d : \text{dist}(x, \mathcal{I}(t)) < \delta\}$  of  $\mathcal{I}(t)$ . For fixed  $t \in [0, T]$ , we have

$$\begin{aligned} & \int_{\Omega \setminus B_\delta(\mathcal{I}(t))} |c_0 \chi - \psi_\varepsilon| |v_\varepsilon - v| \, dx \\ & \stackrel{(30)}{\leq} \frac{1}{\delta} \int_{\Omega} \min\{\text{dist}(\cdot, \mathcal{I}), 1\} |c_0 \chi - \psi_\varepsilon| |v_\varepsilon - v| \, dx \\ & \stackrel{(12b)}{\leq} \frac{C}{\delta} \int_{\Omega} \sqrt{\min\{\text{dist}(\cdot, \mathcal{I}), 1\}} |c_0 \chi - \psi_\varepsilon|^{1/2} |v_\varepsilon - v| \, dx \\ & \stackrel{(30), (21)}{\leq} \frac{C}{\delta} (E_{\text{vol}}[\varphi_\varepsilon | \chi] + E[\varphi_\varepsilon, v_\varepsilon | \chi, v]). \end{aligned}$$

It remains to estimate the integral over  $B_\delta(\mathcal{I}(t))$ , and we shall use the following elementary inequality

$$\left( \int_0^\tau g(r) \, dr \right)^2 \leq 2 \|g\|_{L^\infty(0, \tau)} \int_0^\tau r g(r) \, dr, \quad \forall g(r) \geq 0, \tau \geq 0. \quad (32)$$

Recall that  $\mathbf{n}_\mathcal{I}(y, t)$  is the inward normal vector at  $y \in \mathcal{I}(t)$ . By the area formula and the one-dimensional Gagliardo–Nirenberg–Sobolev inequality,

$$\begin{aligned} & \int_{B_\delta(\mathcal{I}(t))} |c_0 \chi - \psi_\varepsilon| |v_\varepsilon - v| \, dx \\ & \leq C \int_{\mathcal{I}(t)} \int_{-\delta}^\delta |c_0 \chi - \psi_\varepsilon| |v_\varepsilon - v| (y + \mathbf{n}_\mathcal{I}(y, t)r) \, dr d\mathcal{H}^{d-1}(y) \\ & \leq C \int_{\mathcal{I}(t)} \sup_{|r| \leq \delta} |v_\varepsilon - v| (y + \mathbf{n}_\mathcal{I}(y, t)r) \left( \int_{-\delta}^\delta |c_0 \chi - \psi_\varepsilon| (y + \mathbf{n}_\mathcal{I}(y, t)r) \, dr \right) d\mathcal{H}^{d-1}(y) \\ & \stackrel{(32)}{\leq} C \int_{\mathcal{I}(t)} \|(v_\varepsilon - v)(y + \mathbf{n}_\mathcal{I}r)\|_{L^2((-\delta, \delta); dr)}^{\frac{1}{2}} \|(v_\varepsilon - v)(y + \mathbf{n}_\mathcal{I}r)\|_{H^1((-\delta, \delta); dr)}^{\frac{1}{2}} \\ & \quad \cdot \sqrt{\int_{-\delta}^\delta |r| |c_0 \chi - \psi_\varepsilon| (y + \mathbf{n}_\mathcal{I}r) \, dr} \, d\mathcal{H}^{d-1}(y). \end{aligned}$$

By Hölder's and Young's inequalities, the terms in the last step can be controlled by the right hand side of (31).  $\square$

#### 4. ESTIMATE ON THE TIME EVOLUTION OF THE RELATIVE ENERGY

The aim of this section is to derive an estimate representing the key ingredient for the derivation of the asserted stability estimate (8) for the relative energy.

**Proposition 6.** *Let the assumptions and notation of Theorem 1 be in place, let  $(\xi, B)$  be a pair of vector fields subject to (20a)–(20c) as well as (20e), and let the relative energy  $E[\varphi_\varepsilon, v_\varepsilon | \chi, v]$  be given by (17). Define the quantity*

$$H_\varepsilon := -\varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} W'(\varphi_\varepsilon) \quad \text{in } \Omega \times (0, T_0). \quad (33)$$

(This notation serves as a reminder that  $H_\varepsilon$  plays the role of a scalar curvature.) Then, the following relative energy inequality holds true

$$E[\varphi_\varepsilon, v_\varepsilon | \chi, v](T)$$

$$\begin{aligned}
&\leq E[\varphi_\varepsilon, v_\varepsilon | \chi, v](0) - \int_0^T \int_\Omega |\nabla v_\varepsilon - \nabla v|^2 dxdt \\
&\quad - \int_0^T \int_\Omega \frac{1}{2\varepsilon} \left| H_\varepsilon + \sqrt{2W(\varphi_\varepsilon)}(\nabla \cdot \xi) \right|^2 dxdt \\
&\quad - \int_0^T \int_\Omega \frac{1}{2\varepsilon} \left| H_\varepsilon - ((B-v) \cdot \xi) \varepsilon |\nabla \varphi_\varepsilon| \right|^2 dxdt \\
&\quad - \int_0^T \int_\Omega (v_\varepsilon - v) \cdot ((v_\varepsilon - v) \cdot \nabla) v dxdt \\
&\quad - \int_0^T \int_\Omega (c_0 \chi - \psi_\varepsilon) ((v_\varepsilon - v) \cdot \nabla) (\nabla \cdot \xi) dxdt \\
&\quad + \int_0^T \int_\Omega |(B-v) \cdot \xi + \nabla \cdot \xi|^2 \varepsilon |\nabla \varphi_\varepsilon|^2 dxdt \\
&\quad + \int_0^T \int_\Omega |\nabla \cdot \xi|^2 \left( \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| \right)^2 dxdt \\
&\quad - \int_0^T \int_\Omega \frac{1}{\sqrt{\varepsilon}} (H_\varepsilon + \sqrt{2W(\varphi_\varepsilon)}(\nabla \cdot \xi)) ((v-B) \cdot (\mathbf{n}_\varepsilon - \xi)) \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| dxdt \\
&\quad - \int_0^T \int_\Omega (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot (\mathbf{n}_\varepsilon - \xi) |\nabla \psi_\varepsilon| dxdt \\
&\quad - \int_0^T \int_\Omega \xi \cdot (\partial_t \xi + (B \cdot \nabla) \xi) |\nabla \psi_\varepsilon| dxdt \\
&\quad - \int_0^T \int_\Omega \nabla B : (\xi - \mathbf{n}_\varepsilon) \otimes (\xi - \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| dxdt \\
&\quad + \int_0^T \int_\Omega (\nabla \cdot B) (1 - \xi \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| dxdt \\
&\quad + \int_0^T \int_\Omega (\nabla \cdot B) \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_\varepsilon)} \right)^2 dxdt \\
&\quad - \int_0^T \int_\Omega (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \xi \otimes \xi) : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) dxdt \\
&\quad - \int_0^T \int_\Omega \xi \otimes \xi : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) dxdt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$ .

*Proof.* We split the proof into six steps.

*Step 1: Time evolution of kinetic energy terms.* We claim that

$$\begin{aligned}
&\int_\Omega \frac{1}{2} |(v_\varepsilon - v)(\cdot, T)|^2 dx - \int_\Omega \frac{1}{2} |v_{\varepsilon,0} - v_0|^2 dx \\
&= \int_\Omega \frac{1}{2} |v_\varepsilon(\cdot, T)|^2 dx - \int_\Omega \frac{1}{2} |v_{\varepsilon,0}|^2 dx + \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 dxdt \\
&\quad - \int_0^T \int_\Omega |\nabla(v_\varepsilon - v)|^2 dxdt - \int_0^T \int_\Omega (v_\varepsilon - v) \cdot ((v_\varepsilon - v) \cdot \nabla) v dxdt
\end{aligned} \tag{34}$$

$$- \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) v \, dx dt - \int_0^T \int_{\Omega} c_0 \chi((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) \, dx dt$$

for a.e.  $T \in (0, T_*)$ . For a proof of (34), we first observe that by (13f) and the regularity of the sharp interface limit solution it holds

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} \eta \cdot (\partial_t + (v \cdot \nabla)) v \, dx dt + \int_0^T \int_{\Omega} \nabla v : \nabla \eta \, dx dt \\ &\quad - c_0 \int_0^T \int_{\mathcal{I}} H_{\mathcal{I}} \mathbf{n}_{\mathcal{I}} \cdot \eta \, d\mathcal{H}^{d-1} dt \end{aligned} \quad (35)$$

for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^{\infty}(\Omega \times [0, T]; \mathbb{R}^d)$  with  $\nabla \cdot \eta = 0$ . Instead of directly exploiting the information provided by (35), it is convenient to post-process the curvature term before. More precisely, for any admissible solenoidal test vector field in (35), it follows from plugging in the second identity of (20e) as well as integrating by parts that

$$\begin{aligned} -c_0 \int_0^T \int_{\mathcal{I}} H_{\mathcal{I}} \mathbf{n}_{\mathcal{I}} \cdot \eta \, d\mathcal{H}^{d-1} dt &= c_0 \int_0^T \int_{\mathcal{I}} (\nabla \cdot \xi) \mathbf{n}_{\mathcal{I}} \cdot \eta \, d\mathcal{H}^{d-1} dt \\ &= - \int_0^T \int_{\Omega} c_0 \chi(\eta \cdot \nabla) (\nabla \cdot \xi) \, dx dt. \end{aligned}$$

Hence, (35) updates to

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} \eta \cdot (\partial_t + (v \cdot \nabla)) v \, dx dt + \int_0^T \int_{\Omega} \nabla v : \nabla \eta \, dx dt \\ &\quad - \int_0^T \int_{\Omega} c_0 \chi(\eta \cdot \nabla) (\nabla \cdot \xi) \, dx dt \end{aligned} \quad (36)$$

for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^{\infty}(\Omega \times [0, T]; \mathbb{R}^d)$  with  $\nabla \cdot \eta = 0$ . By means of standard mollification arguments, one may plug in the choice  $\eta = v_{\varepsilon} - v$  into (36) resulting in the identity

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} (v_{\varepsilon} - v) \cdot (\partial_t + (v \cdot \nabla)) v \, dx dt + \int_0^T \int_{\Omega} \nabla v : \nabla (v_{\varepsilon} - v) \, dx dt \\ &\quad - \int_0^T \int_{\Omega} c_0 \chi((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) \, dx dt \end{aligned} \quad (37)$$

for a.e.  $T \in (0, T_*)$ . Next to the previous display, we also trivially have by the regularity of the sharp interface limit solution as well as the solenoidality of the velocity fields  $v_{\varepsilon}$  and  $v$ , respectively, that

$$\begin{aligned} &\int_{\Omega} \frac{1}{2} |v(\cdot, T)|^2 \, dx - \int_{\Omega} \frac{1}{2} |v_0|^2 \, dx \\ &= \int_0^T \int_{\Omega} v \cdot \partial_t v \, dx dt = \int_0^T \int_{\Omega} v \cdot \partial_t v \, dx dt + \int_0^T \int_{\Omega} v \cdot (v_{\varepsilon} \cdot \nabla) v \, dx dt \end{aligned} \quad (38)$$

for a.e.  $T \in (0, T_*)$ . Finally, up to a standard mollification argument one may insert  $\eta = v$  as a test vector field in (12d) entailing

$$\begin{aligned} &\int_{\Omega} v_{\varepsilon}(\cdot, T) \cdot v(\cdot, T) \, dx - \int_{\Omega} v_{\varepsilon,0} \cdot v_0 \, dx \\ &= \int_0^T \int_{\Omega} v_{\varepsilon} \cdot \partial_t v \, dx dt + \int_0^T \int_{\Omega} v_{\varepsilon} \cdot (v_{\varepsilon} \cdot \nabla) v \, dx dt \end{aligned} \quad (39)$$

$$- \int_0^T \int_{\Omega} \nabla v_{\varepsilon} : \nabla v \, dxdt + \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \otimes \nabla \varphi_{\varepsilon} : \nabla v \, dxdt$$

for a.e.  $T \in (0, T_*)$ . Hence, adding first (37) to (38) and then subtracting (39) from the resulting identity produces the claim (34).

*Step 2: Time evolution of interfacial energy terms.* We next claim that

$$\begin{aligned} & \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon}(\cdot, T)|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(\cdot, T)) - (\xi \cdot \nabla \psi_{\varepsilon})(\cdot, T) \, dx \\ & - \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon,0}|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon,0}) - (\xi(\cdot, 0) \cdot \nabla \psi_{\varepsilon,0}) \, dx \\ & = \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon}(\cdot, T)|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon}(\cdot, T)) \, dx - \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi_{\varepsilon,0}|^2 + \frac{1}{\varepsilon} W(\varphi_{\varepsilon,0}) \, dx \quad (40) \\ & + \int_0^T \int_{\Omega} \frac{1}{\varepsilon} |H_{\varepsilon}|^2 \, dxdt \\ & - \int_0^T \int_{\Omega} \frac{1}{\varepsilon} |H_{\varepsilon}|^2 \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \, dxdt \\ & - \int_0^T \int_{\Omega} (\nabla \cdot \xi)(v_{\varepsilon} \cdot \nabla) \psi_{\varepsilon} \, dxdt - \int_0^T \int_{\Omega} \partial_t \xi \cdot \nabla \psi_{\varepsilon} \, dxdt \end{aligned}$$

for a.e.  $T \in (0, T_*)$ . For a proof of (40), we start with an integration by parts which combined with the fundamental theorem of calculus, the regularity of the phase-field  $\varphi_{\varepsilon}$ , a standard mollifier approximation (w.r.t. the spatial variable)  $\xi_k := \theta_k * \xi$  for the vector field  $\xi$  (recall for this the regularity requirements (20a) and (20c)), the condition (12g), and finally another integration by parts ensures

$$\begin{aligned} & \int_{\Omega} (\xi \cdot \nabla \psi_{\varepsilon})(\cdot, T) \, dx - \int_{\Omega} \xi(\cdot, 0) \cdot \nabla \psi_{\varepsilon,0} \, dx \\ & = - \left( \int_{\Omega} (\nabla \cdot \xi)(\cdot, T) \psi_{\varepsilon}(\cdot, T) \, dx - \int_{\Omega} (\nabla \cdot \xi)(\cdot, 0) \psi_{\varepsilon,0} \, dx \right) \quad (41) \\ & = - \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} (\nabla \cdot \partial_t \xi_k) \psi_{\varepsilon} \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) \partial_t \psi_{\varepsilon} \, dxdt \\ & = \int_0^T \int_{\Omega} \partial_t \xi \cdot \nabla \psi_{\varepsilon} \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) \partial_t \psi_{\varepsilon} \, dxdt. \end{aligned}$$

By the chain rule, the condition (12f), and the definition (33), it also follows

$$\partial_t \psi_{\varepsilon} = - \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} - (v_{\varepsilon} \cdot \nabla) \psi_{\varepsilon}. \quad (42)$$

Hence, one obtains the claim (40) from plugging in (42) into (41) and adding (in a self-evident way) several zeros.

*Step 3: Exploiting sharp energy dissipation.* The combination of the two identities (34) and (40) from the previous two steps together with the sharp energy dissipation estimate (12i), the condition (12f), and finally the definition (33) entail

$$\begin{aligned} & E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](T) \\ & \leq E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](0) - \int_0^T \int_{\Omega} |\nabla v_{\varepsilon} - \nabla v|^2 \, dxdt \end{aligned}$$



$$\begin{aligned}
& - \int_0^T \int_{\Omega} (v_{\varepsilon} - v) \cdot ((v_{\varepsilon} - v) \cdot \nabla) v \, dxdt \\
& - \int_0^T \int_{\Omega} \frac{1}{\varepsilon} |H_{\varepsilon}|^2 \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \, dxdt \\
& - \int_0^T \int_{\Omega} c_0 \chi ((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) (v_{\varepsilon} \cdot \nabla) \psi_{\varepsilon} \, dxdt \\
& - \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) v \, dxdt - \int_0^T \int_{\Omega} \partial_t \xi \cdot \nabla \psi_{\varepsilon} \, dxdt \\
& =: E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](0) - \int_0^T \int_{\Omega} |\nabla v_{\varepsilon} - \nabla v|^2 \, dxdt \tag{43} \\
& - \int_0^T \int_{\Omega} (v_{\varepsilon} - v) \cdot ((v_{\varepsilon} - v) \cdot \nabla) v \, dxdt + \text{Res}^{(1)}(T).
\end{aligned}$$

The remainder of the proof takes care of suitably rewriting the residual term  $\text{Res}^{(1)}$ . To this end, we start with an auxiliary result.

*Step 4: A well-known identity for the phase-field curvature operator.* We claim that it holds

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) \eta \, dxdt \\
& = - \int_0^T \int_{\Omega} H_{\varepsilon} (\eta \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \eta) |\nabla \psi_{\varepsilon}| \, dxdt \tag{44} \\
& - \int_0^T \int_{\Omega} (\nabla \cdot \eta) \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right)^2 \, dxdt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^{\infty}(\Omega \times [0, T]; \mathbb{R}^d)$ . In particular, we record that

$$- \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) \eta \, dxdt = - \int_0^T \int_{\Omega} H_{\varepsilon} (\eta \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| \, dxdt \tag{45}$$

for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^{\infty}(\Omega \times [0, T]; \mathbb{R}^d)$  with  $\nabla \cdot \eta = 0$ .

Indeed, as we may compute by an integration by parts and (16)

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) \eta \, dxdt \\
& = \int_0^T \int_{\Omega} \varepsilon \Delta \varphi_{\varepsilon} (\eta \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| \, dxdt + \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \otimes \eta : \nabla^2 \varphi_{\varepsilon} \, dxdt \\
& = \int_0^T \int_{\Omega} \varepsilon \Delta \varphi_{\varepsilon} (\eta \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| \, dxdt - \int_0^T \int_{\Omega} \frac{1}{2} (\nabla \cdot \eta) \varepsilon |\nabla \varphi_{\varepsilon}|^2 \, dxdt
\end{aligned}$$

it follows in combination with completing a square that

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \varepsilon \nabla \varphi_{\varepsilon} \cdot (\nabla \varphi_{\varepsilon} \cdot \nabla) \eta \, dxdt \\
& = \int_0^T \int_{\Omega} \varepsilon \Delta \varphi_{\varepsilon} (\eta \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| \, dxdt + \int_0^T \int_{\Omega} (\nabla \cdot \eta) \frac{1}{\varepsilon} W(\varphi_{\varepsilon}) \, dxdt \\
& - \int_0^T \int_{\Omega} (\nabla \cdot \eta) |\nabla \psi_{\varepsilon}| \, dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \eta) \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_{\varepsilon})} \right)^2 \, dxdt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$  and all  $\eta \in C_c^\infty(\Omega \times [0, T]; \mathbb{R}^d)$ . Further integrating by parts in the second term on the right hand side of the last display, we obtain the claim (44) from recalling the definition (33) of  $H_\varepsilon$ .

*Step 5: Exploiting control on the time evolution of  $\xi$ .* We next claim that

$$\begin{aligned}
\text{Res}^{(1)}(T) &= - \int_0^T \int_\Omega (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot (\mathbf{n}_\varepsilon - \xi) |\nabla \psi_\varepsilon| \, dxdt \quad (46) \\
&\quad - \int_0^T \int_\Omega \xi \cdot (\partial_t + (B \cdot \nabla)) \xi |\nabla \psi_\varepsilon| \, dxdt \\
&\quad - \int_0^T \int_\Omega \nabla B : (\xi - \mathbf{n}_\varepsilon) \otimes (\xi - \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dxdt \\
&\quad + \int_0^T \int_\Omega (\nabla \cdot B) (1 - \xi \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dxdt \\
&\quad + \int_0^T \int_\Omega (\nabla \cdot B) \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_\varepsilon)} \right)^2 \, dxdt \\
&\quad - \int_0^T \int_\Omega (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \xi \otimes \xi) : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \, dxdt \\
&\quad - \int_0^T \int_\Omega \xi \otimes \xi : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \, dxdt \\
&\quad + \text{Res}^{(2)}(T)
\end{aligned}$$

for a.e.  $T \in (0, T_*)$ , where

$$\begin{aligned}
\text{Res}^{(2)}(T) &:= - \int_0^T \int_\Omega \frac{1}{\varepsilon} |H_\varepsilon|^2 \, dxdt - \int_0^T \int_\Omega (\nabla \cdot \xi) \frac{H_\varepsilon}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} \, dxdt \quad (47) \\
&\quad - \int_0^T \int_\Omega c_0 \chi((v_\varepsilon - v) \cdot \nabla) (\nabla \cdot \xi) \, dxdt \\
&\quad - \int_0^T \int_\Omega (\nabla \cdot \xi) ((v_\varepsilon - B) \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dxdt \\
&\quad - \int_0^T \int_\Omega H_\varepsilon ((v - B) \cdot \mathbf{n}_\varepsilon) |\nabla \varphi_\varepsilon| \, dxdt.
\end{aligned}$$

For a proof of (46) and (47), we first record that it follows from plugging in  $\eta = v$  as a test function in (45), which is indeed an admissible choice after a standard mollification argument, that

$$- \int_0^T \int_\Omega \varepsilon \nabla \varphi_\varepsilon \cdot (\nabla \varphi_\varepsilon \cdot \nabla) v \, dxdt = - \int_0^T \int_\Omega H_\varepsilon (v \cdot \mathbf{n}_\varepsilon) |\nabla \varphi_\varepsilon| \, dxdt \quad (48)$$

for a.e.  $T \in (0, T_*)$ . The next step in the computation takes care of the term involving the time derivative of the vector field  $\xi$ . Adding and subtracting the anticipated PDE for the time evolution of the vector field  $\xi$ , cf. the left hand side of (20f), and analogously for the anticipated equation for the time evolution of its length, cf. the left hand side of (20g), we get making also use of (16)

$$- \int_0^T \int_\Omega \nabla \psi_\varepsilon \cdot \partial_t \xi \, dxdt$$

$$\begin{aligned}
&= - \int_0^T \int_{\Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx dt \\
&\quad + \int_0^T \int_{\Omega} ((B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx dt \\
&= - \int_0^T \int_{\Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot (\mathbf{n}_\varepsilon - \xi) |\nabla \psi_\varepsilon| \, dx dt \quad (49) \\
&\quad - \int_0^T \int_{\Omega} \xi \cdot (\partial_t + (B \cdot \nabla)) \xi |\nabla \psi_\varepsilon| \, dx dt \\
&\quad - \int_0^T \int_{\Omega} \nabla B : \xi \otimes (\xi - \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dx dt \\
&\quad + \int_0^T \int_{\Omega} (\mathbf{n}_\varepsilon \cdot (B \cdot \nabla) \xi) |\nabla \psi_\varepsilon| \, dx dt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$ . We further rewrite the last term in the preceding identity by repeatedly adding zero and also using that  $(B \cdot \nabla) \xi = \nabla \cdot (\xi \otimes B) - (\nabla \cdot B) \xi$  as follows

$$\begin{aligned}
&\int_0^T \int_{\Omega} (\mathbf{n}_\varepsilon \cdot (B \cdot \nabla) \xi) |\nabla \psi_\varepsilon| \, dx dt \\
&= \int_0^T \int_{\Omega} (\nabla \cdot B) (1 - \xi \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dx dt - \int_0^T \int_{\Omega} (\text{Id} - \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) : \nabla B |\nabla \psi_\varepsilon| \, dx dt \\
&\quad + \int_0^T \int_{\Omega} (\nabla \cdot (\xi \otimes B)) \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx dt - \int_0^T \int_{\Omega} \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon : \nabla B |\nabla \psi_\varepsilon| \, dx dt.
\end{aligned}$$

In order to combine the last two terms, we compute based on a standard mollifier approximation (w.r.t. the spatial variable)  $\xi_k := \theta_k * \xi$  and  $B_k := \theta_k * B$  for the vector fields  $\xi$  and  $B$ , respectively (recall the regularity assumptions (20a)–(20c)), another integration by parts as well as (16) that

$$\begin{aligned}
&\int_0^T \int_{\Omega} (\nabla \cdot (\xi \otimes B)) \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx dt \\
&= - \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \psi_\varepsilon \nabla \cdot (\nabla \cdot (\xi_k \otimes B_k)) \, dx dt \\
&= - \lim_{k \rightarrow \infty} \int_0^T \int_{\Omega} \psi_\varepsilon \nabla \cdot (\nabla \cdot (B_k \otimes \xi_k)) \, dx dt \\
&= \int_0^T \int_{\Omega} (\nabla \cdot \xi) B \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx dt + \int_0^T \int_{\Omega} \mathbf{n}_\varepsilon \otimes \xi : \nabla B |\nabla \psi_\varepsilon| \, dx dt.
\end{aligned}$$

All in all, feeding back into (49) the previous two displays we consequently obtain

$$\begin{aligned}
&- \int_0^T \int_{\Omega} \nabla \psi_\varepsilon \cdot \partial_t \xi \, dx dt \\
&= - \int_0^T \int_{\Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^\top \xi) \cdot (\mathbf{n}_\varepsilon - \xi) |\nabla \psi_\varepsilon| \, dx dt \quad (50) \\
&\quad - \int_0^T \int_{\Omega} \xi \cdot (\partial_t + (B \cdot \nabla)) \xi |\nabla \psi_\varepsilon| \, dx dt
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \nabla B : (\xi - \mathbf{n}_\varepsilon) \otimes (\xi - \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dxdt \\
& + \int_0^T \int_{\Omega} (\nabla \cdot B)(1 - \xi \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dxdt \\
& + \int_0^T \int_{\Omega} (\nabla \cdot \xi) B \cdot \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dxdt \\
& - \int_0^T \int_{\Omega} (\text{Id} - \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) : \nabla B |\nabla \psi_\varepsilon| \, dxdt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$ . Finally appealing to (44) with the admissible choice of the test vector field  $\eta = B$  (again after a standard mollification argument) in form of

$$\begin{aligned}
& - \int_0^T \int_{\Omega} (\text{Id} - \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) : \nabla B |\nabla \psi_\varepsilon| \, dxdt \\
& = \int_0^T \int_{\Omega} H_\varepsilon(B \cdot \mathbf{n}_\varepsilon) |\nabla \varphi_\varepsilon| \, dxdt \\
& + \int_0^T \int_{\Omega} (\nabla \cdot B) \frac{1}{2} \left( \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| \, dxdt - \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(\varphi_\varepsilon)} \right)^2 \\
& - \int_0^T \int_{\Omega} (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon - \xi \otimes \xi) : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \, dxdt \\
& - \int_0^T \int_{\Omega} \xi \otimes \xi : \nabla B (\varepsilon |\nabla \varphi_\varepsilon|^2 - |\nabla \psi_\varepsilon|) \, dxdt
\end{aligned} \tag{51}$$

for a.e.  $T \in (0, T_*)$ , the claims (46) and (47) therefore follow from the combination of the identities (48), (50) and (51).

*Step 6: Generating dissipation squares.* For the final step, we claim that

$$\begin{aligned}
\text{Res}^{(2)} & \leq - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_\varepsilon + \sqrt{2W(\varphi_\varepsilon)} (\nabla \cdot \xi) \right|^2 \, dxdt \\
& - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_\varepsilon - ((B - v) \cdot \xi) \varepsilon |\nabla \varphi_\varepsilon| \right|^2 \, dxdt \\
& - \int_0^T \int_{\Omega} (c_0 \chi - \psi_\varepsilon) ((v_\varepsilon - v) \cdot \nabla) (\nabla \cdot \xi) \, dxdt \\
& + \int_0^T \int_{\Omega} |(B - v) \cdot \xi + \nabla \cdot \xi|^2 \varepsilon |\nabla \varphi_\varepsilon|^2 \, dxdt \\
& + \int_0^T \int_{\Omega} |\nabla \cdot \xi|^2 \left( \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| \right)^2 \, dxdt \\
& - \int_0^T \int_{\Omega} \frac{1}{\sqrt{\varepsilon}} (H_\varepsilon + \sqrt{2W(\varphi_\varepsilon)} (\nabla \cdot \xi)) ((v - B) \cdot (\mathbf{n}_\varepsilon - \xi)) \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| \, dxdt
\end{aligned} \tag{52}$$

for a.e.  $T \in (0, T_*)$ . Indeed, once we established (52) the asserted relative energy inequality from Proposition 6 is a consequence of collecting the estimates and identities from (43), (46) and (47), respectively.

For a proof of (52), we start by spending half of the available interfacial energy dissipation in form of

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \frac{1}{\varepsilon} |H_{\varepsilon}|^2 dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} dxdt \\
& = - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_{\varepsilon} + \sqrt{2W(\varphi_{\varepsilon})} (\nabla \cdot \xi) \right|^2 dxdt \\
& \quad - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} |H_{\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega} \frac{1}{2} |\nabla \cdot \xi|^2 \left( \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right)^2 dxdt.
\end{aligned} \tag{53}$$

We proceed by combining the remaining terms. Adding zero, recalling (16), and integrating by parts (using in the process the solenoidality of the velocity fields) results in

$$\begin{aligned}
& - \int_0^T \int_{\Omega} c_0 \chi ((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) dxdt - \int_0^T \int_{\Omega} (\nabla \cdot \xi) ((v_{\varepsilon} - B) \cdot \mathbf{n}_{\varepsilon}) |\nabla \psi_{\varepsilon}| dxdt \\
& = - \int_0^T \int_{\Omega} (c_0 \chi - \psi_{\varepsilon}) ((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) dxdt \\
& \quad - \int_0^T \int_{\Omega} (\nabla \cdot \xi) ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) |\nabla \psi_{\varepsilon}| dxdt \\
& \quad - \int_0^T \int_{\Omega} (\nabla \cdot \xi) ((v - B) \cdot \xi) |\nabla \psi_{\varepsilon}| dxdt.
\end{aligned} \tag{54}$$

Moreover, adding zero as well as spending the remaining half of the available interfacial energy dissipation leads to

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} |H_{\varepsilon}|^2 dxdt - \int_0^T \int_{\Omega} H_{\varepsilon} ((v - B) \cdot \mathbf{n}_{\varepsilon}) |\nabla \varphi_{\varepsilon}| dxdt \\
& = - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_{\varepsilon} - ((B - v) \cdot \xi) \varepsilon |\nabla \varphi_{\varepsilon}| \right|^2 dxdt \\
& \quad + \int_0^T \int_{\Omega} \frac{1}{2} |(B - v) \cdot \xi|^2 \varepsilon |\nabla \varphi_{\varepsilon}|^2 dxdt \\
& \quad - \int_0^T \int_{\Omega} H_{\varepsilon} ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) |\nabla \varphi_{\varepsilon}| dxdt.
\end{aligned} \tag{55}$$

Two simple steps remain until we reach the desired final form (52). The first makes use of  $|\nabla \psi_{\varepsilon}| = \sqrt{2W(\varphi_{\varepsilon})} |\nabla \varphi_{\varepsilon}|$  which in turn allows to combine

$$\begin{aligned}
& - \int_0^T \int_{\Omega} H_{\varepsilon} ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) |\nabla \varphi_{\varepsilon}| dxdt \\
& \quad - \int_0^T \int_{\Omega} (\nabla \cdot \xi) ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) |\nabla \psi_{\varepsilon}| dxdt \\
& = - \int_0^T \int_{\Omega} \frac{1}{\sqrt{\varepsilon}} (H_{\varepsilon} + \sqrt{2W(\varphi_{\varepsilon})} (\nabla \cdot \xi)) ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| dxdt.
\end{aligned} \tag{56}$$

The second consists of collecting the square, adding zero and exploiting the simple estimate  $(a + b)^2 \leq 2(a^2 + b^2)$  in form of

$$\int_0^T \int_{\Omega} \frac{1}{2} |(B - v) \cdot \xi|^2 \varepsilon |\nabla \varphi_{\varepsilon}|^2 dxdt + \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| \sqrt{2W(\varphi_{\varepsilon})} (\nabla \cdot \xi) \right|^2 dxdt$$

$$\begin{aligned}
& + \int_0^T \int_{\Omega} (\nabla \cdot \xi) ((B - v) \cdot \xi) |\nabla \psi_{\varepsilon}| \, dx dt \\
& = \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| ((B - v) \cdot \xi) \varepsilon |\nabla \varphi_{\varepsilon}| + \sqrt{2W(\varphi_{\varepsilon})} (\nabla \cdot \xi) \right|^2 \, dx dt \\
& = \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| ((B - v) \cdot \xi + \nabla \cdot \xi) \varepsilon |\nabla \varphi_{\varepsilon}| + (\sqrt{2W(\varphi_{\varepsilon})} - \varepsilon |\nabla \varphi_{\varepsilon}|) (\nabla \cdot \xi) \right|^2 \, dx dt \\
& \leq \int_0^T \int_{\Omega} |(B - v) \cdot \xi + \nabla \cdot \xi|^2 \varepsilon |\nabla \varphi_{\varepsilon}|^2 \, dx dt \tag{57} \\
& \quad + \int_0^T \int_{\Omega} |\nabla \cdot \xi|^2 \left( \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} - \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \right)^2 \, dx dt.
\end{aligned}$$

Hence, the combination of (53)–(57) implies the claim (52) and thus concludes the derivation of the relative energy inequality.  $\square$

## 5. TIME EVOLUTION OF THE ERROR IN THE PHASE INDICATORS

**Lemma 7.** *Let the assumptions and notation of Theorem 1 be in place, let  $(\xi, B)$  be a pair of vector fields subject to (20a) and (20b), respectively, let  $\vartheta$  be a time-dependent weight satisfying (29a) as well as (29d)–(29e), and recall finally the notation (15) and (33). The time evolution of the error in the phase indicators (28) may then be represented as follows: it holds*

$$\begin{aligned}
& E_{\text{vol}}[\varphi_{\varepsilon} | \chi](T) \\
& = E_{\text{vol}}[\varphi_{\varepsilon} | \chi](0) + \int_0^T \int_{\Omega} (\psi_{\varepsilon} - c_0 \chi) (\partial_t \vartheta + (B \cdot \nabla) \vartheta) \, dx dt \\
& \quad + \int_0^T \int_{\Omega} (\psi_{\varepsilon} - c_0 \chi) \vartheta \nabla \cdot B \, dx dt \\
& \quad + \int_0^T \int_{\Omega} \vartheta ((B - v) \cdot (\mathbf{n}_{\varepsilon} - \xi)) |\nabla \psi_{\varepsilon}| \, dx dt \\
& \quad - \int_0^T \int_{\Omega} (\psi_{\varepsilon} - c_0 \chi) ((v - v_{\varepsilon}) \cdot \nabla) \vartheta \, dx dt \\
& \quad + \int_0^T \int_{\Omega} \vartheta ((B - v) \cdot \xi) (|\nabla \psi_{\varepsilon}| - \varepsilon |\nabla \varphi_{\varepsilon}|)^2 \, dx dt \\
& \quad + \int_0^T \int_{\Omega} \left( ((B - v) \cdot \xi) \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \right) \vartheta \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \, dx dt \\
& \quad + \int_0^T \int_{\Omega} \vartheta \left( \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} + \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} (\nabla \cdot \xi) \right) \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right) \, dx dt \\
& \quad + \int_0^T \int_{\Omega} \vartheta (\nabla \cdot \xi) \left| \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right|^2 \, dx dt \\
& \quad - \int_0^T \int_{\Omega} \vartheta \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right) \, dx dt
\end{aligned}$$

for a.e.  $T \in (0, T_*)$ .

*Proof.* First, since for a.e.  $t \in (0, T_0)$  we have  $|\varphi_\varepsilon(\cdot, t)| \leq 1$   $\mathcal{L}^d$ -a.e. in  $\Omega$ , which in turn is guaranteed by a maximum principle argument and the analogous condition for the initial phase-field  $\varphi_{\varepsilon,0}$ , it follows that  $\psi_\varepsilon \in [0, c_0]$  and thus from the conditions (29d) and (29e) that

$$E_{\text{vol}}[\varphi_\varepsilon|\chi](T) = \int_{\Omega} (\psi_\varepsilon - c_0\chi)(\cdot, T) \vartheta(\cdot, T) dx \quad (58)$$

for a.e.  $T \in (0, T_*)$ . Second, as a consequence of  $\vartheta \equiv 0$  along  $\mathcal{I}$ , cf. again to this end (29d) and (29e), we also observe that it holds  $c_0 \int_0^T \int_{\Omega} \vartheta \partial_t \chi dx dt = 0$  for a.e.  $T \in (0, T_*)$ . Hence, based on these two properties we may compute by an application of the fundamental theorem of calculus, the regularity of the phase-field  $\varphi_\varepsilon$ , the condition (12g), adding zero, as well as the product rule that

$$\begin{aligned} E_{\text{vol}}[\varphi_\varepsilon|\chi](T) &= E_{\text{vol}}[\varphi_\varepsilon|\chi](0) + \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) \partial_t \vartheta dx dt + \int_0^T \int_{\Omega} \vartheta \partial_t \psi_\varepsilon dx dt \\ &= E_{\text{vol}}[\varphi_\varepsilon|\chi](0) + \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) (\partial_t \vartheta + (B \cdot \nabla) \vartheta) dx dt \\ &\quad - \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) \nabla \cdot (B \vartheta) dx dt + \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) \vartheta \nabla \cdot B dx dt \\ &\quad + \int_0^T \int_{\Omega} \vartheta \partial_t \psi_\varepsilon dx dt \end{aligned} \quad (59)$$

for a.e.  $T \in (0, T_*)$ . Integrating by parts and exploiting again that  $\vartheta \equiv 0$  along  $\mathcal{I}$ , we may upgrade (59) to

$$\begin{aligned} E_{\text{vol}}[\varphi_\varepsilon|\chi](T) &= E_{\text{vol}}[\varphi_\varepsilon|\chi](0) + \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) (\partial_t \vartheta + (B \cdot \nabla) \vartheta) dx dt \\ &\quad + \int_0^T \int_{\Omega} (\psi_\varepsilon - c_0\chi) \vartheta \nabla \cdot B dx dt \\ &\quad + \int_0^T \int_{\Omega} \vartheta (\partial_t \psi_\varepsilon + (B \cdot \nabla) \psi_\varepsilon) dx dt \end{aligned} \quad (60)$$

for a.e.  $T \in (0, T_*)$ . The remainder of the proof takes care of suitably post-processing the last right hand side term from the previous display incorporating an advective derivative for the  $BV$ -approximation  $\psi_\varepsilon$  of  $c_0\chi$ .

To this end, inserting in a first step the identity (42), making use of (16) in a second step, and finally adding zero yields

$$\begin{aligned} &\int_0^T \int_{\Omega} \vartheta (\partial_t \psi_\varepsilon + (B \cdot \nabla) \psi_\varepsilon) dx dt \\ &= \int_0^T \int_{\Omega} \vartheta (B \cdot \nabla) \psi_\varepsilon dx dt - \int_0^T \int_{\Omega} \vartheta \left( \frac{H_\varepsilon}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} + (v_\varepsilon \cdot \nabla) \psi_\varepsilon \right) dx dt \\ &= \int_0^T \int_{\Omega} \vartheta ((B - v) \cdot (\mathbf{n}_\varepsilon - \xi)) |\nabla \psi_\varepsilon| dx dt \\ &\quad + \int_0^T \int_{\Omega} \vartheta ((B - v) \cdot \xi) |\nabla \psi_\varepsilon| dx dt - \int_0^T \int_{\Omega} \vartheta \frac{H_\varepsilon}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} dx dt \\ &\quad + \int_0^T \int_{\Omega} \vartheta ((v - v_\varepsilon) \cdot \nabla) \psi_\varepsilon dx dt. \end{aligned} \quad (61)$$

Note next that we may rewrite

$$\begin{aligned} & \int_0^T \int_{\Omega} \vartheta((v - v_{\varepsilon}) \cdot \nabla) \psi_{\varepsilon} \, dxdt \\ &= - \int_0^T \int_{\Omega} \psi_{\varepsilon}((v - v_{\varepsilon}) \cdot \nabla) \vartheta \, dxdt = - \int_0^T \int_{\Omega} (\psi_{\varepsilon} - c_0 \chi)((v - v_{\varepsilon}) \cdot \nabla) \vartheta \, dxdt. \end{aligned} \quad (62)$$

Indeed, the first equality simply relies on the solenoidality of the velocity fields  $v_{\varepsilon}$  and  $v$ , respectively, in combination with an integration by parts. The second equality in turn exploits  $\vartheta \equiv 0$  along  $\mathcal{I}$  which—by the solenoidality of the velocity fields—allows to smuggle in the same term but with  $\psi_{\varepsilon}$  being replaced by  $c_0 \chi$ . Furthermore, adding zero several times entails

$$\begin{aligned} & \int_0^T \int_{\Omega} \vartheta((B - v) \cdot \xi) |\nabla \psi_{\varepsilon}| \, dxdt - \int_0^T \int_{\Omega} \vartheta \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \, dxdt \\ &= \int_0^T \int_{\Omega} \vartheta((B - v) \cdot \xi) (|\nabla \psi_{\varepsilon}| - \varepsilon |\nabla \varphi_{\varepsilon}|^2) \, dxdt \\ & \quad + \int_0^T \int_{\Omega} \left( ((B - v) \cdot \xi) \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} \right) \vartheta \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \, dxdt \\ & \quad + \int_0^T \int_{\Omega} \vartheta \left( \frac{H_{\varepsilon}}{\sqrt{\varepsilon}} + \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} (\nabla \cdot \xi) \right) \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right) \, dxdt \\ & \quad + \int_0^T \int_{\Omega} \vartheta (\nabla \cdot \xi) \left| \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right|^2 \, dxdt \\ & \quad - \int_0^T \int_{\Omega} \vartheta \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| \left( \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| - \frac{\sqrt{2W(\varphi_{\varepsilon})}}{\sqrt{\varepsilon}} \right) \, dxdt. \end{aligned} \quad (63)$$

Feeding back (61)–(63) into (60) thus yields the desired result.  $\square$

## 6. PROOF OF MAIN RESULTS

*Proof of Theorem 1.* The proof is split into six steps.

*Step 1: Construction of the triple  $(\xi, B, \vartheta)$ .* Fix  $T \in [0, T_*)$ . Due to the assumed regularity of the evolving geometry underlying the strong solutions  $(\chi, v)$ , cf. item *iii*) of Definition 4, there exists a small scale  $\delta = \delta(\chi, T) \in (0, \frac{1}{2}]$  such that the signed distance  $s_{\mathcal{I}}(\cdot, t)$  to  $\mathcal{I}(t)$  as well as the nearest point projection  $P_{\mathcal{I}}(\cdot, t)$  onto  $\mathcal{I}(t)$ ,  $t \in [0, T]$ , are regular in the sense that

$$s_{\mathcal{I}} \in (C_t^1 C_x^1 \cap C_t C_x^3)(\overline{B_{2\delta}(\mathcal{I})}), \quad (64)$$

$$((x, t) \mapsto P_{\mathcal{I}}(x, t) := (x - (s_{\mathcal{I}} \nabla s_{\mathcal{I}})(x, t), t) \in (C_t^1 C_x \cap C_t C_x^2)(\overline{B_{2\delta}(\mathcal{I})}). \quad (65)$$

We remark in this context that the signed distance is oriented by the requirement  $\nabla s_{\mathcal{I}} = \mathbf{n}_{\mathcal{I}}$  on  $\mathcal{I}$ , and that for every  $r \in (0, 1]$  the associated space-time tubular neighborhood  $B_r(\mathcal{I})$  of  $\mathcal{I}$  is defined by

$$B_r(\mathcal{I}) := \bigcup_{t \in [0, T]} B_r(\mathcal{I}(t)) \times \{t\}. \quad (66)$$

Due to (13h) and (13i), we may further choose  $\delta$  small enough such that

$$\overline{B_{2\delta}(\mathcal{I}(t))} \subset \Omega \quad \text{for all } t \in [0, T]. \quad (67)$$



Next, let  $\bar{\eta}: \mathbb{R} \rightarrow [0, 1]$  be a smooth and even profile with  $\text{supp } \bar{\eta} \subset [-1, 1]$  and with quadratic decay at the origin in the sense of

$$c_{\bar{\eta}} r^2 \leq 1 - \bar{\eta}(r) \leq C_{\bar{\eta}} r^2 \quad \text{for all } r \in [-1, 1], \quad (68)$$

for some constants  $0 < c_{\bar{\eta}} < C_{\bar{\eta}} < \infty$ . We further choose a smooth and even profile  $\tilde{\eta}: \mathbb{R} \rightarrow [0, 1]$  such that  $\text{supp } \tilde{\eta} \subset [-2, 2]$  and

$$\tilde{\eta} \equiv 1 \quad \text{on } [-1, 1]. \quad (69)$$

We then define two quadratic cutoffs

$$\eta_{\mathcal{I}}(x, t) := \bar{\eta}(s_{\mathcal{I}}(x, t)/\delta), \quad (x, t) \in \mathbb{R}^d \times [0, T], \quad (70)$$

$$\tilde{\eta}_{\mathcal{I}}(x, t) := \tilde{\eta}(s_{\mathcal{I}}(x, t)/\delta), \quad (x, t) \in \mathbb{R}^d \times [0, T]. \quad (71)$$

Note that

$$\text{supp } \eta_{\mathcal{I}}(\cdot, t) \subset \overline{B_{\delta}(\mathcal{I}(t))} \quad \text{for all } t \in [0, T], \quad (72)$$

$$\text{supp } \tilde{\eta}_{\mathcal{I}}(\cdot, t) \subset \overline{B_{2\delta}(\mathcal{I}(t))} \quad \text{for all } t \in [0, T], \quad (73)$$

$$\tilde{\eta}_{\mathcal{I}}(\cdot, t) \equiv 1 \quad \text{on } \overline{B_{\delta}(\mathcal{I}(t))} \quad \text{for all } t \in [0, T]. \quad (74)$$

With these ingredients in place, we define the vector fields  $\xi$  and  $B$  by means of

$$\xi := \eta_{\mathcal{I}} \nabla s_{\mathcal{I}} \quad \text{on } \Omega \times [0, T], \quad (75)$$

$$B := ((v \circ P_{\mathcal{I}}) \cdot \nabla s_{\mathcal{I}}) - (\Delta s_{\mathcal{I}}) \circ P_{\mathcal{I}} \tilde{\eta}_{\mathcal{I}} \nabla s_{\mathcal{I}} \quad \text{on } \Omega \times [0, T], \quad (76)$$

where  $v$  is the velocity field of the strong solution.

For a construction of a suitable weight  $\vartheta$ , we first fix a smooth and odd map  $\bar{\vartheta}: \mathbb{R} \rightarrow [-1, 1]$  representing a suitable truncation of the (negative of the) identity in the sense that

$$\bar{\vartheta} \equiv 1 \quad \text{on } (-\infty, -1] \quad \text{and} \quad \bar{\vartheta} \equiv -1 \quad \text{on } [1, \infty), \quad (77)$$

$$\bar{\vartheta} > 0 \quad \text{in } (-1, 0) \quad \text{and} \quad \bar{\vartheta} < 0 \quad \text{in } (0, 1), \quad (78)$$

$$c_{\bar{\vartheta}} r \leq |\bar{\vartheta}(r)| \leq C_{\bar{\vartheta}} r \quad \text{for all } r \in [-1, 1], \quad (79)$$

for some constants  $0 < c_{\bar{\vartheta}} < C_{\bar{\vartheta}} < \infty$ . We may then define

$$\vartheta := \bar{\vartheta}(s_{\mathcal{I}}/\delta) \quad \text{on } \Omega \times [0, T]. \quad (80)$$

*Step 2: Proof of the conditions (20a)–(20i).* The required regularity (20a)–(20c) is immediate from the definitions (75) and (76) as well as the regularity (13d), (64) and (65) of the associated building blocks. Furthermore, because of (72), (73) and (67) it follows that the vector fields  $\xi$  and  $B$  are indeed compactly supported within  $\Omega$ .

We next note that the coercivity estimate (20d) directly follows from the definition (75) and the lower bound from (68). For a proof of the consistency conditions (20e),  $\xi = \mathbf{n}_{\mathcal{I}}$  along  $\mathcal{I}$  simply follows from the definition (75) as well as  $\bar{\eta}(0) = 0$  whereas  $\nabla \cdot \xi = -H_{\mathcal{I}}$  along  $\mathcal{I}$  is a consequence of the definition (75),  $\bar{\eta}'(0) = 0$  as well as the well-known identity  $\Delta s_{\mathcal{I}} = -H_{\mathcal{I}}$  along  $\mathcal{I}$ .

We proceed with a proof of the (approximate) evolution equations (20f) and (20g). The argument is based on the claim

$$(\partial_t s_{\mathcal{I}} + (B \cdot \nabla) s_{\mathcal{I}})(\cdot, t) \equiv 0 \quad \text{on } \overline{B_{\delta}(\mathcal{I}(t))}, \quad t \in [0, T]. \quad (81)$$

Since  $\partial_t s_{\mathcal{I}} = -V_{\mathcal{I}}$  along  $\mathcal{I}$ , the identity (81) is indeed valid as a consequence of the definition (76), the property (74), item *iv*) of Definition 4 in form of  $V_{\mathcal{I}} = (B \cdot \nabla) s_{\mathcal{I}}$

along  $\mathcal{I}$ , as well as  $\partial_t s_{\mathcal{I}} = (\partial_t s_{\mathcal{I}}) \circ P_{\mathcal{I}}$  (the latter following from a straightforward computation based on differentiating  $s_{\mathcal{I}} \circ P_{\mathcal{I}} \equiv 0$ ). Note then that (81) together with the chain rule immediately implies

$$(\partial_t \eta_{\mathcal{I}} + (B \cdot \nabla) \eta_{\mathcal{I}})(\cdot, t) \equiv 0 \quad \text{on } \overline{B_{\delta}(\mathcal{I}(t))}, t \in [0, T]. \quad (82)$$

This in turn directly entails (20g) due to the simple observation  $|\xi|^2 = \eta_{\mathcal{I}}^2$ , which itself follows from the definition (75). Furthermore, (82) reduces the proof of (20f) to a proof of

$$(\partial_t \nabla s_{\mathcal{I}} + (B \cdot \nabla) \nabla s_{\mathcal{I}} + (\nabla B)^{\top} \nabla s_{\mathcal{I}})(\cdot, t) \equiv 0 \quad \text{on } \overline{B_{\delta}(\mathcal{I}(t))}, t \in [0, T]. \quad (83)$$

However, (83) simply follows from taking the spatial gradient of (81).

We finally note that (20h) and (20i) are straightforward consequences of the definitions (75) and (76), the regularity of the associated building blocks, as well as the already established property (20e).

*Step 3: Proof of the conditions (29a)–(29f).* The regularity requirements (29a) and (29b) are immediate from the definition (80) and the regularity (64). The coercivity estimate (29c) as well as the sign conditions (29d)–(29e) in turn follow directly from the definition (80) and the properties (77)–(79). Finally, the approximate transport equation (29f) simply results from a combination of (81) and the chain rule.

*Step 4: Derivation of the stability estimates (8) and (9).* Applying (21)–(24) and (20f)–(20g) to the inequality in Proposition 6, we have for a.e.  $T \in (0, T_*)$  that

$$\begin{aligned} & E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](T) \\ & \leq E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](0) + C \int_0^T E[\varphi_{\varepsilon}, v_{\varepsilon} | \chi, v](t) dt - \int_0^T \int_{\Omega} |\nabla v_{\varepsilon} - \nabla v|^2 dx dt \\ & \quad - \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_{\varepsilon} + \sqrt{2W(\varphi_{\varepsilon})}(\nabla \cdot \xi) \right|^2 dx dt \end{aligned} \quad (84)$$

$$- \int_0^T \int_{\Omega} \frac{1}{2\varepsilon} \left| H_{\varepsilon} - ((B - v) \cdot \xi) \varepsilon |\nabla \varphi_{\varepsilon}| \right|^2 dx dt \quad (85)$$

$$- \int_0^T \int_{\Omega} (c_0 \chi - \psi_{\varepsilon}) ((v_{\varepsilon} - v) \cdot \nabla) (\nabla \cdot \xi) dx dt \quad (86)$$

$$+ \int_0^T \int_{\Omega} |(B - v) \cdot \xi + \nabla \cdot \xi|^2 \varepsilon |\nabla \varphi_{\varepsilon}|^2 dx dt \quad (87)$$

$$- \int_0^T \int_{\Omega} \frac{1}{\sqrt{\varepsilon}} (H_{\varepsilon} + \sqrt{2W(\varphi_{\varepsilon})}(\nabla \cdot \xi)) ((v - B) \cdot (\mathbf{n}_{\varepsilon} - \xi)) \sqrt{\varepsilon} |\nabla \varphi_{\varepsilon}| dx dt \quad (88)$$

$$- \int_0^T \int_{\Omega} (\mathbf{n}_{\varepsilon} \otimes \mathbf{n}_{\varepsilon} - \xi \otimes \xi) : \nabla B (\varepsilon |\nabla \varphi_{\varepsilon}|^2 - |\nabla \psi_{\varepsilon}|) dx dt \quad (89)$$

$$- \int_0^T \int_{\Omega} \xi \otimes \xi : \nabla B (\varepsilon |\nabla \varphi_{\varepsilon}|^2 - |\nabla \psi_{\varepsilon}|) dx dt. \quad (90)$$

The integral (86) is estimated by (31) with a sufficiently small  $\lambda > 0$ . The integral (87) can be estimated using (20h) and (25). To estimate (88), we first employ Cauchy–Schwarz’s inequality, followed by Young’s inequality with a sufficiently small prefactor in order to exploit the sign of (84), and then conclude with (25).

To estimate (89), we write

$$\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon : \nabla B = \mathbf{n}_\varepsilon \otimes (\mathbf{n}_\varepsilon - \xi) : \nabla B + (\xi \cdot \nabla) B \cdot (\mathbf{n}_\varepsilon - \xi) + \xi \otimes \xi : \nabla B$$

so that we can estimate

$$|\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon : \nabla B - \xi \otimes \xi : \nabla B| \leq C \sqrt{1 - \mathbf{n}_\varepsilon \cdot \xi}.$$

Because of this, (89) can be estimated by using (26). Finally, (90) can be bounded by means of (20i) and (26). All in all,

$$\begin{aligned} & E[\varphi_\varepsilon, v_\varepsilon | \chi, v](T) + \frac{1}{2} \int_0^T \int_\Omega |\nabla v_\varepsilon - \nabla v|^2 dx dt \\ & + \int_0^T \int_\Omega \frac{1}{4\varepsilon} \left| H_\varepsilon + \sqrt{2W(\varphi_\varepsilon)} (\nabla \cdot \xi) \right|^2 dx dt \\ & + \int_0^T \int_\Omega \frac{1}{4\varepsilon} \left| H_\varepsilon - ((B - v) \cdot \xi) \varepsilon |\nabla \varphi_\varepsilon| \right|^2 dx dt \\ & \leq E[\varphi_\varepsilon, v_\varepsilon | \chi, v](0) + C \int_0^T E[\varphi_\varepsilon, v_\varepsilon | \chi, v](t) + E_{\text{vol}}[\varphi_\varepsilon | \chi](t) dt. \end{aligned}$$

Regarding the evolution of the bulk error in Lemma 7, similar considerations based in addition on (25), (26), (29b), (29c), (29d) and (29f) also lead to

$$\begin{aligned} & E_{\text{vol}}[\varphi_\varepsilon | \chi](T) \\ & \leq E_{\text{vol}}[\varphi_\varepsilon | \chi](0) + C \int_0^T E[\varphi_\varepsilon, v_\varepsilon | \chi, v](t) + E_{\text{vol}}[\varphi_\varepsilon | \chi](t) dt \\ & + \int_0^T \int_\Omega \frac{1}{4} |\nabla v_\varepsilon - \nabla v|^2 dx dt \\ & + \int_0^T \int_\Omega \frac{1}{8} \left( ((B - v) \cdot \xi) \sqrt{\varepsilon} |\nabla \varphi_\varepsilon| - \frac{H_\varepsilon}{\sqrt{\varepsilon}} \right)^2 dx dt \\ & + \int_0^T \int_\Omega \frac{1}{8} \left( \frac{H_\varepsilon}{\sqrt{\varepsilon}} + \frac{\sqrt{2W(\varphi_\varepsilon)}}{\sqrt{\varepsilon}} (\nabla \cdot \xi) \right)^2 dx dt. \end{aligned}$$

The above two inequalities together with Grönwall's inequality therefore allow to conclude the proof.  $\square$

*Proof of Corollary 2.* We proceed in two steps.

*Step 1: From  $E_{\text{vol}}$ -control to  $L^1(\Omega)$ -control of the error in the phase indicators.* We claim that for all  $T \in [0, T_*)$  there exists a constant  $C = C(\chi, v, T) \in (0, \infty)$  such that

$$\|c_0 \chi(\cdot, t) - \psi_\varepsilon(\cdot, t)\|_{L^1(\Omega)}^2 \leq C E_{\text{vol}}[\varphi_\varepsilon | \chi](t) \quad (91)$$

holds true for all  $t \in [0, T]$ . For the simple proof based on a slicing argument and an application of Fubini's theorem, we refer, e.g., to [24, Proof of Theorem 1, Step 2].

*Step 2: Derivation of the sharp convergence rate (11).* This now immediately follows from post-processing the stability estimates (8) and (9) by means of the bound (91) and the assumption (10).  $\square$

## APPENDIX A. WELL-POSEDNESS OF THE SHARP INTERFACE LIMIT MODEL

The goal of this part is to construct a strong solution in the sense of Definition 4. We consider the following system which is equivalent to (2a)–(2h) (with  $c_0 = 1$ ):

$$\partial_t v + (v \cdot \nabla)v = \Delta v + \nabla \pi \quad \text{in } \mathring{\Omega}(t), \quad (92a)$$

$$\nabla \cdot v = 0 \quad \text{in } \mathring{\Omega}(t), \quad (92b)$$

$$[[2Dv - \pi \text{Id}] \cdot \mathbf{n}_{\mathcal{I}} = H_{\mathcal{I}} \mathbf{n}_{\mathcal{I}} \quad \text{on } \mathcal{I}(t), \quad (92c)$$

$$[[v]] = 0 \quad \text{on } \mathcal{I}(t), \quad (92d)$$

$$V_{\mathcal{I}} = \mathbf{n}_{\mathcal{I}} \cdot v + H_{\mathcal{I}} \quad \text{on } \mathcal{I}(t). \quad (92e)$$

Here  $\mathring{\Omega}(t) := \Omega_+(t) \cup \Omega_-(t)$  is the bulk region and  $Dv := \nabla^{\text{sym}} v$  is the symmetric part of the flow gradient. The major difference of (92) to the system studied in [40] is that in the latter the interface is purely transported by the fluid velocity, i.e.,  $V_{\mathcal{I}} = \mathbf{n}_{\mathcal{I}} \cdot v$ .

**Proposition 8.** *Let  $p > d + 2$  and  $\mathcal{I}(0) \subset \Omega$  be a  $C^3$  closed surface. Assume that the initial velocity field  $v_0$  satisfies  $v_0 \in W_p^{2-2/p}(\mathring{\Omega}(0)) \cap W_p^1(\Omega)$  together with the following compatibility conditions:*

$$\text{div } v_0 = 0 \text{ in } \mathring{\Omega}(0), \quad (93)$$

$$v_0 = 0 \text{ on } \partial\Omega, \quad (94)$$

$$[[v_0]] = 0 \text{ on } \mathcal{I}(0), \quad (95)$$

$$\Pi_{\mathcal{I}(0)} [[\nabla^{\text{sym}} v_0 \cdot \mathbf{n}_{\mathcal{I}}]] = 0 \text{ on } \mathcal{I}(0), \quad (96)$$

where  $\Pi_{\mathcal{I}(t)} = \text{Id} - \mathbf{n}_{\mathcal{I}(t)} \otimes \mathbf{n}_{\mathcal{I}(t)}$  is the tangential projection. Then there exists  $T > 0$  such that (92) has a unique solution  $(v, \pi, \mathcal{I})$  with

$$v \in W_p^1(0, T; L^p(\Omega)) \cap L^p\left(0, T; W_p^2(\mathring{\Omega}(t)) \cap W_p^1(\Omega)\right), \quad (97a)$$

$$\pi \in L^p\left(0, T; \dot{W}_p^1(\mathring{\Omega}(t))\right) \text{ with } \int_{\Omega} \pi(\cdot, t) = 0, \quad (97b)$$

$$[[\pi]] \in W_p^{1-\frac{1}{p}, \frac{1}{2}(1-\frac{1}{p})}(\mathcal{I}(t)). \quad (97c)$$

Moreover, the free boundary  $\mathcal{I} = \cup_{t \in (0, T)} \mathcal{I}(t) \times \{t\}$  is parametrized through the diffeomorphism  $\Theta_h(x, t): \mathring{\Omega}(0) \mapsto \mathring{\Omega}(t)$  defined by

$$\Theta_h(x, t) := x + \zeta \left( \frac{\text{dist}(x, \mathcal{I}(0))}{\delta} \right) h(P_{\mathcal{I}(0)}(x), t) \mathbf{n}_{\mathcal{I}(0)}(x) \quad \text{for all } x \in \mathbb{R}^d, \quad (98)$$

where  $P_{\mathcal{I}(t)}$  is the nearest point projection to  $\mathcal{I}(t)$ ,  $\zeta \in C^\infty(\mathbb{R}^d)$  satisfies

$$|\nabla \zeta(s)| \leq 4 \text{ in } \mathbb{R}^d \text{ and } \zeta(s) = 1 \text{ for } |s| \leq 1/2, \zeta(s) = 0 \text{ for } |s| \geq 1, \quad (99)$$

and  $h$  is the height function

$$h \in W_p^{2-\frac{1}{2p}}(J; L^p(\mathcal{I}(0))) \cap L^p(J; W_p^{4-\frac{1}{p}}(\mathcal{I}(0))) \text{ with } h|_{t=0} = 0. \quad (100)$$

Throughout the next three subsection, we sketch the main ingredients for a proof of Proposition 8 and thus conclude with the regularity stated in Definition 4.

**A.1. Preliminaries.** We will employ the following notation:

$$\mathring{W}_p^k(\Omega) = \{\partial^\alpha f \in L^p(\Omega), |\alpha| = k\}, \quad (101a)$$

$$W_p^k(\Omega) = \{\partial^\alpha f \in L^p(\Omega), |\alpha| \leq k\}, \quad (101b)$$

$$L_{(0)}^p(\Omega) = \left\{ f \in L^p(\Omega), \int_{\Omega} f \, dx = 0 \right\}. \quad (101c)$$

We need elementary results from interpolation theory and maximal regularity theory of parabolic system, see to this end, for instance, [15, Section 4.10] and [16]. For a Banach space  $(X, \|\cdot\|_X)$  and  $s \in (0, 1), p \geq 1$ , we also recall the Sobolev–Slobodeckij space  $W_p^s(J; X)$  normed by

$$\|f\|_{W_p^s(J; X)} := \|f\|_{L^p(J; X)} + [f]_{s,p} < \infty, \quad (102)$$

where  $J = [0, T]$  and

$$[f]_{s,p,X} := \left( \int_J \int_J \frac{\|f(t) - f(\tau)\|_X^p}{|t - \tau|^{ps+1}} dt d\tau \right)^{\frac{1}{p}}. \quad (103)$$

For  $s \in (m, m+1)$  with  $m$  being a positive integer, we finally define

$$\|f\|_{W_p^s(J; X)} := \|f\|_{W_p^m(J; X)} + \max_{|\alpha|=m} [\partial^\alpha f]_{s-m,p,X}. \quad (104)$$

Now we state the required interpolation inequalities involving these spaces. We denote by  $(E_0, E_1)_{\theta,p}$  the real interpolation of Banach spaces  $E_0$  and  $E_1$ . For  $\theta \in [0, 1]$  and Banach spaces  $X_0, X_1$ , we then have

$$W_p^1(J; X_0) \cap L_p(J; X_1) \hookrightarrow W_p^{1-\theta}(J; (X_0, X_1)_{\theta,p}), \quad (105)$$

$$W_p^1(J; X_0) \cap L_p(J; X_1) \hookrightarrow C^{1-\theta-\frac{1}{p}}(J; (X_0, X_1)_{\theta,p}), \quad (106)$$

where the second embedding holds provided  $1 - \theta \geq \frac{1}{p}$ . In particular,

$$W_p^1(J; X_0) \cap L_p(J; X_1) \hookrightarrow C^0(J; (X_0, X_1)_{1-1/p,p}). \quad (107)$$

We will also rely on the following inequality

$$\|f\|_{W_p^s(J; X)} \leq C_{s,s',p} T^{(s'-s)+\frac{1}{p}} \|f\|_{C^{s'}(J; X)} \quad (108)$$

provided that  $0 < s < s' \leq 1, 0 < T \leq 1$ . Indeed,

$$\begin{aligned} [f]_{s,p,X}^p &\stackrel{(103)}{=} \int_J \int_J \frac{\|f(t) - f(\tau)\|_X^p}{|t - \tau|^{ps+1}} dt d\tau \\ &\leq \int_J \int_J |t - \tau|^{p(s'-s)-1} dt d\tau \|f\|_{C^{s'}(J; X)}^p \\ &\leq C_{s',s,p} T^{p(s'-s)+1} \|f\|_{C^{s'}(J; X)}^p \end{aligned}$$

for all  $0 < s < s' \leq 1$ , and this implies (108).

**A.2. The work of Abels and Moser [8].** We shall first reduce (92) to a system with fixed domains. Assume  $\text{dist}(\mathcal{I}, \partial\Omega) > 4\delta$ . For  $h \in C^2(\mathcal{I}(0) \times J)$  with  $\|h\|_{L_{x,t}^\infty} < 2\delta$ , the Hanzawa transformation (98) is a family of diffeomorphisms

$$\Theta_h(\cdot, t): \mathcal{I}(0) \mapsto \mathcal{I}(t) \quad \text{and} \quad \mathring{\Omega}(0) \mapsto \mathring{\Omega}(t).$$

Recalling (99), we see that for a fixed  $t$ ,  $\Theta_h(\cdot, t)$  is the identity map on  $\mathbb{R}^d \setminus B_\delta(\mathcal{I}(t))$ ; in particular near  $\partial\Omega$ . Moreover  $\det \nabla \Theta_h \geq c > 0$  and

$$\|\nabla \Theta_h\|_{L_{x,t}^\infty} + \|\nabla \Theta_h^{-1}\|_{L_{x,t}^\infty} \leq C(1 + \|h\|_{C^1(\mathcal{I}(0) \times J)})$$

with  $c, C > 0$  independent of  $h$ . For technical reasons, it is simpler to replace  $h \circ P_{\mathcal{I}(0)}$  in (98) by an extension  $Eh$  where

$$E: X_0 := W_p^{1-1/p}(\mathcal{I}(0)) \mapsto W_p^1(B_\delta(\mathcal{I}(0))) \quad (109)$$

with  $E$  being a bounded and linear extension operator of class

$$E \in \mathcal{L}\left(W_p^{k-1/p}(\mathcal{I}(0)), W_p^k(B_\delta(\mathcal{I}(0)))\right), \quad \forall k \in \{1, \dots, 4\}. \quad (110)$$

Based on this extension operator, we define a modified Hanzawa transform by

$$y = \tilde{\Theta}_h(x, t) := x + \zeta \left( \frac{\text{dist}(x, \mathcal{I}(0))}{\delta} \right) (Eh)(x, t) \mathbf{n}_{\mathcal{I}(0)}(x), \quad \forall x \in \mathbb{R}^d. \quad (111)$$

Let  $(v, \pi)$  be a solution of (92). We then define the transformed solution by

$$\tilde{v}(x, t) := v(\tilde{\Theta}_h(x, t), t), \quad \tilde{\pi}(x, t) := \pi(\tilde{\Theta}_h(x, t), t), \quad (112)$$

for  $(x, t) \in \Omega \times (0, T)$ . Equivalently for  $y \in \tilde{\Omega}(t)$ , we have

$$v(y, t) := \tilde{v}(\tilde{\Theta}_h^{-1}(y, t), t), \quad \pi(y, t) := \tilde{\pi}(\tilde{\Theta}_h^{-1}(y, t), t). \quad (113)$$

To identify the system of PDEs satisfied by  $(\tilde{v}, \tilde{\pi})$ , we introduce the following notation:

$$\nabla^h := ((\nabla \tilde{\Theta}_h)^{-1})^\top \nabla, \quad (114a)$$

$$D^h \mathbf{v} := \frac{1}{2} (\nabla^h \mathbf{v} + (\nabla^h \mathbf{v})^\top), \quad (114b)$$

$$\text{div}^h \tilde{v} := \text{tr} \nabla^h \tilde{v}, \quad (114c)$$

$$T^h(\tilde{v}, \tilde{\pi}) := 2D^h \tilde{v} - \tilde{\pi} \text{Id}, \quad (114d)$$

$$H^h := H_{\mathcal{I}} \circ \tilde{\Theta}_h(x, t): \mathcal{I}(0) \times J \rightarrow \mathbb{R}^d, \quad (114e)$$

$$\mathbf{n}^h := \mathbf{n}_{\mathcal{I}} \circ \tilde{\Theta}_h(x, t): \mathcal{I}(0) \times J \rightarrow \mathbb{R}^d. \quad (114f)$$

Then it follows from (114a) and (113) that

$$(\nabla_y v)|_{y=\tilde{\Theta}_h(x,t)} = \nabla^h \tilde{v}(x). \quad (115)$$

Similarly, by (112) we obtain  $\partial_t \tilde{v}(x) = (\partial_t v + \partial_t \tilde{\Theta}_h \cdot \nabla v)|_{y=\tilde{\Theta}_h(x,t)}$ . With these definitions and formulas, the system (92) can be rewritten as one over a fixed domain. For simplicity we split it into two parts, one describing the hydrodynamics, and one for the evolution of the interface:

$$\partial_t \tilde{v} + \tilde{v} \cdot \nabla^h \tilde{v} = \text{div}^h \nabla^h \tilde{v} + \nabla^h \tilde{\pi} + \nabla^h \tilde{v} \partial_t \tilde{\Theta}_h \quad \text{in } \mathring{\Omega}(0) \times (0, T), \quad (116a)$$

$$\text{div}^h \tilde{v} = 0 \quad \text{in } \mathring{\Omega}(0) \times (0, T), \quad (116b)$$

$$\llbracket 2D^h \tilde{v} - \tilde{\pi} \text{Id} \rrbracket \cdot \mathbf{n}^h = H^h \mathbf{n}^h \quad \text{on } \mathcal{I}(0) \times (0, T), \quad (116c)$$

$$\llbracket \tilde{v} \rrbracket = 0 \quad \text{on } \mathcal{I}(0) \times (0, T), \quad (116d)$$

$$\tilde{v}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (116e)$$

$$\tilde{v}|_{t=0} = v_0 \quad \text{in } \Omega, \quad (116f)$$

as well as

$$\partial_t h = H^h + \tilde{v} \cdot \mathbf{n}^h \quad \text{on } \mathcal{I}(0) \times (0, T), \quad (117a)$$

$$h|_{t=0} = 0 \quad \text{on } \mathcal{I}(0). \quad (117b)$$

To state the regularity of solutions to the above system, we finally introduce the following function spaces:

$$\mathbb{V}_p := W_p^1(J; L^p(\Omega)) \cap L^p\left(J; W_p^2(\mathring{\Omega}(0)) \cap W_p^1(\Omega)\right), \quad (118a)$$

$$\mathbb{P}_p := \left\{ L^p\left(J; \mathring{W}_p^1(\mathring{\Omega}(0))\right) \mid \int_{\Omega} \tilde{\pi}(\cdot, t) = 0, \right. \\ \left. [\tilde{\pi}] \in W_p^{1-1/p, \frac{1}{2}(1-1/p)}(\mathcal{I}(0) \times (0, T)) \right\}, \quad (118b)$$

$$\mathbb{H}_p := W_p^1\left(J; W_p^{1-1/p}(\mathcal{I}(0))\right) \cap L^p\left(J; W_p^{3-1/p}(\mathcal{I}(0))\right). \quad (118c)$$

**Lemma 9.** *The following embeddings hold true:*

$$\mathbb{V}_p \hookrightarrow C(J; W_p^{2-2/p}(\mathring{\Omega}(0))), \quad (119a)$$

$$\mathbb{H}_p \hookrightarrow C\left(J; W_p^{3-3/p}(\mathcal{I}(0))\right) \hookrightarrow C(J; C^2(\mathcal{I}(0))). \quad (119b)$$

*Proof.* To compute the interpolation of Sobolev–Slobodeckij spaces we recall

$$(W_p^{m_1}(U), W_p^{m_2}(U))_{\theta, p} = W_p^s(U), \quad s = (1 - \theta)m_1 + \theta m_2, \quad (120)$$

where  $\theta \in (0, 1)$  and  $U$  is an open set or a closed compact manifold. We deduce from (120) that

$$\left(W_p^{1-1/p}(U), W_p^{3-1/p}(U)\right)_{1-1/p, p} = W_p^{3-\frac{3}{p}}(U) \hookrightarrow C^2(U), \quad (121)$$

$$(L_p(U), W_p^2(U))_{1-1/p, p} = W_p^{2-\frac{2}{p}}(U). \quad (122)$$

These combined with the interpolation inequality (107) leads to (119a) and the first embedding in (119b). Concerning the second embedding in (119b), as  $p > d + 2$  and  $\mathcal{I}(0)$  is  $(d-1)$ -dimensional, we have  $3 - \frac{3}{p} - \frac{d-1}{p} > 2$ . Then the result follows from Morrey’s embedding.  $\square$

The following result is a simpler version of Abels and Moser [8, Theorem 4.1].

**Proposition 10.** *Let the assumptions of Proposition 8 be in place. Let  $p > d + 2$  and  $2 < q < 3$  with  $1 + \frac{d+2}{p} > \frac{d+2}{q}$ . Then there exists  $T > 0$  such that the system consisting of (116) and (117) has a unique solution with*

$$(\tilde{v}, \tilde{\pi}, h) \in \mathbb{V}_q \times \mathbb{P}_q \times \mathbb{H}_p. \quad (123)$$

We make a comment on the regularity of  $(\tilde{v}, \tilde{\pi})$ .

**Remark 11.** According to the regularity of  $\tilde{v}$  by (118a) and the embedding (119a), we must have  $v_0(\cdot) = \tilde{v}(\cdot, 0) \in W_q^{2-2/q}(\mathring{\Omega}(0))$ . If  $q > 3$ , then the trace estimate implies  $\nabla v_0 \in W_q^{1-2/q}(\mathring{\Omega}(0)) \hookrightarrow W_q^{1-3/q}(\mathcal{I}(0))$ . In order to guarantee that the restrictions from outer and inner domains  $\Omega^\pm(0)$  give the same trace, an additional compatibility condition must be added. Such a condition is caused by the possible jump of  $\nabla v_0$  across  $\mathcal{I}(0)$ . To avoid such a compatibility condition, Abels and Moser [8] assume  $q < 3$ , and this causes unbalanced regularities (123). However, in

the current system (116), we work in the regime of coinciding shear viscosities of the two phases. It turns out that this assumption necessitates continuity of the flow gradient across the interface, see Lemma 12 below. In particular, we can simply assume  $v_0 \in W_q^{2-2/q}(\Omega)$ .

**Lemma 12.** *Under the assumption of coinciding shear viscosities  $\mu_+ = \mu_-$  of the two fluid phases, it follows that the flow gradient  $\nabla v$  is continuous across  $\mathcal{I}$ , i.e.,  $[[\nabla v]] = 0$ . As a result, we can improve (97a) to*

$$v \in W_p^1(J; L^p(\Omega)) \cap L^p(J; W_p^2(\Omega)). \quad (124)$$

*Proof.* In the proof, we shall abbreviate  $\mathbf{n}_{\mathcal{I}}$  by  $\mathbf{n}$ . It follows from (97a) that

$$\nabla v \in W_p^1(\dot{\Omega}(t)) \text{ for a.e. } t \in [0, T], \quad (125)$$

and thus  $[[\nabla v]]$  makes sense. By (92d), we deduce that the tangential derivative of  $v$  do not jump across  $\mathcal{I}$ , i.e.,

$$[[\nabla^{\text{tan}} v_i]] = 0 \text{ on } \mathcal{I} \text{ for } 1 \leq i \leq d, \quad (126)$$

where  $\nabla^{\text{tan}} v_i := (\text{Id} - \mathbf{n} \otimes \mathbf{n}) \nabla v_i$  is the tangential gradient of  $v_i$ . Thus, one can verify that  $\nabla \cdot v = 0$  in  $\Omega$  in the sense of distribution (see [13, Lemma 2.5] for an even more general situation), i.e., for any  $\varphi \in C_c^1(\Omega)$  it holds  $\int_{\Omega} v \cdot \nabla \varphi \, dx = 0$ . This in turn implies by the regularity of  $v$  that  $\nabla \cdot v = 0$  a.e. in  $\Omega$ , and therefore by (126)

$$[[\mathbf{n} \otimes \mathbf{n} : \nabla v]] = [[(\mathbf{n} \otimes \mathbf{n} - \text{Id}) : \nabla v]] + [[\nabla \cdot v]] = 0. \quad (127)$$

Next, multiplying (92c) by any vector field  $\mathbf{t}$  orthogonal to  $\mathbf{n}$ , we deduce from (126)

$$0 = [[2\nabla^{\text{sym}} v : \mathbf{n} \otimes \mathbf{t}]] = [[\nabla v : \mathbf{t} \otimes \mathbf{n}]]. \quad (128)$$

Hence, the claim of Lemma 12 follows from (126), (127) and (128).  $\square$

**A.3. Proof of Proposition 8.** The regularities of  $(\tilde{v}, \tilde{\pi})$  and  $\tilde{h}$  in (123) are not balanced. It remains to show further integrability of  $(\tilde{v}, \tilde{\pi})$  by solving (116) separately under some additional compatibility conditions. A complete proof of this will be quite lengthy and technical, and we thus only give a sketch of the proof. To this end, we write the system (116) in an abstract form

$$\mathcal{L} \begin{pmatrix} \tilde{v} \\ \tilde{\pi} \end{pmatrix} = \mathcal{N} \begin{pmatrix} \tilde{v} \\ \tilde{\pi} \end{pmatrix} \quad (129)$$

where

$$\mathcal{L} \begin{pmatrix} \tilde{v} \\ \tilde{\pi} \end{pmatrix} := \begin{pmatrix} \partial_t \tilde{v} - \Delta \tilde{v} - \nabla \tilde{\pi} \\ \text{div } \tilde{v} \\ [[2D\tilde{v} - \tilde{\pi}\text{Id}]] \mathbf{n}_{\mathcal{I}} \\ \tilde{v}|_{t=0} \end{pmatrix} \quad (130)$$

is the linearized operator and  $\mathcal{N}$  is the nonlinear one:

$$\mathcal{N} \begin{pmatrix} \tilde{v} \\ \tilde{\pi} \end{pmatrix} := \begin{pmatrix} \mathbf{a}(\tilde{v}, \tilde{\pi}, h) - \tilde{v} \cdot \nabla^h \tilde{v} + \nabla^h \tilde{v} \partial_t \tilde{\Theta}_h \\ (\text{div } \tilde{v} - \text{div}^h \tilde{v}) - \int_{\Omega} (\text{div } \tilde{v} - \text{div}^h \tilde{v}) \, dx \\ \mathbf{b}(\tilde{v}, \tilde{\pi}, h) + H^h \mathbf{n}^h \\ v_0 \end{pmatrix}. \quad (131)$$

In (131), we used the definitions from (114) and

$$\mathbf{a}(\tilde{v}, \tilde{\pi}, h) := (\text{div}^h \nabla^h \tilde{v} - \Delta \tilde{v}) + (\nabla^h - \nabla) \tilde{\pi}, \quad (132a)$$



$$\mathbf{b}(\tilde{v}, \tilde{\pi}, h) := \llbracket 2D\tilde{v} - \tilde{\pi} \text{Id} \rrbracket (\mathbf{n}_{\mathcal{I}} - \mathbf{n}^h) + 2\llbracket D\tilde{v} - D^h\tilde{v} \rrbracket \mathbf{n}^h. \quad (132b)$$

**Remark 13.** It is easy to verify that the operator equation (129) is equivalent to (116) except that the second equation in the latter, i.e.,  $\text{div}^h \tilde{v} = 0$ , is replaced by

$$\text{div}^h \tilde{v} = \rlap{-}\int_{\Omega} \text{div}^h \tilde{v} \, dx \quad (133)$$

after simplification. It is obvious that the former equation implies the latter one. The opposite direction is proved in [12, p. 51]:

$$\begin{aligned} & \rlap{-}\int_{\Omega} \text{div}^h \tilde{v} \, dx \int_{\Omega} |\det \nabla \tilde{\Theta}(x, t)| \, dx \\ & \stackrel{(133)}{=} \int_{\Omega} \text{div}^h \tilde{v} |\det \nabla \tilde{\Theta}(x, t)| \, dx \\ & \stackrel{(115)}{=} \int_{\Omega} (\text{div}_y v)|_{y=\tilde{\Theta}_h(x,t)} |\det \nabla \tilde{\Theta}(x, t)| \, dx \\ & = \int_{\Omega} \text{div}_y v(y) \, dy = 0. \end{aligned}$$

Recall from (117) that  $h$  starts from 0. By the regularity  $h \in \mathbb{H}_p$ , cf. (118c), it can be shown that, within a short time period  $[0, T]$ , the nonlinear operator  $\mathcal{N}$  defined by (131) is locally Lipschitz continuous between a pair of Banach spaces. Moreover, we shall show that  $\mathcal{L}$  is an isomorphism between this pair. These two results together with Banach's fixed point theorem then lead to the proof of Proposition 8.

The invertibility of the linear operator  $\mathcal{L}$  from (130) corresponds to the solvability of the following linear system:

$$\partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{\pi} = f \quad \text{in } \mathring{\Omega}(0) \times (0, T), \quad (134a)$$

$$\text{div } \tilde{v} = g \quad \text{in } \mathring{\Omega}(0) \times (0, T), \quad (134b)$$

$$\llbracket 2D\tilde{v} - \tilde{\pi} \text{Id} \rrbracket \mathbf{n}_{\mathcal{I}} = w \quad \text{on } \mathcal{I}(0) \times (0, T), \quad (134c)$$

$$\llbracket \tilde{v} \rrbracket = 0 \quad \text{on } \mathcal{I}(0) \times (0, T), \quad (134d)$$

$$\tilde{v}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (134e)$$

$$\tilde{v}|_{t=0} = \tilde{v}_0 \quad \text{in } \Omega. \quad (134f)$$

Here, we assume  $(f, g, w) \in L^p_{x,t} \times \mathbb{G}_p \times \mathbb{W}_p$  where

$$\mathbb{G}_p := \left\{ g \in L^p \left( 0, T; W_p^1(\mathring{\Omega}(0)) \right) \mid \int_{\Omega} g(t, \cdot) = 0 \text{ for a.e. } t \in (0, T) \right\}, \quad (135a)$$

$$\mathbb{W}_p := W_p^{\frac{1}{2}(1-\frac{1}{p})} \left( 0, T; L^p(\mathcal{I}(0)) \right) \cap L^p \left( 0, T; W_p^{1-\frac{1}{p}}(\mathcal{I}(0)) \right), \quad (135b)$$

and we assume that  $(g, w, \tilde{v}_0)$  satisfy the following compatibility conditions:

$$\tilde{v}_0 \in W_p^{2-2/p}(\mathring{\Omega}(0)), \quad (136a)$$

$$\text{div } \tilde{v}_0 = g|_{t=0} \quad \text{in } \mathring{\Omega}(0) \quad (136b)$$

$$\tilde{v}_0|_{\partial\Omega} = 0, \quad (136c)$$

$$\llbracket \tilde{v}_0 \rrbracket = 0 \quad \text{on } \mathcal{I}(0), \quad (136d)$$

$$\Pi_{\mathcal{I}} (\llbracket 2D\tilde{v}_0 \rrbracket \cdot \mathbf{n}_{\mathcal{I}}) \Big|_{t=0} = \Pi_{\mathcal{I}} w \Big|_{t=0}. \quad (136e)$$

**Remark 14.** We note that the integral constraint in (135a) follows by integrating (134b) and using (134d) and (134e). The compatibility condition (136b) follows by taking  $t = 0$  in (134b). Finally, (136e) is a consequence of taking the tangential projection of (134c) and then restricting it to  $t = 0$ .

The solvability of the system (134) is due to the following maximal regularity result in [54].

**Lemma 15.** *Assuming the compatibility conditions (136), the system (134) has a unique solution with the following estimate under notations in (118):*

$$\|(\tilde{v}, \tilde{\pi})\|_{\mathbb{V}_p \times \mathbb{P}_p} \leq C \left( \|f\|_{L^p_{x,t}} + \|(g, w)\|_{\mathbb{G}_p \times \mathbb{W}_p} + \|\tilde{v}_0\|_{W_p^{2-2/p}(\dot{\Omega}(0))} \right) \quad (137)$$

where  $C$  is a constant depending on the geometry of  $\dot{\Omega}(0)$ .

Note that the resolvent estimate for (134) in bent half-spaces is given in [53, Theorem 6.1], which implies the maximal regularity of the instationary system when  $p = 2$  (see [12] for a short proof). However, the  $L^p$  maximal regularity of the instationary system does not follow directly from the corresponding resolvent estimate.

To show the contraction property of (131), we need to study the regularity of the mapping  $(\nabla \tilde{\Theta}_h)^{-1} - \text{Id}$ .

**Lemma 16.** *Under the regularity assumption  $h \in \mathbb{H}_p$ , cf. (118c), we have*

$$\left\| (\nabla \tilde{\Theta}_h)^{-1} - \text{Id} \right\|_{C([0,T], W_p^{2-\frac{2}{p}}(\Omega))} + \left\| (\nabla \tilde{\Theta}_h)^{-1} - \text{Id} \right\|_{W_p^{\frac{1}{2}}(0,T; L^p(\Omega))} \xrightarrow{T \rightarrow 0} 0. \quad (138)$$

*Proof.* Recalling (118c) and (110), we have

$$Eh \in W_p^1(0, T; W_p^1(B_\delta(\mathcal{I}(0)))) \cap L^p(0, T; W_p^3(B_\delta(\mathcal{I}(0)))) . \quad (139)$$

By (111), we obtain the same regularity for  $\tilde{\Theta}$ . So using (105) and (107) yields

$$\tilde{\Theta} \in C([0, T]; W_p^{3-\frac{2}{p}}(\Omega)) \cap W_p^{\frac{1}{2}}(0, T; W_p^2(\Omega)). \quad (140)$$

As  $Eh|_{t=0} = 0$ , for a sufficiently small  $T$ , we deduce from (111) and (140) that

$$\sup_{x \in \Omega, t \in [0, T]} |\nabla \tilde{\Theta}_h - \text{Id}| \leq 1/2. \quad (141)$$

By a Taylor expansion, we have

$$(\nabla \tilde{\Theta}_h)^{-1} - \text{Id} = \mathfrak{F}(\nabla \tilde{\Theta}_h - \text{Id}) \quad (142)$$

where  $\mathfrak{F}(B) = \sum_{k=1}^{\infty} (-1)^k B^k$  is a smooth matrix-valued function for  $|B| < 1$ . This combined with (140) and (111) yields

$$\begin{aligned} & \left\| (\nabla \tilde{\Theta}_h)^{-1} - \text{Id} \right\|_{C([0,T], W_p^{2-2/p}(\Omega))} + \left\| (\nabla \tilde{\Theta}_h)^{-1} - \text{Id} \right\|_{W_p^{1/2}(0,T; L^p(\Omega))} \\ & \leq \|Eh\|_{C([0,T], W_p^{3-2/p}(B_\delta(\mathcal{I}(0))))} + \|Eh\|_{W_p^{1/2}(0,T; W_p^1(B_\delta(\mathcal{I}(0))))} . \end{aligned}$$

The first term on the right hand side vanishes as  $T \downarrow 0$  because of (140) and  $Eh|_{t=0} = 0$ . Concerning the second one, we have for  $s' = \frac{1}{2} + \epsilon$ ,  $0 < \epsilon \ll 0$  and  $p > d + 2$  that

$$\|Eh\|_{W_p^{\frac{1}{2}}(0,T; W_p^1)} \stackrel{(108)}{\leq} C_{s', p} T^{s' - \frac{1}{2} + \frac{1}{p}} \|Eh\|_{C^{s'}([0,T]; W_p^1)}$$

$$\begin{aligned} &\leq C_{s',p} T^{s' - \frac{1}{2} + \frac{1}{p}} \|Eh\|_{W_p^{s'+\epsilon+1/p}([0,T];W_p^1)} \\ &\stackrel{(139)}{\leq} C_{s',p} T^{s' - \frac{1}{2} + \frac{1}{p}} C_{Eh}. \end{aligned}$$

With this, we may conclude with our sketch of the proof.  $\square$

**Lemma 17.** *Under the regularity assumption  $h \in \mathbb{H}_p$ , cf. (118c), the mapping (131)*

$$\mathcal{N} : \mathbb{V}_p \times \mathbb{P}_p \mapsto L_{x,t}^p \times \mathbb{G}_p \times \mathbb{W}_p \times W_p^{2-2/p}(\mathring{\Omega}(0)) =: \mathbb{N}_p \quad (143)$$

is locally Lipschitz. Moreover, for

$$(v_i, \pi_i) \in B_R(\mathbb{V}_p \times \mathbb{P}_p) \quad \text{with } 1 \leq i \leq 2, \quad (144)$$

we have the following estimate:

$$\left\| \mathcal{N} \begin{pmatrix} v_1 \\ \pi_1 \end{pmatrix} - \mathcal{N} \begin{pmatrix} v_2 \\ \pi_2 \end{pmatrix} \right\|_{\mathbb{N}_p} \leq C(R, T) \|(v_1 - v_2, \pi_1 - \pi_2)\|_{\mathbb{V}_p \times \mathbb{P}_p}, \quad (145)$$

where  $\lim_{T \downarrow 0} C(R, T) = 0$  for each fixed  $R > 0$ .

*Proof.* A full proof of (145) will be quite lengthy and technical, so we again only sketch a few key steps here.

We first deduce from (139) and  $Eh|_{t=0} = 0$  that

$$\mathfrak{h}(\tau) := \|Eh\|_{C([0,\tau];W_p^{3-2/p}(B_\delta(\mathcal{I}(0))))} \quad (146)$$

is continuous on  $[0, T)$  and that  $\lim_{\tau \downarrow 0} \mathfrak{h}(\tau) = 0$ . To verify the Lipschitz continuity (145), we start from the only nonlinear term  $\tilde{v} \cdot \nabla^h \tilde{v}$  of it. Let  $\tilde{v} = v_1 - v_2$ . Then

$$\begin{aligned} &\|v_1 \cdot \nabla^h v_1 - v_2 \cdot \nabla^h v_2\|_{L_{x,t}^p} \\ &= \|\tilde{v} \cdot \nabla^h v_1 - v_2 \cdot \nabla^h \tilde{v}\|_{L_{x,t}^p} \\ &\leq C_1(R) (1 + \mathfrak{h}(T)) T^{\frac{1}{p}} \|\tilde{v}\|_{C([0,T],W_p^1)} \quad \text{by (144) and (138)} \\ &\leq C_2(R) T^{\frac{1}{p}} \|\tilde{v}\|_{\mathbb{V}_p}. \end{aligned}$$

All the remaining terms defining (131) are linear. We shall merely estimate  $\llbracket D\tilde{v} - D^h \tilde{v} \rrbracket \mathbf{n}^h$  in (132b), since all the other terms can be treated in a similar way. Note that  $W_p^1(\Omega)$  is a Banach algebra for  $p > d + 2$ . Then

$$\begin{aligned} &\|\llbracket Dv_1 - D^h v_1 \rrbracket - \llbracket Dv_2 - D^h v_2 \rrbracket\|_{\mathbb{W}_p} \\ &\stackrel{(114b)}{\lesssim} \left\| ((\nabla \tilde{\Theta}_h)^{-1} - \text{Id}) \llbracket \nabla \tilde{v} \rrbracket \right\|_{\mathbb{W}_p} \\ &\stackrel{(135b)}{\lesssim} \left\| ((\nabla \tilde{\Theta}_h)^{-1} - \text{Id}) \nabla \tilde{v} \right\|_{L^p(0,T;W_p^1(\mathring{\Omega}(0))) \cap W_p^{1/2}(0,T;L^p(\Omega))} \\ &\leq \|(\nabla \tilde{\Theta}_h)^{-1} - \text{Id}\|_{C([0,T],W_p^{2-2/p}(\Omega))} \|\tilde{v}\|_{L^p(0,T;W_p^2(\mathring{\Omega}(0))) \cap W_p^{1/2}(0,T;W_p^1(\mathring{\Omega}(0)))} \\ &\quad + \left\| (\nabla \tilde{\Theta}_h)^{-1} - \text{Id} \right\|_{W_p^{1/2}(0,T;L^p(\Omega(0)))} \|\nabla \tilde{v}\|_{L_{x,t}^\infty} \\ &\stackrel{(138),(119a)}{\leq} C(T) \|\tilde{v}\|_{W_p^1(0,T;L^p(\mathring{\Omega}(0))) \cap L^p(0,T;W_p^2(\mathring{\Omega}(0)))} \end{aligned}$$

with  $C(T) \xrightarrow{T \rightarrow 0} 0$ .  $\square$

*Proof of Proposition 8.* Combining the above two lemmas, we deduce  $(\tilde{v}, \tilde{\pi}) \in \mathbb{V}_p \times \mathbb{P}_p$  (cf. (118)) by a fixed point argument. To obtain further regularity of  $h$ , we recall from (118a) that

$$\tilde{v} \in \mathbb{V}_p \hookrightarrow W_p^{1-\frac{1}{2p}}(J; L^p(\mathcal{I}(0))) \cap L^p(J; W_p^{2-\frac{1}{p}}(\mathcal{I}(0))). \quad (147)$$

Hence, by solving the quasilinear parabolic equation (117) we can improve the regularity of  $h \in \mathbb{H}_p$ , cf. (118c), to (100). This in turn is sufficient to conclude with the regularity stated in Definition 4.  $\square$

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#### REFERENCES

- [1] H. Abels, *(Non-)convergence of solutions of the convective Allen–Cahn equation*, Partial Differ. Equ. Appl. **3** (2022), no. 1.
- [2] H. Abels and M. Fei, *Sharp interface limit for a Navier–Stokes/Allen–Cahn system with different viscosities*, arXiv preprint (2022). [arXiv:2201.09343](https://arxiv.org/abs/2201.09343).
- [3] H. Abels, H. Garcke, and G. Grün, *Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities*, Math. Models Methods Appl. Sci. **22** (2012), no. 3, 1150013 (40 pages).
- [4] H. Abels and D. Lengeler, *On sharp interface limits for diffuse interface models for two-phase flows*, Interfaces Free Bound. **16** (2014), no. 3, 395–418. MR3264795
- [5] H. Abels and Y. Liu, *Sharp interface limit for a Stokes/Allen–Cahn System*, Arch. Ration. Mech. Anal. **229** (2018), no. 1, 417–502.
- [6] H. Abels and A. Marquardt, *Sharp interface limit of a Stokes/Cahn–Hilliard system. Part I: Convergence result*, Interfaces Free Bound. **23** (2021), no. 3, 353–402.
- [7] ———, *Sharp interface limit of a Stokes/Cahn–Hilliard system, Part II: Approximate solutions*, J. Math. Fluid Mech. **23** (2021mar), no. 2.
- [8] H. Abels and M. Moser, *Well-posedness of a Navier–Stokes/mean curvature flow system*, Mathematical analysis in fluid mechanics: selected recent results. international conference on vorticity, rotation and symmetry (iv) – complex fluids and the issue of regularity, cirm, luminy, marseille, france, may 8–12, 2017. proceedings, 2018, pp. 1–23 (English).
- [9] ———, *Convergence of the Allen–Cahn equation to the mean curvature flow with 90°-contact angle in 2D*, Interfaces Free Bound. **21** (2019), no. 3, 313–365.
- [10] ———, *Convergence of the Allen–Cahn equation with a nonlinear Robin boundary condition to mean curvature flow with contact angle close to 90°*, SIAM J. Math. Anal. **54** (2022), no. 1, 114–172.
- [11] H. Abels and M. Röger, *Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 6, 2403–2424.
- [12] H. Abels and M. Wilke, *Well-posedness and qualitative behaviour of solutions for a two-phase Navier–Stokes–Mullins–Sekerka system*, Interfaces Free Bound. **15** (2013), no. 1, 39–75. MR3062573
- [13] G. Alberti, G. Bouchitté, and G. Dal Maso, *The calibration method for the Mumford–Shah functional and free-discontinuity problems*, Calc. Var. Partial Differential Equations **16** (2003), no. 3, 299–333.
- [14] N. D. Alikakos, P. W. Bates, and X. Chen, *Convergence of the Cahn–Hilliard equation to the Hele–Shaw model*, Arch. Ration. Mech. Anal. **128** (1994), no. 2, 165–205.
- [15] H. Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory. MR1345385
- [16] ———, *Compact embeddings of vector-valued Sobolev and Besov spaces*, 2000, pp. 161–177. Dedicated to the memory of Branko Najman. MR1783238

- [17] G. Barles and F. Da Lio, *A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions*, *Interfaces Free Bound.* **5** (2003), no. 3, 239–274.
- [18] G. Barles and P. E. Souganidis, *A new approach to front propagation problems: Theory and applications*, *Arch. Ration. Mech. Anal.* **141** (1998), no. 3, 237–296.
- [19] X. Chen, *Global asymptotic limit of solutions of the Cahn-Hilliard equation*, *J. Differential Geom.* **44** (1996), no. 2, 262–311.
- [20] P. De Mottoni and M. Schatzman, *Geometrical evolution of developed interfaces*, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1533–1589.
- [21] L. C. Evans, H. M. Soner, and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, *Comm. Pure Appl. Math.* **45** (1992), no. 9, 1097–1123.
- [22] J. Fischer and S. Hensel, *Weak-strong uniqueness for the Navier–Stokes equation for two fluids with surface tension*, *Arch. Ration. Mech. Anal.* **236** (2020), no. 2, 967–1087.
- [23] J. Fischer, S. Hensel, T. Laux, and T. M. Simon, *The local structure of the energy landscape in multiphase mean curvature flow: Weak-strong uniqueness and stability of evolutions*, arXiv preprint (2020). [arXiv:2003.05478v2](https://arxiv.org/abs/2003.05478v2).
- [24] J. Fischer, T. Laux, and T. M. Simon, *Convergence rates of the Allen–Cahn equation to mean curvature flow: A short proof based on relative entropies*, *SIAM J. Math. Anal.* **52** (2020), no. 6, 6222–6233.
- [25] J. Fischer and A. Marveggio, *Quantitative convergence of the vectorial Allen–Cahn equation towards multiphase mean curvature flow*, arXiv preprint (2022). [arXiv:2203.17143](https://arxiv.org/abs/2203.17143).
- [26] C. G. Gal and M. Grasselli, *Longtime behavior for a model of homogeneous incompressible two-phase flows*, *Discrete Contin. Dyn. Syst.* **28** (2010), no. 1, 1–39.
- [27] M. E. Gurtin, D. Polignone, and J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, *Math. Models Methods Appl. Sci.* **6** (1996), no. 6, 815–831.
- [28] S. Hensel and T. Laux, *BV solutions to mean curvature flow with constant contact angle: Allen–Cahn approximation and weak-strong uniqueness*, In (minor) revision at Indiana Univ. Math. J. (2021). [arXiv:2112.11150](https://arxiv.org/abs/2112.11150).
- [29] ———, *A new varifold solution concept for mean curvature flow: Convergence of the Allen–Cahn equation and weak-strong uniqueness*, arXiv preprint (2021). [arXiv:2109.04233](https://arxiv.org/abs/2109.04233).
- [30] ———, *Weak-strong uniqueness for the mean curvature flow of double bubbles*, In (minor) revision at *Interfaces Free Bound.* (2021). [arXiv:2108.01733](https://arxiv.org/abs/2108.01733).
- [31] S. Hensel and A. Marveggio, *Weak-strong uniqueness for the Navier–Stokes equation for two fluids with ninety degree contact angle and same viscosities*, In (minor) revision at *J. Math. Fluid Mech.* (2021). [arXiv:2112.11154](https://arxiv.org/abs/2112.11154).
- [32] S. Hensel and M. Moser, *Convergence rates for the Allen–Cahn equation with boundary contact energy: The non-perturbative regime*, arXiv preprint (2021). [arXiv:2112.11173](https://arxiv.org/abs/2112.11173).
- [33] P. C. Hohenberg and B. I. Halperin, *Theory of dynamic critical phenomena*, *Rev. Mod. Phys.* **49** (1977), 435–479.
- [34] T. Ilmanen, *Convergence of the Allen–Cahn equation to Brakke’s motion by mean curvature*, *J. Differential Geom.* **38** (1993), no. 2, 417–461.
- [35] R. L. Jerrard and D. Smets, *On the motion of a curve by its binormal curvature*, *J. Eur. Math. Soc.* **17** (2015), no. 6, 1487–1515.
- [36] J. Jiang, Y. Li, and C. Liu, *Two-phase incompressible flows with variable density: An energetic variational approach*, *Discrete Contin. Dyn. Syst. - A* **37** (2017), no. 6, 3243–3284.
- [37] S. Jiang, X. Su, and F. Xie, *Remarks on sharp interface limit for an incompressible Navier–Stokes and Allen–Cahn coupled system*, arXiv preprint (2022). [arXiv:2205.01301](https://arxiv.org/abs/2205.01301).
- [38] T. Kagaya, *Convergence of the Allen–Cahn equation with a zero Neumann boundary condition on non-convex domains*, *Math. Ann.* **373** (2018), no. 3-4, 1485–1528.
- [39] M. Katsoulakis, G. T. Kossioris, and F. Reitich, *Generalized motion by mean curvature with Neumann conditions and the Allen–Cahn model for phase transitions*, *J. Geom. Anal.* **5** (1995), no. 2, 255–279.
- [40] M. Köhne, J. Prüss, and M. Wilke, *Qualitative behaviour of solutions for the two-phase Navier–Stokes equations with surface tension*, *Math. Ann.* **356** (2013), no. 2, 737–792.
- [41] M. Kroemer and T. Laux, *The Hele–Shaw flow as the sharp interface limit of the Cahn–Hilliard equation with disparate mobilities*, arXiv preprint (2021). [arXiv:2111.14505](https://arxiv.org/abs/2111.14505).

- [42] T. Laux and Y. Liu, *Nematic–isotropic phase transition in liquid crystals: A variational derivation of effective geometric motions*, Arch. Ration. Mech. Anal. **241** (2021), no. 3, 1785–1814.
- [43] T. Laux and T. M. Simon, *Convergence of the Allen–Cahn equation to multiphase mean curvature flow*, Comm. Pure Appl. Math. **71** (2018), no. 8, 1597–1647.
- [44] N. Q. Le, *A gamma-convergence approach to the Cahn–Hilliard equation*, Calc. Var. Partial Differential Equations **32** (2008), no. 4, 499–522.
- [45] F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math. **48** (1995), no. 5, 501–537. MR1329830
- [46] C. Liu and J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Phys. D **179** (2003), no. 3-4, 211–228.
- [47] Y. Liu, *Sharp interface limit of an anisotropic Ginzburg–Landau equation*, arXiv preprint (2021). [arXiv:2111.15061](https://arxiv.org/abs/2111.15061).
- [48] M. Mizuno and Y. Tonegawa, *Convergence of the Allen–Cahn equation with Neumann boundary conditions*, SIAM J. Math. Anal. **47** (2015), no. 3, 1906–1932.
- [49] L. Mugnai and M. Röger, *The Allen–Cahn action functional in higher dimensions*, Interfaces Free Bound. **10** (2008), no. 1, 45–78.
- [50] N. Owen and P. Sternberg, *Gradient flow and front propagation with boundary contact energy*, Proc. Math. Phys. Sci. **437** (1992), 715–728.
- [51] E. Sandier and S. Serfaty, *Gamma-convergence of gradient flows with applications to Ginzburg–Landau*, Comm. Pure Appl. Math. **57** (2004), no. 12, 1627–1672.
- [52] S. Serfaty, *Gamma-convergence of gradient flows on Hilbert and metric spaces and applications*, Discrete Contin. Dyn. Syst. - A **31** (2011), no. 4, 1427–1451.
- [53] Y. Shibata and S. Shimizu, *On a resolvent estimate of the interface problem for the Stokes system in a bounded domain*, J. Differential Equations **191** (2003), no. 2, 408–444. MR1978384
- [54] S. Shimizu, *Maximal regularity and viscous incompressible flows with free interface*, Parabolic and Navier-Stokes equations. Part 2, 2008, pp. 471–480. MR2548875

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